On an inverse crack problem in a linearized elasticity by the enclosure method

Hiromichi Itou

(Tokyo University of Science, Japan)

Collaborators:

- Masaru Ikehata (Hiroshima University, Japan)

第45回偏微分方程式論札幌シンポジウム

2020年8月19日

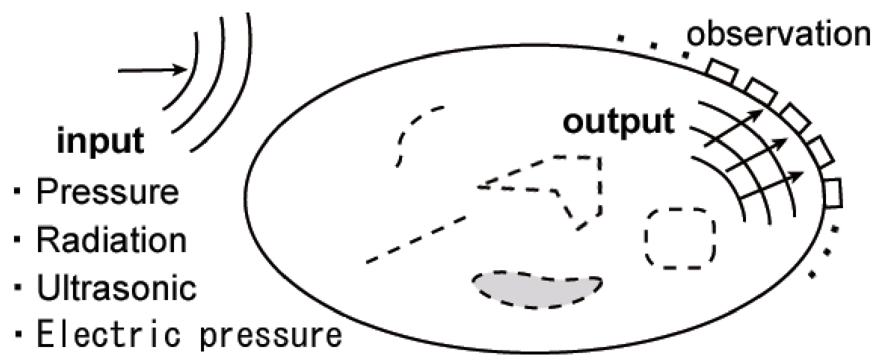
Contents

1. Introduction

● Motivation

逆問題とは、様々な物理量からなる観測データから未知の対象に関する情報を抽出する問題。多くの場合、数学的には偏微分方程式の境界値逆問題として定式化されるが、ほとんどが非適切問題や非線形問題であるので、解析は困難である。逆問題の研究は、材料や建築（非破壊検査）の分野だけでなく、医療診断や医用画像（ＣＴやＭＲＩなど）、資源探査、地球科学（地球の内部構造の決定）など広汎な応用がある。

このように逆問題の研究は、実用上重要であるだけでなく、偏微分方程式の解の構造の深い理解を必要とするので、純粋な数学としても興味深い問題である。



- **Inverse crack problems:**

Single crack case

$\Omega \subset \mathbb{R}^2$: a bounded, homogeneous, isotropic, linearized elasticity,

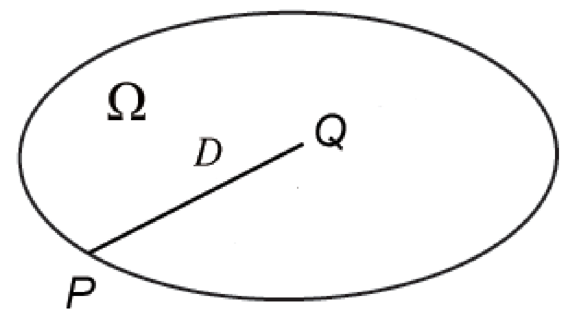
$\partial\Omega$: Lipschitz

D : an unknown linear crack PQ

$P \in \partial\Omega$ (known),

$Q \in \Omega$ (unknown)

"Find Q from boundary data"



Isotropic: [5] M. I. & H. I. 2007 Inverse Problems 23 589-607

Anisotropic: [6] M. I. & H. I. 2008 Inverse Problems 24 025005

The linearized elasticity eq. (the Navier eq.) :

For the displacement vector $u = (u_1, u_2)$

$$Au := \mu \triangle u + (\lambda + \mu)\nabla(\nabla \cdot u) = 0$$

λ, μ : *Lamé* constants in Ω satisfying $\mu > 0$ and $\lambda + \mu > 0$. And we define $\kappa = (\lambda + 3\mu)/(\lambda + \mu)$.

- The linearized strain tensor:
  $\varepsilon = (\varepsilon_{ij})_{i,j=1,2} = 1/2\{\nabla u + (\nabla u)^{\mathrm{T}}\}$
- The stress tensor: $\sigma = (\sigma_{ij})_{i,j=1,2} = \lambda(\mathrm{tr}\varepsilon)I + \mu\varepsilon$
  The stress vector: $Tu := \sigma\nu$,
  $\nu = (\nu_1, \nu_2)$: the unit outward normal to $\partial(\Omega \setminus \overline{D})$
- Rigid displacements: $\rho(x) = (k_1 + k_0 x_2, k_2 - k_0 x_1) \in \mathcal{R}$
  with a constant vector $k = (k_1, k_2, k_0)$.

For given $g \in L^2(\partial\Omega)$

$$(*)_1 \begin{cases} Au = 0 & \text{in } \Omega \setminus \overline{D} \\ Tu = 0 & \text{on } D \\ Tu = g & \text{on } \partial\Omega \end{cases}$$

Key :

Behaviour of the solution of $(*)_1$ near a crack tip

Proposition 1 $\exists A_n, B_n \in \mathbb{R}$ s. t.

$$u(r,\theta) = \sum_{n=0}^{\infty} \frac{1}{2\mu} r^{\frac{n}{2}} \left(A_n \varphi_n(\theta) - B_n \psi_n(\theta) \right),$$

$$\varphi_n(\theta) = \begin{pmatrix} \kappa \cos \frac{n}{2}\theta - \frac{n}{2} \cos\left(\frac{n}{2} - 2\right)\theta + \left\{\frac{n}{2} + (-1)^n\right\} \cos \frac{n}{2}\theta \\ \kappa \sin \frac{n}{2}\theta + \frac{n}{2} \sin\left(\frac{n}{2} - 2\right)\theta - \left\{\frac{n}{2} + (-1)^n\right\} \sin \frac{n}{2}\theta \end{pmatrix}$$

$$\psi_n(\theta) = \begin{pmatrix} \kappa \sin \frac{n}{2}\theta - \frac{n}{2} \sin\left(\frac{n}{2} - 2\right)\theta + \left\{\frac{n}{2} - (-1)^n\right\} \sin \frac{n}{2}\theta \\ -\kappa \cos \frac{n}{2}\theta - \frac{n}{2} \cos\left(\frac{n}{2} - 2\right)\theta + \left\{\frac{n}{2} - (-1)^n\right\} \cos \frac{n}{2}\theta \end{pmatrix}.$$

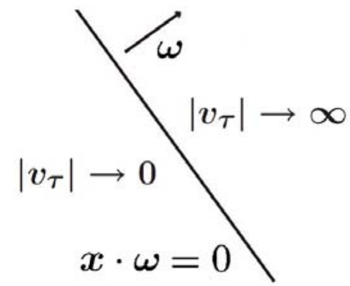
The enclosure method

([2] M. Ikehata 1999 Inverse Problems 15 1231-1241)

- the support function of D:

$$h_D(\omega) := \sup_{x \in D} x \cdot \omega, \quad \omega = (\omega_1, \omega_2) \in S^1.$$

- Special sol. of $Av = 0$ in $\mathbb{R}^2$: for $\tau > 0$



$$v_\tau(x;\omega) := (\omega + i\omega^\perp) e^{\tau x \cdot (\omega + i\omega^\perp)}.$$

$\Rightarrow$ As $\tau \to \infty$, $|v_\tau| \to 0$ $(x \cdot \omega < 0)$, $|v_\tau| \to \infty$ $(x \cdot \omega > 0)$

- the indicator function: Let u be a weak sol. of $(*)_1$

$$I_\omega(\tau, t) = e^{-\tau t} \left\{ \int_{\partial\Omega} (g \cdot v_\tau - u \cdot T v_\tau) \; \mathrm{d}\sigma \right\}$$

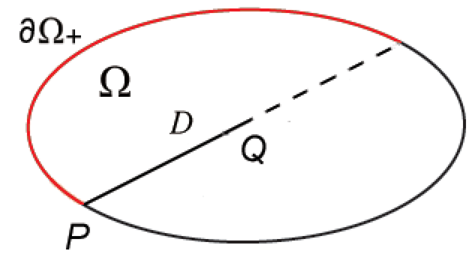
for $\tau > 0$ and $t \in \mathbb{R}$.

Theorem 2 Let u be not a rigid displacement

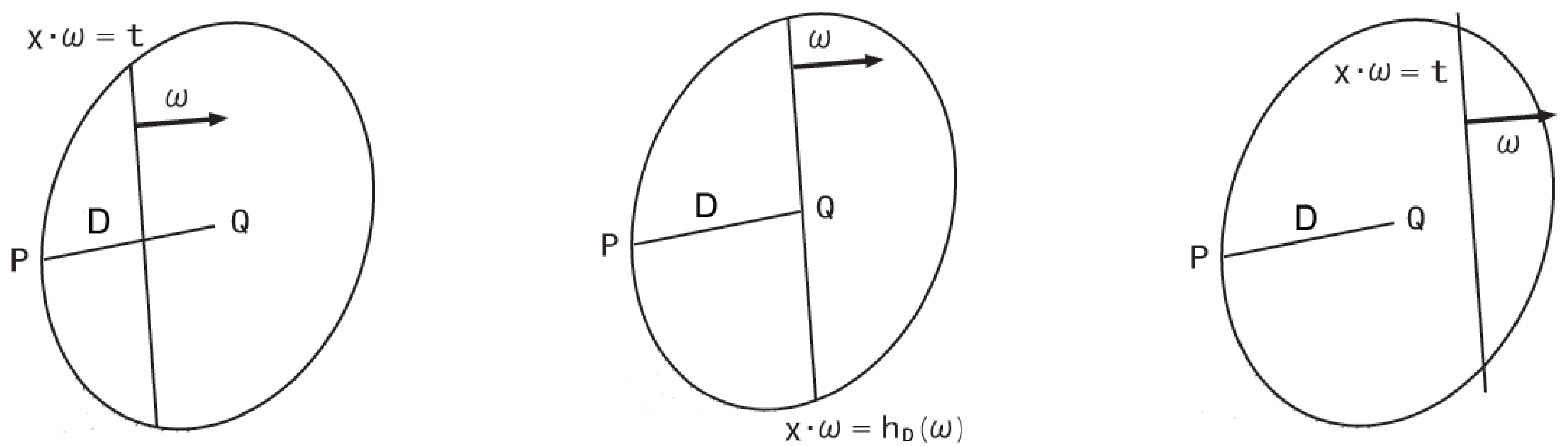
Assumptions: Ω is covex, $\partial\Omega \setminus \{P\}$ is C^2,

$g \in C^1(\partial\Omega \setminus \{P\})$ is well controlled (ex. uniform load)

$$\overset{\text{def}}{\Longleftrightarrow} \quad \exists \rho_0(x) \in \mathcal{R}, \int_{\partial\Omega_+} g \cdot \rho_0(x) \, \mathrm{d}s \neq 0.$$



$$\Rightarrow \quad h_D(\omega) = \lim_{\tau \longrightarrow \infty} \frac{1}{\tau} \log |I_\omega(\tau, 0)| .$$



$t < h_D(\omega)$ $\qquad$ $t = h_D(\omega)$ $\qquad$ $t > h_D(\omega)$

Multiple cracks case (Today's Problem)

$\Omega \subset \mathbb{R}^2$: a rectangular domain, $]0,a[\times]0,b[$.

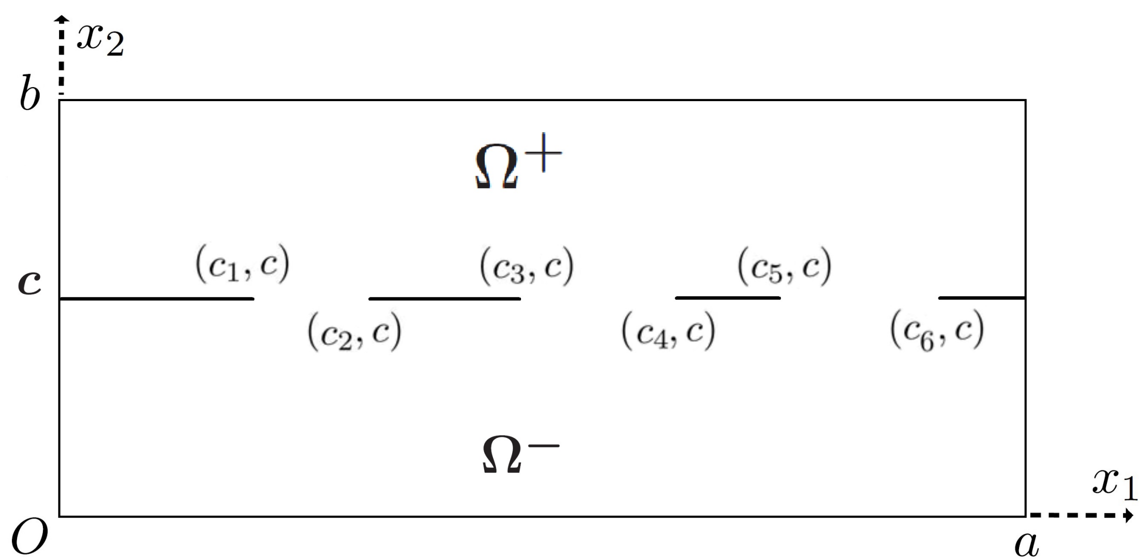
Let $c \in]0,b[$, $m \geq 1$ and $0 = c_0 < c_1 < \cdots < c_{2m+1} = a$.

linear cracks Σ:

$$\Sigma := ([c_0, c_1] \cup [c_2, c_3] \cup [c_4, c_5] \cup \cdots \cup [c_{2m}, c_{2m+1}]) \times \{c\}$$

$$W := ([0,\ a] \times \{c\}) \setminus \Sigma = (]c_1, c_2[\cup \cdots \cup]c_{2m-1}, c_{2m}[) \times \{c\}$$

$g \in L^2(\partial\Omega)$ satisfy $\forall \rho(x) \in \mathcal{R}$, $\int_{\partial\Omega} g \cdot \rho \, \mathrm{d}s_x = 0$ (2.6).



Definition 3 Let $g \in L^2(\partial\Omega)$ satisfy (2.6). We say that $u \in H^1(\Omega \setminus \overline{\Sigma}) \setminus \mathcal{R}$ is a weak solution of the following boundary value problem

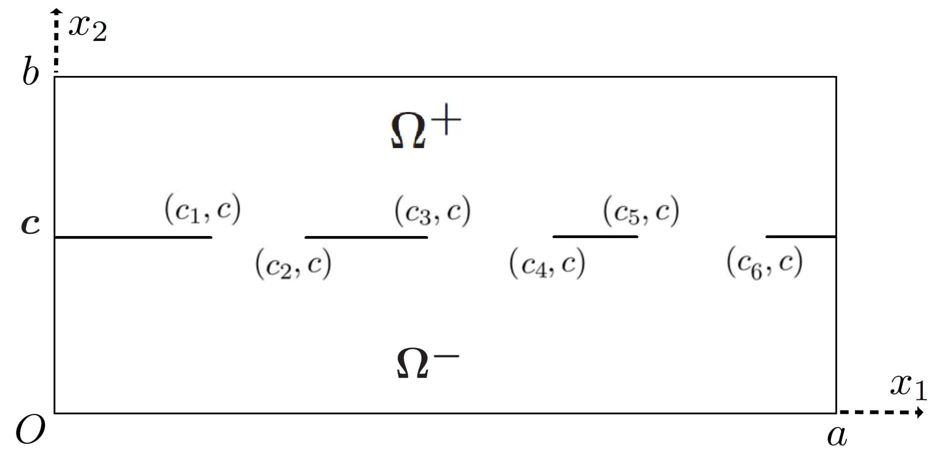
$$(*)_2 \begin{cases} \mu \triangle u + (\lambda + \mu)\nabla(\nabla \cdot u) = 0 \quad \text{in} \quad \Omega \setminus \overline{\Sigma}, \\ \sigma^{\pm}\nu = 0 \quad \text{on} \quad \Sigma^{\pm}, \\ \sigma\nu = g \quad \text{on} \quad \partial\Omega \end{cases}$$

if $\forall \varphi \in H^1(\Omega \setminus \overline{\Sigma}) \setminus \mathcal{R}$ it holds

$$\int_{\Omega \setminus \overline{\Sigma}} \sigma(u) : \varepsilon(\varphi)\, \mathrm{d}x = \int_{\partial\Omega} g \cdot \varphi \, \mathrm{d}s_x. \qquad (1)$$

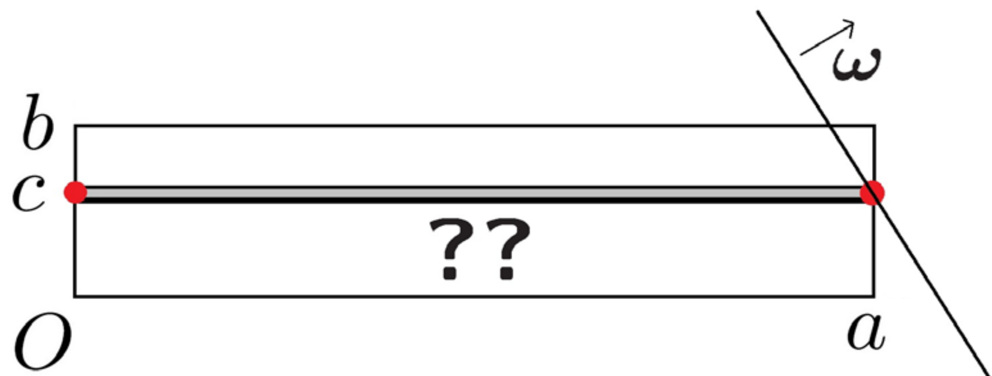
Problem : "Fix $g \neq 0$ satisfying (2.6). Extract information about the location of Σ from the knowledge of a single set of data (u, g) on $\partial\Omega$"

Today's case $\Rightarrow$



$\Rightarrow$ $\forall t < h_\Sigma(\omega)$, $\omega \in S_1$, $|e^{-\tau t} I(\tau;\omega)| \to \infty$ as $\tau \to \infty$

Since $h_W(\omega) < h_\Sigma(\omega)$, usual enclosure method ×



Case of Laplace eq.

[7] M.I., H.I. & A.S. MMAS 39(2016) 3565-75

[1] A.H., M.I., H.I. & S.S. Inv. Prob. 35(2019) 025004

- **Kelvin transformation**:

$$v_\tau = v_\tau(y;x) := (e_1 + ie_2)\exp\left\{-\tau\frac{y-x}{|y-x|^2}\cdot(e_2+ie_1)\right\}.$$

$\Rightarrow$ $\triangle_y v_\tau = 0$ and $\nabla_y \cdot v_\tau = 0$ in $\mathbb{R}^2 \setminus \{x\}$

Moreover, let $s > 0$ and write

$$e^{-\frac{\tau}{2s}}v_\tau = (e_1+ie_2)\exp\left\{-\tau\left(\frac{y-x}{|y-x|^2}\cdot e_2 + \frac{1}{2s}\right)\right\}\exp\left\{-i\tau\frac{y-x}{|y-x|^2}\cdot e_1\right\}$$

Since $\frac{y-x}{|y-x|^2}\cdot e_2 + \frac{1}{2s} = \frac{|y-(x-se_2)|^2-s^2}{2s|y-x|^2}$, we see that

i. $|y-(x-se_2)| > s \Rightarrow \lim\limits_{\tau\longrightarrow\infty} e^{-\tau/(2s)}|v_\tau(y;x)| = 0$;

ii. $|y-(x-se_2)| < s \Rightarrow \lim\limits_{\tau\longrightarrow\infty} e^{-\tau/(2s)}|v_\tau(y;x)| = \infty$;

iii. $|y-(x-se_2)| = s \Rightarrow e^{-\tau/(2s)}v_\tau(y;x)$ is highly oscillating as $\tau \longrightarrow \infty$.

Modification

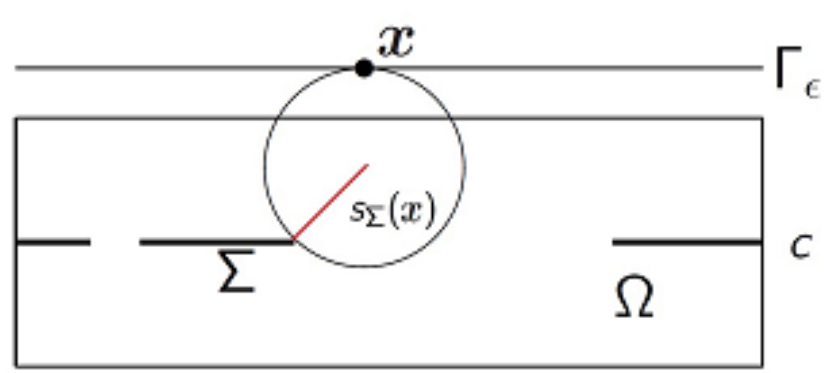
Definition 4 Let u be a weak solution of $(*)_2$. Given $x \in \Gamma_\epsilon := [0,a] \times \{b+\epsilon\}$ $(\epsilon > 0)$ and $\tau > 0$ define

$$I(\tau; x) := \int_{\partial\Omega} g(y) \cdot v_\tau(y;x) - u(y) \cdot \sigma(v_\tau(y;x))\nu \, \mathrm{d}s_y.$$

Define a function of $x \in \Gamma_\epsilon$:

$$s_\Sigma(x) = \sup\{s > 0 \,|\, B_s(x - se_2) \subset \mathbb{R}^2 \setminus \Sigma\}.$$

The value $s_\Sigma(x)$ at $x \in \Gamma_\epsilon$ coincides with the radius of the largest disc whose exterior encloses Σ.



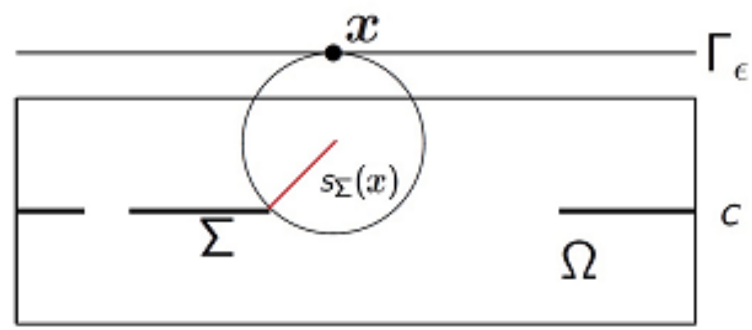
Theorem

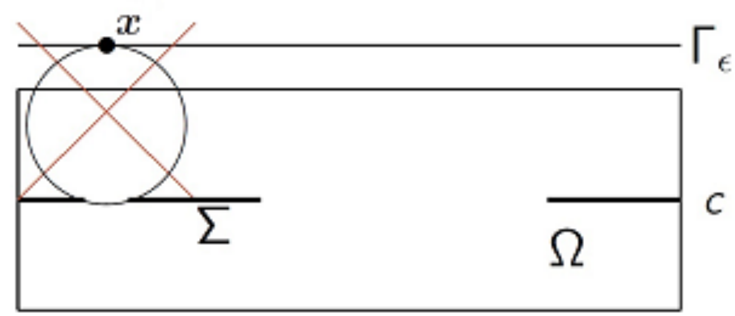
Assumptions :

Let x satisfy that

$\exists j \in \{1, \cdots, 2m\}$ s. t.

$$\overline{B_{s_\Sigma(x)}(x - s_\Sigma(x)e_2)} \cap \Sigma = \{(c_j, c)\}.$$





Let the non-trivial $g \in L^2(\partial\Omega)$ satisfy (2.6) and

(†) $\mathrm{supp}(g) \subset (\partial\Omega \cap \{|x_2 - c| > \gamma\}) \setminus \big(B_\gamma(O) \cup B_\gamma(0, b) \cup B_\gamma(a, b) \cup B_\gamma(a, 0)\big)$ for some $\gamma > 0$ and $\exists \rho_0 \in \mathcal{R}$ s.t.

$$\int_{\partial\Omega\cap\{x_2>c\}} g \cdot \rho_0 \, \mathrm{d}s_x \neq 0 \quad \text{or} \quad \int_{\partial\Omega\cap\{x_2<c\}} g \cdot \rho_0 \, \mathrm{d}s_x \neq 0.$$

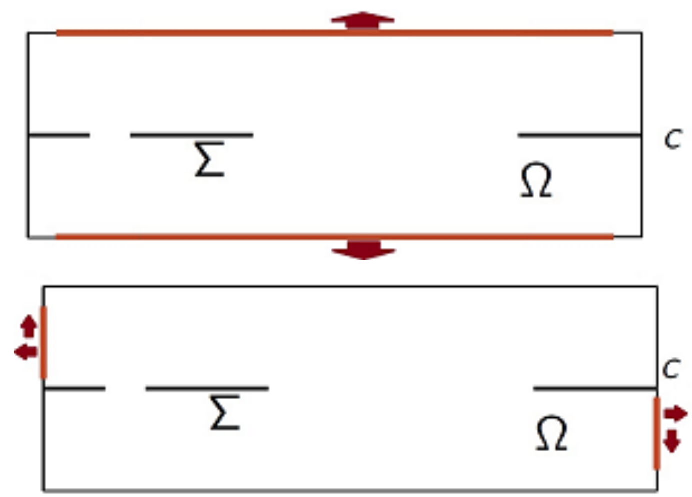
Then, the function $e^{-\tau/(2s_\Sigma(x))}I(\tau;x)$ is truly algebraically decaying as $\tau \to \infty$ and it holds that

$$\lim_{\tau\to\infty} \frac{\log |I(\tau;x)|}{\tau} = \frac{1}{2s_\Sigma(x)}.$$

Example (2 examples of g satisfying (2.6) & (†))

Set $\gamma' = \min\{c-2\gamma, b-c-2\gamma\}$.

For β_1, $\beta_2 \neq 0$,



$$g_1 = \begin{cases} \beta_1 e_2 \text{ on }]\gamma,\, a-\gamma[\, \times \{b\}, \\ -\beta_1 e_2 \text{ on }]\gamma,\, a-\gamma[\, \times \{0\}, \\ 0, \quad \text{otherwise.} \end{cases}$$

$$g_2 = \begin{cases} \beta_2(ae_1 - (2\gamma+\gamma')e_2) \text{ on } \{a\}\times\,]c-\gamma-\gamma',\, c-\gamma[, \\ -\beta_2(ae_1 - (2\gamma+\gamma')e_2) \text{ on } \{0\}\times\,]c+\gamma,\, c+\gamma+\gamma'[, \\ 0, \quad \text{otherwise.} \end{cases}$$

2. Proof of Main Theorem

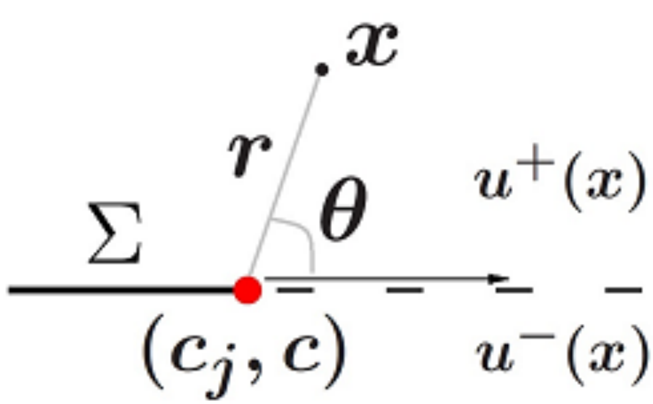
• **Asymptotic behavior of the indicator function** $I(\tau;x)$

Proposition 5 The formula

$$I(\tau;x) = -\int_{\Sigma}(u^{+}(y)-u^{-}(y))\cdot(\sigma(v_{\tau}(y;x))e_2)\ \mathrm{d}s_y$$

is valid.

Polar coordinates system w.r.t. the origin (c_j, c)



j : *odd*

$$x = (c_j + r\cos\theta,\ c + r\sin\theta),$$

$$-\pi < \theta < \pi,$$

$$0 < r < \eta_0 < \min\{b-c, c\},$$

$$\eta_0 < \min_{j=1,\cdots,2m+1}\{c_j - c_{j-1}\},$$

$$u^{+}(x) = u(r,\theta),\ 0 < \theta < \pi.$$

$$u^{-}(x) = u(r,\theta),\ -\pi < \theta < 0.$$

- **Convergent series expansion of u around (c_j, c)**

Proposition 6 Fix $0 < \eta < \eta_0/2$. $\exists\{A_k^{(j)}\}$, $\{B_k^{(j)}\} \subset \mathbb{R}$ $(k = 0, 1, 2, \cdots)$ s.t. in $B_{2\eta}((c_j, c)) \setminus \Sigma$

$$u(r,\theta) = \sum_{k=0}^{\infty} \frac{1}{2\mu} r^{\frac{k}{2}} \left(A_k^{(j)} \varphi_k(\theta) - B_k^{(j)} \psi_k(\theta) \right).$$

$$\varphi_k(\theta) = \begin{pmatrix} \kappa \cos \frac{k}{2}\theta - \frac{k}{2} \cos\left(\frac{k}{2} - 2\right)\theta + \left\{\frac{k}{2} + (-1)^k\right\} \cos \frac{k}{2}\theta \\ \kappa \sin \frac{k}{2}\theta + \frac{k}{2} \sin\left(\frac{k}{2} - 2\right)\theta - \left\{\frac{k}{2} + (-1)^k\right\} \sin \frac{k}{2}\theta \end{pmatrix}$$

$$\psi_k(\theta) = \begin{pmatrix} \kappa \sin \frac{k}{2}\theta - \frac{k}{2} \sin\left(\frac{k}{2} - 2\right)\theta + \left\{\frac{k}{2} - (-1)^k\right\} \sin \frac{k}{2}\theta \\ -\kappa \cos \frac{k}{2}\theta - \frac{k}{2} \cos\left(\frac{k}{2} - 2\right)\theta + \left\{\frac{k}{2} - (-1)^k\right\} \cos \frac{k}{2}\theta \end{pmatrix}.$$

This is absolutely convergent in $H^1(B_\eta((c_j, c)) \cap \Omega^+)$ and $H^1(B_\eta((c_j, c)) \cap \Omega^-)$, and uniformly in $B_{2\eta}((c_j, c))$.

Moreover, $\exists \rho_j \in \mathcal{R}$ s.t. for each $n = 1, 2, \cdots$ the following estimate is valid uniformly for $0 < r < \eta$:

$$\left| u(r,\pi) - \rho_j - \sum_{k=1}^{n} \frac{r^{\frac{k}{2}}}{2\mu} \left(A_k^{(j)} \varphi_k(\pi) - B_k^{(j)} \psi_k(\pi) \right) \right| + \left| u(r,-\pi) - \rho_j - \sum_{k=1}^{n} \frac{r^{\frac{k}{2}}}{2\mu} \left(A_k^{(j)} \varphi_k(-\pi) - B_k^{(j)} \psi_k(-\pi) \right) \right| \leq K_n r^{\frac{n+1}{2}}.$$

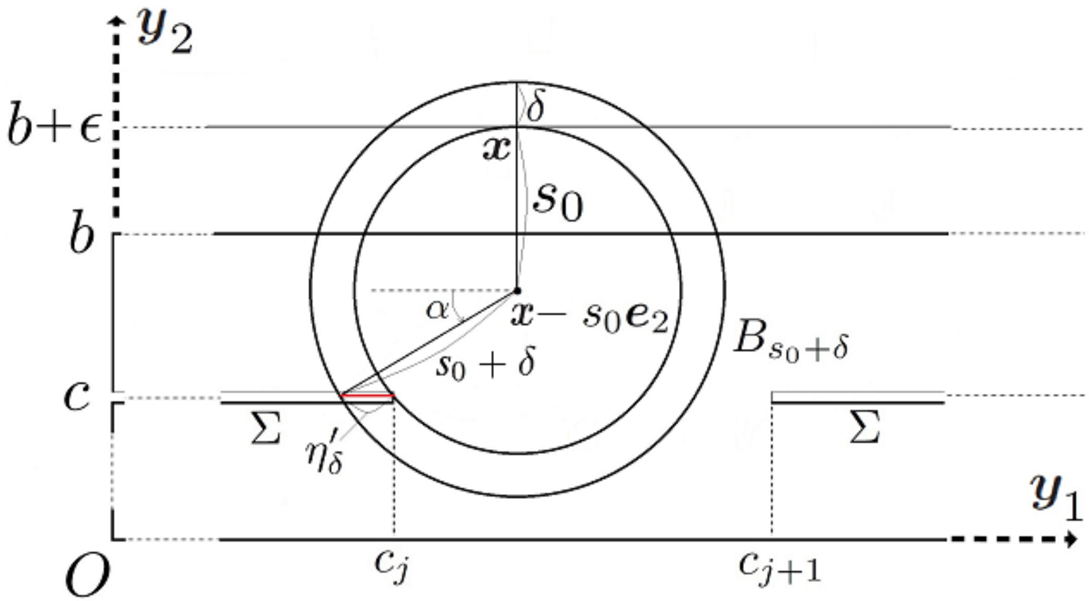
$\Rightarrow$ for each $n = 1, 2, \cdots$

$$u(c_j - r, c+0) - u(c_j - r, c-0) = \frac{\kappa+1}{\mu} \sum_{k=1}^{n} (-1)^k r^{\frac{2k-1}{2}} \begin{pmatrix} -B_{2k-1}^{(j)} \\ A_{2k-1}^{(j)} \end{pmatrix} + O\left(r^{\frac{2n+1}{2}} \right) k$$

with a constant vector k.

For simplicity, $s_0 := s_\Sigma(x)$.

Choose $\delta > 0$ in such a way that



$$\overline{B_{s_0+\delta}(x - s_0 e_2)} \cap \Sigma \subset [c_{j-1}, c_j] \times \{c\},$$

$\eta_\delta < \eta$ (fixed in Prop. 6)

$$\eta_\delta := \sqrt{(s_0+\delta)^2 - s_0^2}.$$

Set $\eta'_\delta = \sqrt{(s_0+\delta)^2 - (x_2 - s_0 - c)^2} - |x_1 - c_j|$.

Since $\forall y \in \Sigma \setminus B_{s_0+\delta}(x - s_0 e_2)\ \ |y - (x - s_0 e_2)| \geq s_0 + \delta$,

$$e^{-\frac{\tau}{2s_0}} \int_{\Sigma \setminus B_{s_0+\delta}(x - s_0 e_2)} \Big(u^+(y) - u^-(y)\Big) \cdot (\sigma(v_\tau(y;x))e_2)\ \mathrm{d}s_y = O(\tau e^{-c_3 \tau}),$$

where c_3 is a positive constant.

And $\Sigma \cap B_{s_0+\delta}(x - s_0 e_2) =]c_j - \eta'_\delta, c_j[\times \{c\} \Rightarrow$

$$\begin{aligned}
&-e^{-\frac{\tau}{2s_0}} \int_{\Sigma\cap B_{s_0+\delta}(x-s_0e_2)} (u^+(y) - u^-(y)) \cdot (\sigma(v_\tau(y;x))e_2) \, \mathrm{d}s_y \\
&= -e^{-\frac{\tau}{2s_0}} \int_{\Sigma\cap B_{s_0+\delta}(x-s_0e_2)} \left\{ \frac{\kappa+1}{\mu} \sum_{k=1}^{n} (-1)^k r^{\frac{2k-1}{2}} \begin{pmatrix} -B_{2k-1}^{(j)} \\ A_{2k-1}^{(j)} \end{pmatrix} + O\left(r^{\frac{2n+1}{2}}\right) k \right\} \\
&\quad \cdot \left\{ 2\mu\tau \left(\frac{y-x}{|y-x|^2} \cdot (e_2 + ie_1) \right)^2 \exp\left\{ -\tau \frac{y-x}{|y-x|^2} \cdot (e_2 + ie_1) \right\} (e_1 + ie_2) \right\} \, \mathrm{d}s_y \\
&= 2(\kappa+1)\tau e^{-\frac{\tau}{2s_0}} \sum_{k=1}^{n} (-1)^k \begin{pmatrix} -B_{2k-1}^{(j)} \\ A_{2k-1}^{(j)} \end{pmatrix} \cdot (e_1 + ie_2) I_k(\tau) + O\left(\tau^{-\frac{n+1}{2}} \tau^{\frac{3}{4}}\right). \qquad (2)
\end{aligned}$$

Here $z_\alpha = -(e^{-\frac{\pi}{2}i} + ie^{-(\frac{\pi}{2}+\alpha)i}) = -\cos\alpha + i(1 + \sin\alpha)$,

$$I_k(\tau) = \int_0^{\eta'_\delta} \frac{r^{\frac{2k-1}{2}}}{(r - s_0\overline{z_\alpha})^2} \exp\left(\frac{i\tau}{r - s_0\overline{z_\alpha}} \right) \, \mathrm{d}r,$$

and $\alpha \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ is the unique solution of the equation

$$e^{i\alpha} = \frac{x_1 - c_j}{s_0} + i\frac{x_2 - s_0 - c}{s_0}.$$

Lemma 7 Let $g \neq 0$ satisfy (2.6) and (†).
Then, $\exists n \geq 1$ s.t. $\left(A^{(j)}_{2n-1}\right)^2 + \left(B^{(j)}_{2n-1}\right)^2 \neq 0$.

Contradiction arguments.

Assume $\forall n \in \mathbb{N}$, $A^{(j)}_{2n-1} = B^{(j)}_{2n-1} = 0$.

Step 1. Prop. 6 $\Rightarrow u = \displaystyle\sum_{k=0}^{\infty} \frac{r^k}{2\mu}\left(A^{(j)}_{2k}\varphi_{2k}(\theta) - B^{(j)}_{2k}\psi_{2k}(\theta)\right)$

Step 2. u is real analytic near (c_j, c).

$\sigma(u)e_2 = 0$ on $]c_j, c_j + \epsilon[\times\{c\}$ for a small $\epsilon > 0$.

Step 3. Construction of stress functions $\phi_\pm(z)$, $\omega_\pm(z)$:
holomorphic $(z = z_1 + iz_2 = c_j + r\cos\theta + i(c + r\sin\theta))$
in $B_\pm = \{z \mid z_1 \in]0, a[,\ z_2 \in]c, c \pm \min\{c, b - c\}[\}$.

Step 3. $2\mu(u_1 + iu_2) = \kappa\phi_\pm(z) - \overline{\omega_\pm(z)} + (\overline{z} - z)\overline{\phi'_\pm(z)}$,

$$\sigma_{22} - i\sigma_{12} = \phi'_\pm(z) + \overline{\omega'_\pm(z)} + (z - \overline{z})\overline{\phi''_\pm(z)}.$$

On Σ, $\sigma_{12} = \sigma_{22} = 0 \Rightarrow$

$$\phi'_+(z_1) + \overline{\omega'_+(z_1)} = 0, \quad \phi'_-(z_1) + \overline{\omega'_-(z_1)} = 0 \quad \text{on} \quad \Sigma.$$

$\Rightarrow$ Define sectionally holomorphic functions $\Psi_1(z)$ and $\Psi_2(z)$ cut along W

$$\Psi_1(z) = \begin{cases} \phi'_+(z) & \text{in } B_+, \\ -\overline{\omega'_+(\overline{z})} & \text{in } B_-, \end{cases} \qquad \Psi_2(z) = \begin{cases} -\overline{\omega'_-(\overline{z})} & \text{in } B_+, \\ \phi'_-(z) & \text{in } B_-. \end{cases}$$

On W, $u^+ = u^- \Rightarrow$

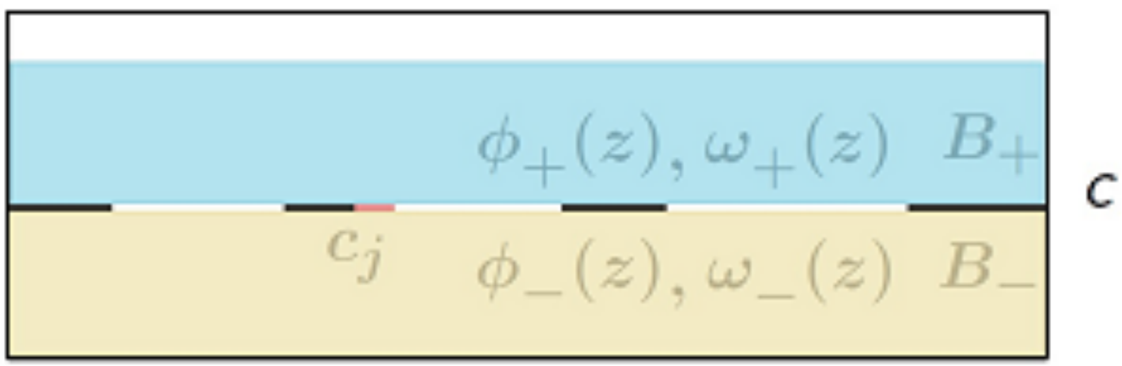
$$\frac{\kappa}{\mu}\phi'_+(z_1) - \frac{1}{\mu}\overline{\omega'_+(z_1)} = \frac{\kappa}{\mu}\phi'_-(z_1) - \frac{1}{\mu}\overline{\omega'_-(z_1)} \quad \text{on} \quad W.$$

From Step 2. on $]c_j, c_j+\epsilon[\times\{c\}$, $\sigma(u)e_2 = 0$ & $u^+ = u^-$.

$\Rightarrow$ $\phi'_+(z_1) = \phi'_-(z_1)$, $\omega'_+(z_1) = \omega'_-(z_1)$, $\phi'_\pm(z_1) = -\overline{\omega'_\mp(z_1)}$.

$$\Rightarrow \quad \phi'(z) = \begin{cases} \phi'_+(z) & \text{in} \quad B_+, \\ \phi'_-(z) & \text{in} \quad B_-, \end{cases} \qquad \omega'(z) = \begin{cases} \omega'_+(z) & \text{in} \quad B_+, \\ \omega'_-(z) & \text{in} \quad B_-. \end{cases}$$

$\Rightarrow$ $\phi'(z) = -\overline{\omega'(\overline{z})}$ $(\Psi_1 = \Psi_2)$ $\Rightarrow$ $\sigma(u)e_2 = 0$ on $\{x_2 = c\}$



Since g satisfies $(\dagger)$, $\sigma(u)\nu = 0$ on the edges near corner points O, $(a, 0)$, $(0, c)$, (a, c), $(0, b)$, (a, b).

Step 4. $Q^-_{\epsilon'} := \Omega^- \setminus \overline{R_{\epsilon'}}$ $(R_{\epsilon'} = B_{\epsilon'}(O) \cup B_{\epsilon'}(0,c) \cup B_{\epsilon'}(a,0) \cup B_{\epsilon'}(a,c))$

For a sufficiently small ϵ', $u \in H^2(Q^-_{\epsilon'})$ satisfies

$$\begin{cases} \mu \triangle u + (\lambda + \mu)\nabla(\nabla \cdot u) = 0 & \text{in} \quad Q^-_{\epsilon'}, \\ \sigma(u)\nu = 0 & \text{on} \quad]\epsilon', a - \epsilon'[\times\{c\}, \\ \sigma(u)\nu = g & \text{on} \quad (\partial\Omega \cap \{x_2 < c\}) \setminus R_{\epsilon'}. \end{cases}$$

Step 5. The divergence theorem $\Rightarrow$ $\forall \rho \in \mathcal{R}$

$$\begin{aligned} 0 &= \int_{Q^-_{\epsilon'}} \rho \cdot (\mu \triangle u + (\lambda + \mu)\nabla(\nabla \cdot u))\, \mathrm{d}x \\ &= \int_{(\partial\Omega \cap \{x_2 < c\}) \setminus R_{\epsilon'}} \rho \cdot g \, \mathrm{d}s_x + \int_{\partial R_{\epsilon'} \cap \Omega^-} \rho \cdot \sigma(u)\nu \, \mathrm{d}s_x. \end{aligned}$$

Step 6. As $\epsilon' \to 0$, $\left| \int_{\partial R_{\epsilon'} \cap \Omega^-} \rho \cdot \sigma(u)\nu \, \mathrm{d}s_x \right| \to 0.$

$\Rightarrow$ $\forall \rho \in \mathcal{R}$, $\int_{\partial\Omega \cap \{x_2 < c\}} g \cdot \rho \, \mathrm{d}s_x = 0.$

$\Rightarrow$ Contradiction that g satisfies $(\dagger)$.

(The above argument is valid for the case of Ω^+.)

By Lemma 7, $\exists N := \min\left\{n \geq 1 \mid \left(A_{2n-1}^{(j)}\right)^2 + \left(B_{2n-1}^{(j)}\right)^2 \neq 0\right\}$.

Lemma 8 **(Lemma 3.5 in [1])**

Let $n = 1, \cdots$ **and** $\alpha \in]-\frac{\pi}{2}, \frac{\pi}{2}[$. **It holds**

$$\lim_{\tau \longrightarrow \infty} e^{-\frac{\tau}{2s_0}} \tau^{\frac{2n+1}{2}} e^{-\frac{i\tau\cos\alpha}{2s_0(1+\sin\alpha)}} I_n(\tau)$$
$$= -is_0^{2n-1} 2^{\frac{2n-1}{2}} (1+\sin\alpha)^{\frac{2n-1}{2}} e^{i\frac{(2n-1)\alpha}{2}} \Gamma\left(\frac{2n+1}{2}\right).$$

(2) $\Rightarrow \lim_{\tau \longrightarrow \infty} \tau^{\frac{2N+1}{2}-1} e^{-\frac{\tau}{2s_0}} |I(\tau; x)|$

$$= (\kappa+1) s_0^{2N-1} 2^{\frac{2N+1}{2}} (1+\sin\alpha)^{\frac{2N-1}{2}} \Gamma\left(\frac{2N+1}{2}\right) \left| \begin{pmatrix} -B_{2N-1}^{(j)} \\ A_{2N-1}^{(j)} \end{pmatrix} \cdot (e_1 + ie_2) \right| \neq 0.$$

$\Rightarrow e^{-\frac{\tau}{2s_0}} I(\tau; x)$ **is truly algebraically decaying as** $\tau \longrightarrow \infty$

$\Rightarrow$ **Theorem**

• Proof of Lemma 8

Asymptotic behavior of

$$I_n(\tau) := \int_0^{\eta'_\delta} \frac{r^{\frac{2n-1}{2}}}{(r - s_0\overline{z_\alpha})^2} e^{\frac{i\tau}{r - s_0\overline{z_\alpha}}}\, \mathrm{d}r.$$

Proof (j: odd)   Method of steepest descent

· Laplace's method ($f(x)$ attains min. at x_0, $g(x) > 0$)

$$\int_a^b e^{-nf(x)} g(x)\, \mathrm{d}x \approx g(x_0) \sqrt{\frac{2\pi}{n|f''(x_0)|}} e^{-nf(x_0)} \quad (n \to \infty)$$

· Fourier type integral

Integration by parts & Riemann-Lebesgue Lem.

· Oscillatory integrals $\int e^{inf(x)} g(x)\, \mathrm{d}x$

Stationary phase method

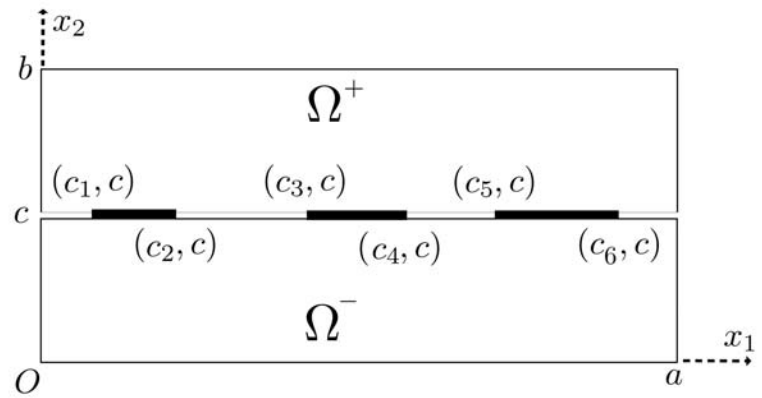
3. Summary

Assumptions:

1. $\Omega \subset \mathbb{R}^2$ is a rectangular domain $]0,a[\times]0,b[$.
2. c is known.
3. $g(\neq 0) \in L^2(\partial\Omega)$

   satisfies $(\dagger)$ and

   $\forall \rho(x) \in \mathcal{R}$, $\int_{\partial\Omega} g \cdot \rho \, \mathrm{d}s_x = 0$.
4. u is a weak solution of $(*)_2$.



$\Downarrow$ The enclosure method (modified)

Result:

We can extract the positions of $c_1, \cdots, c_{2m}$ from a single set of boundary data (u, g) on $\partial\Omega$!!

4. Future works

- **The numerical test of the probing algorithm and checked the performance (e.g. [1] in the conductivity case)**

- **Multilayered materials with different material constants (e.g. [3] in the conductivity case)**

- **Extension to 3 dimensions**

4. Future works

- The numerical test of the probing algorithm and checked the performance
(e.g. [1] in the conductivity case)

- Multilayered materials with different material constants (e.g. [3] in the conductivity case)

- Extension to 3 dimensions

Thank you for your kind attention !!