1 Introduction

In this talk we discuss blow-up mechanisms for semilinear parabolic equations whose typical form is:

\[
\begin{align*}
t u_t & = \Delta u + |u|^{p-1}u, \quad x \in \mathbb{R}^N, \quad t > 0, \\
u(x, 0) & = u_0, \quad x \in \mathbb{R}^N,
\end{align*}
\]

where \(\Delta\) denotes the Laplace operator in the Euclidean space \(\mathbb{R}^N\) with \(N \geq 1\), \(p > 1\) is a constant and \(u_0\) is a bounded function in \(\mathbb{R}^N\). Local-in-time existence of a unique classical solution of (1.1a)-(1.1b) is well known. As usual, we say that the solution \(u\) of (1.1a)-(1.1b) blows up in a finite time \(T\) if the solution stays bounded for \(0 < t < T\) and

\[
\limsup_{t \to T} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} = +\infty.
\]

Various criteria on given data for blow-up in finite time are known. For example, if \(u_0 \in H^1 \cap L^{p+1}(\mathbb{R}^N)\) and

\[
\frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_0|^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^n} |u_0|^{p+1} dx < 0,
\]

then the solution of (1.1a)-(1.1b) blows up in finite time (cf. [8, 15]).

The main focus of this talk is to describe singularity mechanisms for blow-up solutions. More precisely, we are interested in the blow-up rate of \(\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)}\) as \(t\) approaches the blow-up time.

1.1 Known results on blow-up rates

The following definition is due to [10].
Definition 1.1. Let $u$ be a solution of (1.1a)-(1.1b) that blows up in a finite time $T$. The blow-up is called of type I if there exists a positive constant $K$ such that

$$
\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq K(T-t)^{-1/(p-1)};
$$

(1.4)

whereas the blow-up is called of type II otherwise, i.e.,

$$
\limsup_{t \uparrow T} (T-t)^{1/(p-1)} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} = +\infty.
$$

(1.5)

When a solution blows up in finite time and the blow-up is of type II, we call the solution type II blow-up solution.

We just review some known results on blow-up rates.

1. Sobolev subcritical case: $N = 1, 2$ or $p < (N + 2)/(N - 2) =: p_S$.

Giga, Matsui, and Sasayama [3] proved that blow-ups of all the solutions of (1.1a) are of type I for all subcritical range of $p$, thus improving considerably the result of an earlier work by Giga and Kohn [2].

2. Sobolev critical case: $N \geq 3$ and $p = p_S$.

Nonexistence of type II blow-up was proven for positive radial solutions by Matano and Merle [10], whereas sign-changing type II blow-up solutions exist when $3 \leq N \leq 6$ according to a formal matched asymptotic method in Filippas, Herrero, and Velázquez [1]. It has recently proven by Schweyer [16] that type II blow-up does occur for $N = 4$ in the radial case.


In this case another exponent $p_{JL}$ defined by

$$
p_{JL} := \begin{cases} 
+\infty, & N \leq 10, \\
\frac{N - 2\sqrt{N - 1}}{N - 4 - 2\sqrt{N - 1}}, & N \geq 11.
\end{cases}
$$

plays an essential role. The importance of this exponent was first shown in [6].

(a) Joseph-Lundgren subcritical case: $N \geq 3$ and $p_S < p < p_{JL}$.

Matano and Merle [10, 11] and Mizoguchi [13] proved that type II blow-up cannot occur for radial solutions under some mild assumptions on initial data.

(b) Joseph-Lundgren supercritical case: $N \geq 11$ and $p_{JL} < p$.

Type II blow-up may actually occur as was shown in Herrero, and Velázquez [4, 5]. A matched asymptotic method plays a crucial role in constructing type II blow-up solutions in these articles. The result is described in detail in §3. Based upon these specific solutions constructed in [4, 5], further progress has been established in Matano [9] and Mizoguchi [14].

As for positive radial solutions, we may understand that Joseph-Lundgren exponent divides the range of $p$ into two parts in terms of existence/nonexistence of type II blow-up. It should be noticed that this fact has already been conjectured in [4].
2 Main result

A natural question that arises from these results is whether or not type II blow-up would occur in the Joseph-Lundgren critical case: \( p = p_{\text{JL}} \). As far as the speaker knows, no conjecture has circulated for this open question. The aim of this talk is to give a formal result, based on a matched asymptotic method, that suggests the existence of type II blow-up solutions. The main result may be formally stated as follows:

**Main result.** Let \( N \geq 11 \) and \( p = p_{\text{JL}} \). Then there exist radial solutions that blow up in finite time and the blow-ups are of type II.

The blow-up mechanisms of these solutions are different from those of any type II blow-up solutions having been found for \( p > p_{\text{JL}} \). Further details will be presented in the talk.

3 Herrero–Velázquez’ solutions

We shall recall the result of [4, 5] in detail. Throughout this talk we use the following notation:

\[
\begin{align*}
\beta &= \frac{1}{p - 1}; \\
\gamma &= \frac{N - 2 - \sqrt{16\beta^2 - 4(N - 2)\beta + (N - 2)(N - 10)}}{2}.
\end{align*}
\]

Notice that \( \gamma > 0 \) is a real root of the quadratic equation:

\[\gamma^2 - (N - 2)\gamma + 2(N - 2\beta - 2)(\beta + 1) = 0\] (3.2)

if and only if \( N \geq 11 \) and \( p \geq p_{\text{JL}} \). Quadratic equation (3.2) is related to the asymptotic expansions as \( |x| \to \infty \) of stationary solutions \( U_\ell(|x|) \) to be given in §84.2 below.

**Proposition 3.1.** (Herrero and Velázquez [5, Theorem 1]). Assume that \( N \geq 11 \) and \( p > p_{\text{JL}} \) and let \( T > 0 \) be any constant. Then for every positive integer \( \ell \) such that \( \lambda_\ell := \ell - \gamma/2 + 1/(p - 1) > 0 \), there exists a radial solution \( u_\ell \) of (1.1a)-(1.1b) which blows up at \( t = T, x = 0 \), and satisfies (1.5).

Moreover, the solution satisfies \( \|u(\cdot, t)\|_\infty = u(0, t) \) and:

\[
C_1 (T - t)^{-\beta - 2\omega_\ell} \leq u_\ell(0, t) \leq C_2 (T - t)^{-\beta - 2\omega_\ell}
\]
with \( \omega_\ell := \frac{\lambda_\ell}{\gamma - 2\beta} > 0 \) (3.3a)

for some positive constants \( C_1 \) and \( C_2 \) depending only on \( p, N \) and \( \ell \).

We shall call the solutions Herrero–Velázquez’ solutions or **HV solutions** for short.
4 Preliminary results

Let us consider the radial stationary version of equation (1.1a):

\[
\frac{d^2 U}{dr^2} + \frac{N - 1}{r} \frac{dU}{dr} + U^p = 0 \quad \text{for } r > 0.
\]  

(4.1)

Structures of solutions of (4.1) play important roles in the study of existence/nonexistence of type II blow-ups. Up to now, many important properties on those solutions are available. We just review some of them.

4.1 Singular stationary solutions

**Proposition 4.1.** Assume that \( N \geq 3 \) and \( p > N/(N - 2) \). Then there exists a singular stationary solution \( U_\infty \) of (4.1) given by

\[
U_\infty(r) = c_{p,N} r^{-2\beta}, \quad c_{p,N}^{-1} = 2\beta (N - 2 - 2\beta).
\]

Moreover, function \( x \mapsto U_\infty(|x|) \) belongs to \( H^1_{\text{loc}}(\mathbb{R}^N) \) when \( p > p_S \).

4.2 Regular stationary solutions

We just recall some properties on regular solutions of (4.1). Given a constant \( \alpha > 0 \), we investigate regular solutions \( U_\alpha \) of (4.1) satisfying

\[
U(0) = \alpha, \quad U'(0) = 0.
\]

(4.3)

**Proposition 4.2.** (Infinitely many intersection / ordered structure) Assume that \( p > p_S \). Then for every \( \alpha > 0 \) there exists a unique solution \( U_\alpha \) of (4.1) satisfying (4.3). The solutions \( U_\alpha(r) \) are monotone decreasing in \( r \) and

\[
U_\alpha(r) \to U_\infty(r)
\]

as \( r \to \infty \) and also as \( \alpha \to \infty \). Moreover,

1. If \( p_S < p < p_{JL} \), then the graphs of \( U_\alpha(r) \) and \( U_\infty(r) \) intersect infinitely many times:

\[
Z_{(0,\infty)}(U_\alpha - U_\infty) = +\infty,
\]

(4.5)

where \( Z_{(0,\infty)}(F) \) denotes the number of zeros of function \( F \) in the interval \((0, \infty)\).

2. If \( N \geq 11 \) and \( p \geq p_{JL} \), the solutions are ordered according to their values at \( r = 0 \). Namely, if \( 0 < \alpha_1 < \alpha_2 \), it follows that

\[
U_{\alpha_1}(r) < U_{\alpha_2}(r) < U_\infty(r) \quad \text{for all } r > 0.
\]

(4.6)
The fact of (4.5) is a key property to prove nonexistence of type II blow-up for $p < p_5 < p < p_{IL}$ [10, 11, 13]. On the other hand, for $N \geq 11$ and $p > p_{IL}$, the ordered structure (4.6) and the asymptotic formula (4.7a) below were essentially used to construct type II blow-up solutions in [5]. As for the asymptotic expansions, a logarithmic factor appears in the first corrective term (cf. (4.7b) below) when $p = p_{IL}$, which violates the argument of [5]. This fact gives an essential difficulty in the critical case.

**Proposition 4.3.** (Asymptotic expansions) For every $\alpha > 0$, the following asymptotic expansions as $r \to \infty$ hold:

\[
\begin{align*}
  p > p_{IL} & \implies U_\alpha(r) = U_\infty(r) - h(\alpha)r^{-\gamma} + o(r^{-\gamma}); \\
  p = p_{IL} & \implies U_\alpha(r) = U_\infty(r) - h_1(\alpha)r^{-\gamma} \log r + h_2(\alpha)r^{-\gamma} + o(r^{-\gamma})
\end{align*}
\] (4.7a)

where $\gamma(> 2\beta)$ is the real number given in (3.1b), and where $h(\alpha), h_1(\alpha)$ and $h(\alpha)$ are positive constants depending only on $p, N$ and $\alpha$.

Proposition 4.3 was proven in [7, Lemma 4.3–4.4], where more precise formulas were obtained.

**References**


