An introduction to anisotropic and crystalline mean curvature flow

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1 Introduction

The aim of these notes is to give an elementary introduction to embedded anisotropic mean curvature flow in codimension one, with some attention to crystalline mean curvature flow. We will also discuss very briefly the generalization to a multiphase problem in the plane, namely to crystalline evolutions of planar partitions. For a better understanding of the arguments of the notes, some knowledge on motion by mean curvature in the euclidean setting would be recommended. We refer the reader to the introductory parts of the following references: [53], [117], [85], [86], [118], [17], [18], [89], [90], [91], [92], [119], [72], [106], [8], [84], [100], [123], [29].

We will mostly concentrate on the derivation of the evolution laws, rather than on detailed proofs: one reason for this is to keep the exposition whithin a limited number of pages. Another reason is that the proofs can be found in the original papers.

Apart from the initial section, where we often try to minimize the assumptions on the function ϕ^o, and from the final section on partitions, the view point that we will adopt here is mostly based on the use of the anisotropic signed distance function d_ϕ. As a consequence, we will not consider the evolution problem looking at the maps parametrizing the manifolds (see [132] and references therein for this parametric approach), but instead we will look only at the images of the maps. This approach is closely related to various derivations of mean curvature flow that can be found in the literature on phase transitions [81], [82], [83], [24], [71]. In this

respect, a very quick presentation of the reaction-diffusion approximation to crystalline mean curvature flow is also presented.
Mathematical and physical motivations for anisotropic mean curvature flow([1]) can be found in the large number of papers present in the literature devoted to this subject, as well as detailed reference lists. We apologyze with the reader, since the bibliography in these notes is largely incomplete. We sometimes quote references weakly related to the treated argument, but that we believe to be useful for a more general point of view on that subject.

2 Notation

Since we will consider Finsler norms [146], [22], [135], [134], on $\mathbb{R}^n$ and their duals, we believe that it is more clear to use a notation which distinguishes the base manifold from its tangent space, and as most as possible vectors from covectors.
Therefore we set $M = \mathbb{R}^n$ and $V = T_xM = \mathbb{R}^n$ the tangent space to M at any $x \in M$, and $TM = M \times V$ (resp. $T^\star M = M \times V^\star$, $V^\star$ the dual of V) the tangent (resp. cotangent) bundle to M. We denote by $\cdot$ and $|\cdot|$ the scalar product and the norm in V, respectively, and by $d(\cdot)$ the euclidean distance in M. Recall that V can be identifed with $V^\star$([2]).
$\mathcal{L}^n$ is the Lebesgue measure [50] and $\mathcal{H}^k$ the k-dimensional euclidean Hausdorff measure [94] in M for $k \in \{0, \dots, n\}$. Recall that $\mathcal{H}^n = \mathcal{L}^n$ [14]. If $B \subset M$ is measurable, we often write $\mathcal{L}^n(B) = |B|$. We will use the words orthogonal, unit vector etc. in the euclidean sense. If F is a set, we let $\mathcal{P}(F)$ be the class of all subsets of F.
We denote by $\Lambda^1 V$ (resp. $\Lambda_1 V$) the space of one-covectors (resp. one-vectors) of V. On these two vector spaces, we have the norm $|\cdot|$ induced by the euclidean norm [94], [99]. We sometimes use the symbol $\Lambda^1 V$ and sometimes $V^\star$ (which are thought of as row vectors); similarly for $\Lambda_1 V$ and V (column vectors). We usually omit the symbol ^{T} of transpositon when we write a column vector in components. The duality between $\Lambda^1 V$ and $\Lambda_1 V$ is denoted by $\langle \cdot, \cdot \rangle$.
Recall that any covector $\xi^\star \in \Lambda^1 V$ is a linear map $V \to \mathbb{R}$. If $|\xi^\star| = 1$ (where $|\xi^\star| := \max\{\langle \xi^\star, \xi \rangle : \xi \in \Lambda_1 V,\ |\xi| = 1\}$) we can uniquely associate with $\pm\xi^\star$ its kernel, which is an hyperplane in V. Therefore there is a bijection([3]) between the set of unit covectors and the set of all oriented hyperplanes of V passing through the origin. We denote by $G^\star$ (resp. G) the set of all oriented (resp. unoriented) hyperplanes of V passing through the origin; using the euclidean scalar product, with such a hyperplane we can uniquely associate a unit vector (resp. unit vector up to its sign), orthogonal to the hyperplane. Sometimes we will identify

[1] The theory that we present here is far from being realistic: for instance, it cannot explain at all the growth of a crystal. For this, much more refined models would be necessary, such as the Stefan problem and modifications of it [129], [147]. Some results described here can be generalized to crystalline mean curvature flow with a forcing term: see for instance [41], [42], [36], [39], [104].

[2] There is an isometric isomorphism $T : V \to V^\star$ defined as $\langle \xi^\star, \xi \rangle = T(\xi) \cdot \xi^\star$ for any $\xi^\star \in V^\star$, and similarly, there is an isometric isomorphism $T^\star : V^\star \to V$ defined as $\langle \xi^\star, \xi \rangle = T^\star(\xi^\star) \cdot \xi$ for any $\xi \in V$. Since V can be identified with V^{**}, we have $TT^\star = \mathrm{Id}_{V^*}$, $T^\star T = \mathrm{Id}_V$.

[3] Sometimes it is useful to use another identification: a hyperplane H of V can be identified with the linear map $\pi_H : V \to V$ which is the orthogonal projection of V onto H. If ξ is a unit vector which is orthogonal to H, we have $\pi_V = \mathrm{Id} - \xi \otimes \xi$, where the symbol $\xi \otimes \xi$ stands for the $(0,2)$-tensor that is represented by the $(n \times n)$ matrix having $\xi^i \xi^j$ as its ij-th entry.

$G^\star$ with $\mathbb{S}^{n-1} := \{v \in V : |v| = 1\}$. We refer the reader to [94], [99] for more details on the Grassmann algebra.
Covector fields on M have lower indices

$$\omega : x \in M \to \omega(x) = (\omega_1(x), \ldots, \omega_n(x)) \in \Lambda^1 V.$$

Given a function $f : M \to \mathbb{R}$ of class $\mathcal{C}^1$, we denote by $df_x \in \Lambda^1 V$ the differential of f at $x \in M$.
Vector fields (or contravariant vector fields) on M have upper indices,

$$X : x \in M \to X(x) = (X^1(x), \ldots, X^n(x)) \in V. \tag{2.1}$$

If X is of class $\mathcal{C}^1$, the divergence of X is defined as $\mathrm{div} X := \sum_{i=1}^n \frac{\partial X^i}{\partial x^i}$.
Given a function $f : M \to \mathbb{R}$ of class $\mathcal{C}^1$, $\mathrm{grad} f = (\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n})^T$ is the vector field gradient of f. To simplify the notation, we write ∇f in place of $\mathrm{grad} f$. If necessary, df_x will be identified with $\nabla f(x)$, using the euclidean scalar product. If f is of class $\mathcal{C}^2$, the Laplacian of f is defined as $\Delta f := \mathrm{div}\, \nabla f$.
Given open sets $\Omega \subseteq M$, $\Omega' \subseteq \mathbb{R}^m$, $m \geq 1$, and a map $\psi : M \to \Omega'$ of class $\mathcal{C}^1$, $\psi = (\psi^1, \ldots, \psi^m)$, we denote by $d\psi_x \in \mathcal{L}(V, W)$ the differential of f at x, where W is the tangent space to Ω' at any of its points. The Jacobian $(m \times n)$ matrix representing $d\psi_x$ is indicated by $J\psi(x)$. If $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, the ij-entry $(J\psi(x))_{ij}$ of $J\psi(x)$ is $\frac{\partial \psi^i}{\partial x^j}(x)$. Hence the i-th column of the transposed matrix $(J\psi(x))^T$ is $\nabla \psi^i(x)$.
Given a smooth vector field $X = (X^1, \ldots, X^n)$, we denote by ∇X the matrix $(\nabla X)_{ij} = \nabla_i X^j$. If $v = (v^1, \ldots, v^n)$ is a column vector in $\Lambda^1 V$ and A, B are $(n \times n)$-matrices, we use the notation Av, $v^T A$ to denote respectively the vectors of components $(Av)_i = A_{ij} v^j$, and $(vA)_i = A_{ji} v^j$.
The symbol E (resp. $E(t)$ for t belonging to some real interval, E_i for $i \in \mathbb{N}$) will denote a closed subset of M with compact boundary such that $E = \overline{\mathrm{int}(E)}$ (resp. $E(t) = \overline{\mathrm{int}(E(t))}$, $E_i = \overline{\mathrm{int}(E_i)}$).

Definition 2.1 (Lipschitz boundaries). *We say that E is Lipschitz if the boundary ∂E of E can be written, locally, as the graph of a Lipschitz function with respect to a suitable $(n-1)$-dimensional orthogonal coordinate system. We will write $\partial E \in \mathrm{Lip}(M)$.*

Recall that if $\partial E \in \mathrm{Lip}(M)$, then [14] for $\mathcal{H}^{n-1}$-almost every $x \in \partial E$ it is well defined the tangent plane $T_x(\partial E)$, which is identified with $\pm \nu^E(x)$, where

$$\nu^E(x) = \nu(x) \in \mathbb{S}^{n-1}$$

is the unit covector normal to ∂E at x and points toward the complement $M \setminus E$ of E. Lipschitz and polyhedral boundaries (with a finite number of facets) will be useful in connection with crystalline anisotropies. In this context, if F is a facet of a polyhedral ∂E, we denote by ∂F (resp. $\mathrm{int}(F)$) the relative boundary (resp. the relative interior) of F. We define

$$\widetilde{\nu}^F \tag{2.2}$$

to be the $\mathcal{H}^{n-2}$-almost everywhere defined unit normal to the relative boundary ∂F of F, lying in the hyperplane Π_F containing F, and pointing outside of F.

3 Anisotropic functionals on boundaries

Let $M \times G^\star$ be the unit cotangent bundle of M [22]. Let $\sigma : M \times G^\star \to [0, +\infty]$ be a measurable function. We shall assume that $\sigma(x, \cdot)$ is even([4]), namely $\sigma(x, \xi^\star) = \sigma(x, -\xi^\star)$, so that we can consider σ as defined on $M \times G$. The domain of σ is the set $\{(x, \xi^\star) \in M \times G : \sigma(x, \xi^\star) < +\infty\}$, which coincides with $M \times G$ in case that σ is continuous. Let $a : M \to [0, +\infty]$ be a given function([5]) defined everywhere on M. Associated with σ and a, we can consider the following anisotropic functional [94], [52], [10], [11] defined on boundaries:

$$\mathcal{F}(E) := \int_{\partial E} \sigma(x, \nu(x)) a(x) \ d\mathcal{H}^{n-1}(x) = \int_{\partial E} \sigma(x, \nu) a \ d\mathcal{H}^{n-1}, \qquad \partial E \in \mathrm{Lip}(M). \tag{3.1}$$

The functional $\mathcal{F}$ can be extended to the class of finite perimeter sets (where now the unit normal ν must be intended in a proper measure theoretic sense [14]): we will not need such an extension in these notes. Useful lower semicontinuity properties of this extension([6]) can be found in [14].
The boundary ∂E, also called interface, divides the two sets E and $M \setminus E$, sometimes called phases. In case σ is independent of x and $a \equiv 1$, the quantity $\sigma(v)$ can be considered as a surface tension v [1] associated with the hyperplane passing through the origin, orthogonal to the unit covector v.

3.1 The function ϕ^o

Define the function $\phi^o : T^\star M \to [0, +\infty]$ to be the one-homogeneous extension of $\sigma(x, \cdot)$ on the whole space of one-covectors, i.e.,

$$\phi^o(x, \xi^\star) := |\xi^\star| \sigma\left(x, \frac{\xi^\star}{|\xi^\star|}\right), \qquad (x, \xi^\star) \in T^\star M. \tag{3.2}$$

Then $\phi^o(x, \cdot)$ is one-homogeneous, i.e.,

$$\phi^o(x, \lambda\xi^\star) = |\lambda| \phi^o(x, \xi^\star), \qquad (x, \xi^\star) \in T^\star M, \ \lambda \in \mathbb{R}, \tag{3.3}$$

The function $\phi^o(x, \cdot)$ is even, since we supposed $\sigma(x, \cdot)$ to be even([7]). We consider the function $\phi^o(x, \cdot)$ as acting on differentials df_x of functions $f : M \to \mathbb{R}$ at $x \in M$.
We have([8])

$$\mathcal{F}_{\phi^o}(E) := \int_{\partial E} \phi^o(x, \nu) a(x) \ d\mathcal{H}^{n-1}(x) = \mathcal{F}(E). \tag{3.4}$$

For computational convenience, from now on we will consider the functional $\mathcal{F}_{\phi^o}$ in place of $\mathcal{F}$. The gradient flow of the functional $\mathcal{F}_{\phi^o}$ will lead to anisotropic mean curvature flow:

[4]Various results that we will present could be extended without assuming that $\sigma(x, \cdot)$ is even, but we prefer to keep this assumption in order to make simpler the presentation.

[5]We shall see that if σ does not depend on x, from a geometric point of view it is natural to take a to be a positive constant. For simplicity, the reader can assume $a \equiv 1$. We notice that, by redefining σ, one can also include the function a into the new σ. We prefer however to keep σ and a separate.

[6]Strictly related to the convexity of the function $\phi^o(x, \cdot)$ defined in (3.2) below.

[7]If $\sigma(x, \cdot)$ were not even, we should drop the absolute value on the right hand side of (3.3) and take $\lambda > 0$.

[8]We prefer to skip the dependence on a of the functional.

we will mostly be concerned with the case of a function ϕ^o which is independent of x and $a \equiv 1$([9]).
For all $x \in \partial E$ for which $\nu(x)$ is defined, we introduce [48], [47] the normalized covariant vector field([10])

$$\nu^E_{\phi^o}(x) := \frac{\nu^E(x)}{\phi^o(\nu^E(x))} = \nu_{\phi^o}(x). \tag{3.5}$$

In components $\nu_{\phi^o} = (\nu_{\phi^o 1}, \ldots, \nu_{\phi^o n})$.
If $F \subset \partial E$ is a facet of a polyhedral set ∂E, we set([11])

$$\nu_{\phi^o}(F) := \frac{\nu(F)}{\phi^o(\nu(F))}, \tag{3.6}$$

where $\nu(F)$ is the unit normal to $\mathrm{int}(F)$ pointing toward $M \setminus E$.
We define([12])

$$B_{\phi^o}(x) := \left\{\xi^\star \in \Lambda^1 V : \phi^o(x, \xi^\star) \leq 1\right\}, \qquad x \in M. \tag{3.7}$$

The set $B_{\phi^o}(x)$ uniquely identifies $\phi^o(x, \cdot)$, in view of the homogeneity property (3.3).
Notice that if $\phi^o \in \mathcal{C}^1\left(M \times (\Lambda^1 V \setminus \{0\})\right)$, (3.3) yields

$$\phi^o(x, \xi^\star) = \xi^\star \cdot \phi^o_{\xi^\star}(x, \xi^\star), \qquad (x, \xi^*) \in M \times (\Lambda^1 V \setminus \{0\}), \tag{3.8}$$

where $\phi^o_{\xi^\star}$ denotes the gradient of $\phi^o(x, \cdot)$ with respect to $\xi^\star$.

Definition 3.1 (**Spatial homogeneity).** *We say that σ (resp. ϕ^o) is spatially homogeneous if it is independent of x.*

In this case we write $\phi^o : \Lambda^1 V \to [0, +\infty]$, and the right hand side of formula (3.7) is denoted by B_{ϕ^o} ([13]).

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[9]However it is useful to keep in mind that other cases may be of interest. For instance, $a \equiv 1$, $\phi^o(x, \xi^\star) = b(x)|\xi^\star|$, where $b(x) = e^{\pm \sum_{i=1}^n \alpha_i x_i^2}$, where $\alpha_1, \ldots, \alpha_n$ are positive real numbers [79], [15].
[10]In the quoted references this vector field is denoted by ν_ϕ.
[11]Do not confuse this notation with the notation in (2.2).
[12]The set $B_{\phi^o}(x)$ is sometimes called Frank diagram, at least under some further assumptions on ϕ^o.
[13]The set $\{v : v = \rho\nu, \nu \in \mathbb{S}^{n-1}, \rho = \frac{1}{\sigma(\nu)}\} = \partial B_{\phi^o}$ is sometimes called polar plot of σ.

Example 3.2 (Dual norms). The first examples of spatially homogeneous ϕ^o are the following:

- $\phi^o(\xi^\star) = |\xi^\star|$ (euclidean norm, isotropic case).

- $\phi^o(\xi^\star) = \sqrt{\sum_{i,j=1}^n g^{ij}\xi_i^\star\xi_j^\star}$, where (g^{ij}) is a positive definite symmetric matrix (Riemannian norm). In this case B_{ϕ^o} is an ellipsoid centered at the origin([14]). See Figure 2.

- Let $p \in (1,+\infty)$ and $\phi^o(\xi^\star) := (\sum_{i=1}^n |\xi_i^\star|^p)^{1/p}$ (l^p norms). If $p > 2$ then ∂B_{ϕ^o} is of class $\mathcal{C}^2$ but there are some points of ∂B_ϕ where its second fundamental form vanishes. If $p \in (1,2)$ then ∂B_ϕ is not of class $\mathcal{C}^2$.

- A relevant case in these notes is when B_{ϕ^o} is a (convex) n-dimensional polyhedron centered at the origin, and centrally symmetric [139], [140], [142]. See Figure 1.

- Another interesting case is when $B_{\phi^o} = C \times [-1,1]$, where C is an $(n-1)$-dimensional centrally symmetric convex body [110], [112], [51], [30].

Remark 3.3 (Degenerate cases). Let σ be spatially homogeneous: if there exists $v \in \mathbb{S}^{n-1}$ such that $\sigma(v) = 0$, then the whole line $\mathbb{R}v$ is contained in B_{ϕ^o}. In particular, B_{ϕ^o} is unbounded. On the other hand, if there exists $v \in \mathbb{S}^{n-1}$ such that $\sigma(v) = +\infty$, then $\mathbb{R}v \setminus \{0\}$ is not contained in B_{ϕ^o} (hence the origin is not an interior point of B_{ϕ^o}). For example [21], [25], setting $I := \{(\xi_0^\star, \xi^\star) \in \mathbb{R} \times \Lambda^1 V \simeq \Lambda^1\mathbb{R}^{1+n} : -\xi_0^2 + |\xi^\star|^2 \leq 1\}$, we can take $\phi^o(\xi_0^\star, \xi^\star) := \inf\{\lambda > 0 : (\xi_0^\star, \xi^\star) \in \lambda I\}$ for any $(\xi_0^\star, \xi^\star) \in \Lambda^1\mathbb{R}^{n+1}$. Note that I is star-shaped with respect to the origin, the origin is not in the interior of I, and ϕ^o takes also the value $+\infty$. Examples of unbounded B_{ϕ^o} have been considered in [97], see also [37].

Definition 3.4 (Convexity). *We say that $\phi^o : T^\star M \to [0,+\infty)$ is convex if $\phi^o(x,\cdot)$ is convex for any $x \in M$.*

Remark 3.5. All functions ϕ^o in Example 3.2 are convex. In addition they satisfy([15])

$$\lambda|\xi^\star| \leq \phi^o(x,\xi^\star), \qquad (x,\xi^\star) \in T^\star M, \tag{3.9}$$

for a suitable constant $\lambda > 0$ (depending on ϕ^o).

Definition 3.6 (Metrics on $T^\star M$). *The symbol $\mathcal{M}(T^\star M)$ denotes the class of metrics on $T^\star M$, namely of all continuous functions $\phi^o : T^\star M \to [0,+\infty)$ which are convex, and satisfy* (3.3) *and* (3.9).

Among convex ϕ^o we are mainly interested in the crystalline ones [139].

Definition 3.7 (Crystalline metrics). *If $\phi^o \in \mathcal{M}(T^\star M)$ is spatially homogeneous and B_{ϕ^o} is a polyhedron we say that ϕ^o is crystalline.*

[14] If g^{ij} would depend on x, then the ellipsoid would depend on x, and M would become the simplest example of Riemannian manifold.

[15] If we assume continuity of ϕ^o, in view of (3.3) inequality (3.9) becomes equivalent to the inequalities

$$\lambda|\xi^\star| \leq \phi^o(x,\xi^\star) \leq \Lambda|\xi^\star|, \qquad (x,\xi^\star) \in T^\star M,$$

for two constants $0 < \lambda \leq \Lambda < +\infty$. Relevant consequences are that $B_{\phi^o}(x)$ contains the origin in its interior, and it is star-shaped (with respect to the origin).

3.1.1 The map T_{ϕ^o}

For a fixed x, we now define a map that will play a major role in the analysis of anisotropic mean curvature flow. In order to give the definition, we assume the validity of one of the two following hypotheses: either

$$(\phi^o)^2 \in \mathcal{C}^1 (T^\star M) \tag{3.10}$$

or

$$\phi^o \text{ is convex.} \tag{3.11}$$

Notation. If ϕ^o satisfies (3.10) the symbol $\nabla_{\xi^\star}((\phi^o)^2)$ denotes the gradient vector field of $(\phi^o(x,\cdot))^2$ with respect to $\xi^\star$. Assumption (3.11) is equivalent to the convexity of $(\phi^o(x,\cdot))^2$, and in this case the same symbol $\nabla_{\xi^\star}(\phi^o)^2$ denotes the subdifferential([16]) of $(\phi^o(x,\cdot))^2$ with respect to $\xi^\star$ [133]. Moreover, if (3.11) holds, then $\phi^o_{\xi^\star}$ denotes the subdifferential of $\phi^o(x,\cdot)$ with respect to $\xi^\star$.

Definition 3.8 (The map T_{ϕ^o}). *Let $x \in M$. We define $T_{\phi^o}(x,\cdot) : \Lambda^1 V \to \mathcal{P}(\Lambda_1 V)$ as*

$$T_{\phi^o}(x,\xi^\star) := \frac{1}{2}(\nabla_{\xi^\star}((\phi^o)^2))(x,\xi^\star). \tag{3.12}$$

Under assumption (3.11), $T_{\phi^o}(x,\cdot)$ is sometimes called duality map [55], and it is a possibly multivalued maximal monotone([17]) operator [54]. It is multivalued when ϕ^o is crystalline. Note that $T_{\phi^o}(x,\cdot)$ is one-homogeneous, namely

$$T_{\phi^o}(x,\lambda\xi^\star) = |\lambda| T_{\phi^o}(x,\xi^\star), \qquad (x,\xi^\star) \in T^\star M,\ \lambda \in \mathbb{R}.$$

Example 3.9 (Riemannian case). If $\phi^o(x,\xi) = (\sum_{i,j=1}^n g^{ij}(x)\xi_i^\star\xi_j^\star)^{1/2}$ is a Riemannian metric, then([18]) $(T_{\phi^o}(x,\xi^\star))^i = \sum_{j=1}^n g^{ij}(x)\xi_j^\star$.

In the case considered in Remark 3.3, where $(\phi^o(\xi^\star))^2 = -(\xi_0^\star)^2 + (\xi_1^\star)^2 + \cdots (\xi_n^\star)^2$, the map T_{ϕ^o} takes $\xi^\star = (\xi_0^\star, \xi_1^\star, \dots, \xi_n^\star)$ into $(-\xi_0^\star, \xi_1^\star, \dots, \xi_n^\star)$, exchanging the sign of the zeroth component.

Remark 3.10. If ϕ^o is spatially homogeneous and $\xi^\star \in \partial B_{\phi^o}$, then $T_{\phi^o}(\xi^\star)$ is a suitable normalization([19]) of the exterior normal cone orthogonal to ∂B_{ϕ^o} at $\xi^\star$.

Definition 3.11. *Let ∂E be Lipschitz and let $x \in \partial E$ be a point where $\nu(x)$ is defined. If* (3.10) *holds we define [48], [47] the contravariant vector field $n_\phi^E = n_\phi$ at x as*

$$n_\phi(x) := T_{\phi^o}(x, \nu_{\phi^o}(x)) = \phi^o_{\xi^\star}(x, \nu_{\phi^o}(x)).$$

In components([20]) $n_\phi = (n_\phi^1, \dots, n_\phi^n)$. If ϕ^o is convex, n_ϕ is sometimes called the Cahn-Hoffman vector field.

Notice that, using (3.3), it follows that

$$\langle \nu_{\phi^o}(x), n_\phi(x)\rangle = 1. \tag{3.13}$$

[16]The subdifferential of a convex function $u : M \to \mathbb{R}$ at $x \in M$ is defined as $\{\xi^* \in \Lambda^1 V : u(y) \geq u(x) + \langle \xi^*, y - x\rangle \ \forall y \in M\}$.

[17]Monotone means: $\xi_i \in T_{\phi^o}(x,\xi_i^\star)$ for $i = 1,2 \Rightarrow \langle \xi_1^\star - \xi_2^\star, \xi_1 - \xi_2\rangle \geq 0$. Maximal monotone means that $T_{\phi^o}(x,\cdot)$ is monotone and its graph is not properly included in the graph of any other monotone operator.

[18]The map T_{ϕ^o} is used to exchange the indices from lower to upper.

[19]See the fifth item of Remark 3.18 below.

[20]Pay attention to the notation: n is the dimension of V, n_ϕ is the vector field.

Remark 3.12 (**Cahn-Hoffman selections).** Under the sole assumption (3.11), and supposing also for simplicity that ϕ^o is spatially homogeneous, there are several possible choices of vector fields $\eta : \partial E \to V$ which satisfy $\eta(x) \in T_{\phi^o}(\nu_{\phi^o}(x))$ for $\mathcal{H}^{n-1}$-almost every $x \in \partial E$, since in this case $T_{\phi^o}(\nu_{\phi^o}(x))$ is a (compact) convex set. In Section 4 we will impose further regularity on η in order to define what we will call ϕ-regular sets.

If $F \subset \partial E$ is a facet of a polyhedral ∂E and ϕ^o is crystalline, we set

$$\widetilde{B}_\phi^F := T_{\phi^o}(\nu_{\phi^o}(F)), \tag{3.14}$$

see Figure 5. Note the presence of the symbol ϕ (that we are going to define in the next section) on the left hand side of (3.14).

3.2 The convex function ϕ

Under one of the two assumptions (3.10), (3.11), given $x \in M$ we can consider the image $\Sigma(x)$ of the boundary of the star-shaped set $B_{\phi^o}(x)$ via the map $T_{\phi^o}(x, \cdot)$,

$$\Sigma(x) := T_{\phi^o}(x, \partial B_{\phi^o}(x)).$$

If $B_{\phi^o}(x)$ is not convex then it may happen, for instance in $n = 2$ dimensions and if ∂B_{ϕ^o} is a smooth simple closed curve having the origin in its interior, that $\Sigma(x)$ is a curve with cusps and self-intersections [97]. These kind of singularities cannot occur if ϕ^o is convex([21]), and in this case it is possible to define a function $\phi : TM \to [0, +\infty]$ as follows:

$$\phi(x, \xi) := \inf \{\lambda > 0 : (x, \xi) \in \lambda T_{\phi^o}(x, B_{\phi^o}(x))\}, \qquad (x, \xi) \in TM.$$

Then $\phi(x, \cdot)$ is one-homogeneous, namely

$$\phi(x, \lambda\xi) = |\lambda|\phi(x, \xi), \qquad (x, \xi) \in TM, \ \lambda \in \mathbb{R}. \tag{3.15}$$

Moreover $\Sigma(x) = \partial T_{\phi^o}(x, B_{\phi^o}(x))$. Finally, if we define

$$B_\phi(x) := T_{\phi^o}(x, B_{\phi^o}(x)),$$

then

$$B_\phi(x) = \{(x, \xi) \in TM : \phi(x, \xi) \leq 1\},$$

and ϕ is convex (i.e., $\phi(x, \cdot)$ is convex for any $x \in M$).

Remark 3.13. As it follows from the above presentation, when writing the symbol ϕ we assume that ϕ^o is convex (and, as a consequence, so is ϕ).

Definition 3.14 (**Metrics on TM**). *The symbol $\mathcal{M}(TM)$ denotes the class of metrics on TM, namely of all continuous functions ϕ which are convex and satisfy* (3.15) *and*

$$\phi(x, \xi) \geq \mu|\xi|, \qquad (x, \xi) \in TM,$$

for a suitable constant $\mu > 0$ (depending on ϕ).

[21] It is not the aim of these notes to investigate the interesting case of a nonconvex B_{ϕ^o}.

Definition 3.15 (**Regular metrics on** TM)**.** *Let $\phi \in \mathcal{M}(TM)$. We say that ϕ is regular if for any $x \in M$ the set $B_\phi(x)$ has boundary of class $\mathcal{C}^\infty$ and each principal curvature of $\partial B_\phi(x)$ is strictly positive at each point of $\partial B_\phi(x)$. We denote by $\mathcal{M}_{\mathrm{reg}}(TM)$ the class of all regular metrics in TM.*

It is possible to prove that if $\phi \in \mathcal{M}_{\mathrm{reg}}(TM)$, then $B_{\phi^o}(x)$ has boundary of class $\mathcal{C}^\infty$ and each principal curvature of $\partial B_{\phi^o}(x)$ is strictly positive at each point of $\partial B_{\phi^o}(x)$. Namely, $\phi^o \in \mathcal{M}_{\mathrm{reg}}(T^\star M)$. See also [132] for a list of related propertied.

Example 3.16 (**Minkowski space**)**.** If a metric ϕ on TM is spatially homogeneous, it is a norm on $\Lambda_1 V$, called a Minkowski norm (or Minkowski metric). The normed vector space $(\Lambda_1 V, \phi)$ is called Minkowski space [146] and is the simplest example of a Finsler manifold [22].

The symbol $\nabla_\xi(\phi^2)$ (resp. ϕ_ξ) denotes the subdifferential of $(\phi(x,\cdot))^2$ (resp. of $\phi(x,\cdot)$) with respect to ξ.

Definition 3.17 (**The map** T_ϕ)**.** *Let $x \in M$. We define $T_\phi(x,\cdot) : \Lambda_1 V \to \mathcal{P}(\Lambda^1 V)$ as*

$$T_\phi(x,\xi) := \frac{1}{2}(\nabla_\xi(\phi^2))(x,\xi), \qquad (x,\xi) \in TM. \tag{3.16}$$

$T_\phi(x,\cdot)$ is a one-homogeneous maximal monotone map.

Remark 3.18 (**Duality**)**.** Assume $\phi^o \in \mathcal{M}(T^\star M)$ and $\phi \in \mathcal{M}(TM)$. The following properties hold [134], [146].

- $\phi(x,\xi) = \sup\left\{\langle \xi^\star, \xi\rangle \ : \ \xi^\star \in \Lambda^1 V,\ \phi^o(x,\xi^\star) \le 1\right\}$ for any $(x,\xi^\star) \in TM$([22]);
- $\phi^{oo} = \phi$ (the dual of $\Lambda^1 V$ can be identified with $\Lambda_1 V$);
- if ϕ^o is crystalline then ϕ is crystalline;
- if $T_{\phi^o}(x,\cdot)$ and $T_\phi(x,\cdot)$ are single valued, then [47]
    - for any $x \in M$, $\xi \in \Lambda_1 V \setminus \{0\}$ and $\xi^\star \in \Lambda^1 V \setminus \{0\}$ we have $\phi^o(x, \phi_\xi(x,\xi)) = \phi(x, \phi^o(x,\xi^\star)) = 1$, and $\phi^o(x,\xi^\star)\phi_\xi(x, \phi^o_{\xi^\star}(x,\xi^\star)) = \xi^\star$, $\phi(x,\xi)\phi^o_{\xi^\star}(x, \phi_\xi(x,\xi)) = \xi$;
    - $T_\phi(x,\cdot)T_{\phi^o}(x,\cdot) = \mathrm{Id}_{\Lambda^1 V}$, $T_{\phi^o}(x,\cdot)T_\phi(x,\cdot) = \mathrm{Id}_{\Lambda_1 V}$.
- Assume for simplicity that ϕ^o is spatial homogeneous. Then T_ϕ (resp. T_{ϕ^o}) takes ∂B_ϕ (resp. ∂B_{ϕ^o}) onto ∂B_{ϕ^o} (resp. onto ∂B_ϕ). If $\xi \in \partial B_\phi$, $T_\phi(\xi)$ is the intersection of the closed outward normal cone to ∂B_ϕ with ∂B_{ϕ^o}.

Remark 3.19. Assuming ϕ to be convex, it is equivalent([23]) to develop the theory starting with ϕ and then defining ϕ^o by duality (replace ϕ by ϕ^o and $\Lambda^1 V$ with $\Lambda_1 V$ in the first item of Remark 3.18).

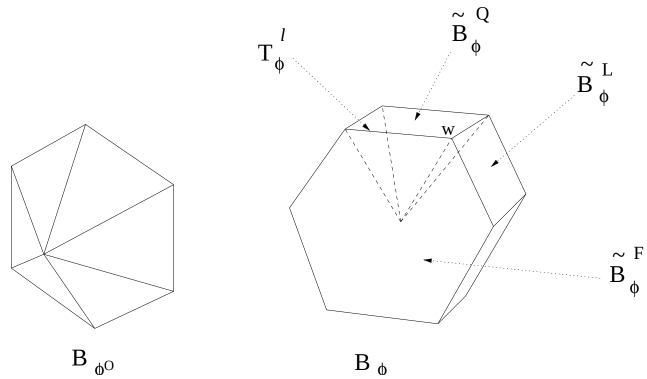

Figure 1: two convex polyhedral bodies one dual of the other. Compare Figure 5 and (3.14): F, L and Q are facets of E in Figure 5, and $\widetilde{B}_\phi^F$, $\widetilde{B}_\phi^L$ and $\widetilde{B}_\phi^Q$ are the corresponding facets of B_ϕ.

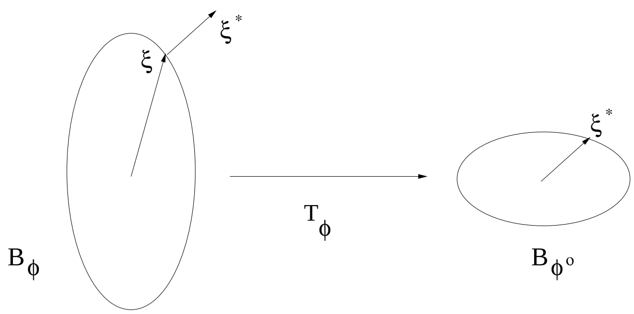

Figure 2: Pick $\xi \in \partial B_\phi$. The covector $T_\phi(\xi) \in \partial B_{\phi^o}$ is orthogonal to ∂B_ϕ, and $|T_\phi(\xi)| = \langle \nu^{B_\phi}(\xi), \xi \rangle^{-1}$.

Example 3.20 (Polyhedral dual bodies). In Figure 1 for a crystalline ϕ^o, we show B_{ϕ^o} and its dual body B_ϕ. If $\xi \in \partial B_\phi$ is a point in the relative interior of a facet, then the normal cone $T_\phi(\xi)$ to ∂B_ϕ at ξ is a vertex in ∂B_{ϕ^o}; if $\xi \in \partial B_\phi$ is a point in the relative interior of an edge, then $T_\phi(\xi)$ is a closed edge in ∂B_{ϕ^o}; if $\xi \in \partial B_\phi$ is a vertex, then $T_\phi(\xi)$ is a closed facet in ∂B_{ϕ^o}.

When ϕ (resp. ϕ^o) is regular and spatially homogeneous, sometimes we simply write $\phi \in \mathcal{M}(\Lambda_1 V)$ (resp. $\phi^o \in \mathcal{M}(\Lambda^1 V)$).

Remark 3.21. Let $\phi \in \mathcal{M}(\Lambda_1 V)$ be spatially homogeneous. We give here a recipe to construct the dual body B_{ϕ^o} of B_ϕ, see Figure 2. Assume for simplicity that $\phi \in \mathcal{C}^1(\Lambda_1 V \setminus \{0\})$. Take a point $\xi \in \partial B_\phi$. Then $\frac{T_\phi(\xi)}{|T_\phi(\xi)|}$ is orthogonal to ∂B_ϕ at ξ, and points out of B_ϕ. Moreover

$$|T_\phi(\xi)| = (\mathrm{dist}(T_\xi(\partial B_\phi), 0))^{-1},$$

where $T_\xi(\partial B_\phi)$ is the the tangent space to ∂B_ϕ at ξ. Indeed, setting $\xi^\star := T_\phi(\xi)$, we have that $\xi^\star$ realizes the supremum in the first item of Remark 3.18, so that $1 = \phi(\xi) = \langle \xi^\star, \xi \rangle$. Therefore $1 = \phi(\xi) = |\xi^\star| \langle \nu^{B_\phi}(\xi), \xi \rangle$, and hence $|\xi^\star| = \langle \nu^{B_\phi}(\xi), \xi \rangle^{-1}$. It is then enough to observe that the euclidean distance $\mathrm{dist}(T_\xi(\partial B_\phi), 0)$ between $T_\xi(\partial B_\phi)$ and the origin equals $\langle \nu^{B_\phi}(\xi), \xi \rangle$. In this way we construct B_{ϕ^o}, starting from B_ϕ, since ∂B_{ϕ^o} consists of all points of the form $T_\phi(\xi)$, with $\xi \in \partial B_\phi$.

Remark 3.22 (The Legendre transform). Some of the above concepts, as it can be seen from formula (3.17) below, can be given in terms of the Legendre transform, that for completeness we recall here. Let $f : TM \to [0, +\infty)$ be a continuous function, such that for any $x \in M$ the map $\xi \to f(x, \xi)$ is convex and([24]) of class $\mathcal{C}^1$. Define $f^\star : T^\star M \to (-\infty, +\infty]$ as

$$f^\star(x, \xi^\star) := \sup \Big\{ \langle \xi^\star, \xi \rangle - f(x, \xi) : \xi \in \Lambda_1 V \Big\}.$$

Let $\mathcal{E} = \mathcal{E}(x) := \{(\xi, \tau) \in \Lambda_1 V \times \mathbb{R} : \tau > f(x, \xi)\}$ be the epigraph of $f(x, \cdot)$, which is a convex set. Given $\xi^\star \in \Lambda^1 V \setminus \{0\}$, consider in $V \times \mathbb{R}$ the set $\mathcal{P}^{\xi^\star}$ of all parallel hyperplanes orthogonal to $(\xi^\star, -1)$. If there is does not exist any point in $\partial\mathcal{E}$ for which the tangent space to $\partial\mathcal{E}$ belongs to $\mathcal{P}^{\xi^\star}$, then $f^\star(x, \xi^\star) = +\infty$. Otherwise, if there exists one point $z \in \partial\mathcal{E}$ having tangent space belonging to $\mathcal{P}^{\xi^\star}$, we take the unique $\pi^{\xi^\star} \in \mathcal{P}^{\xi^\star}$ containing z. Then we consider the intersection of $\pi^{\xi^\star}$ with the vertical axis $\{0\} \times \mathbb{R}$, and we define $f^\star(x, \xi^\star)$ as minus the vertical component of such an intersection, namely

$$f^\star(x, \xi^\star) = -t, \qquad (0, t) = \pi^{\xi^\star} \cap \Big\{ (\xi, \tau) \in \Lambda_1 V \times \mathbb{R} : \xi = 0 \Big\}.$$

For example,

$$\nu \in \Lambda^1 V,\ c \in \mathbb{R},\ f(x, \xi) = \langle \xi, \nu \rangle + c \Rightarrow f^\star(\xi^\star) = \begin{cases} -c & \text{if } \xi^\star = \nu, \\ +\infty & \text{if } \xi^\star \neq \nu, \end{cases}$$

$$\alpha \in \mathbb{R},\ f(x, \xi) = \alpha |\xi|^2 \Rightarrow f^\star(\xi^\star) = \frac{1}{4\alpha} |\xi^\star|^2,$$

and for a non everwywhere differentiable function a similar construction gives

$$f(x, \xi) = \phi(\xi) \Rightarrow f^\star(\xi^\star) = \begin{cases} 0 & \text{if } \phi^o(\xi^\star) \leq 1, \\ +\infty & \text{if } \phi^o(\xi^\star) > 1. \end{cases} \tag{3.17}$$

[22]The function $\phi(x, \cdot)$ is sometimes called the support function of $B_{\phi^o}(x)$, and $B_{\phi^o}(x)$ is called the polar reciprocal of $B_\phi(x)$, [146, pag. 50]. $B_{\phi^o}(x)$ is called the dual body of $B_\phi(x)$. Once we assume ϕ^o to be convex, then the right hand side of the first item in Remark 3.18 can be taken as the definition of ϕ.

[23]There could be, however, geometrical or physical reasons to prefer B_ϕ instead of B_{ϕ^o} as the starting point of the theory.

[24]Even if $f(x, \cdot)$ is $\mathcal{C}^1$ and not convex (or convex but not of class $\mathcal{C}^1$), still $f^\star(x, \cdot)$ is defined and it is convex.

3.3 The distance function dist_ϕ

We shall assume from now on that $\phi : TM \to [0, +\infty)$ is continuous([25]).

Definition 3.23. *Given $x, y \in M$ we set*

$$\mathrm{dist}_\phi(x, y) := \inf \left\{ \int_0^1 \phi(\gamma, \dot\gamma)\ dt : \gamma \in AC([0,1]; M), \gamma(0) = x, \gamma(1) = y \right\}, \tag{3.18}$$

where $AC([0,1]; M)$ denotes the class of all absolutely continuous [14] curves $\gamma : [0,1] \to M$. Notice that if ϕ is spatially homogeneous and convex, then $\mathrm{dist}_\phi(x, y) = \phi(y - x)$. Recall that if $\phi(x, \xi) = |\xi|$, we set $\mathrm{dist}_\phi = d$.
For any $F \subseteq M$ we denote

$$\mathrm{dist}_\phi(x, F) := \inf_{y \in F} \mathrm{dist}_\phi(x, y), \qquad x \in M.$$

The next definition will be applied only to rather regular sets.

Definition 3.24 (**Signed ϕ-distance**). *Assume that $\partial E \in \mathrm{Lip}(M)$. We define the signed ϕ-distance function from ∂E negative in E and positive in $M \setminus E$ as*

$$d_\phi^E(x) = d_\phi(x) := \mathrm{dist}_\phi(x, E) - \mathrm{dist}_\phi(x, M \setminus E), \qquad x \in M. \tag{3.19}$$

3.3.1 ϕ-Volume

Once we have the distance dist_ϕ at our disposal, we can define the n-dimensional Hausdorff measure $\mathcal{H}^n_\phi$ with respect to the distance dist_ϕ [94], i.e., for $S \subseteq \mathbb{R}^n$

$$\mathcal{H}^n_\phi(S) := \frac{\omega_n}{2^n} \lim_{\rho \to 0^+}\ \inf \left\{ \sum_{i=1}^{+\infty} (\mathrm{diam}_{\mathrm{dist}_\phi}(S_i))^n : S \subseteq \bigcup_{i=1}^{+\infty} S_i,\ \mathrm{diam}_{\mathrm{dist}_\phi}(S_i) < \rho \right\}, \tag{3.20}$$

where, if $F \subseteq \mathbb{R}^n$, $\mathrm{diam}_{\mathrm{dist}_\phi}(F) := \sup\{d_\phi(x, y) : x, y \in F\}$, and $\omega_n := \mathcal{L}^n(\{\xi \in M : |\xi| < 1\})$. Notice that if ϕ is spatially homogeneous, then $\mathcal{H}^n_\phi(B_\phi) = \omega_n$, since $\mathrm{diam}_{\mathrm{dist}_\phi}(B_\phi) = 2$.

Example 3.25. Assume that ϕ is spatially homogeneous and riemannian, i.e., $\phi(\xi) = |\sqrt{g}\xi|$ for any $\xi \in V$, where $g = (g_{ij})$ is a symmetric positive definite $(n \times n)$-matrix, and we write $g = \sqrt{g^T}\sqrt{g}$. Then

$$\mathcal{L}^n(B_\phi) = \frac{\omega_n}{\det \sqrt{g}}$$

$$\mathcal{H}^n_\phi(S) = \det \sqrt{g}\ \mathcal{L}^n(S) = \frac{\omega_n}{\mathcal{L}^n(B_\phi)} \mathcal{L}^n(S) = \mathcal{H}^n(T_\phi(S)).$$

We recall the following representation result for the Hausdorff measure [56].
Define

$$\mathrm{vol}_\phi(x) := \frac{\omega_n}{\mathcal{L}^n\big(B_\phi(x)\big)}, \qquad x \in M. \tag{3.21}$$

[25]Discontinuous $\phi(\cdot, \xi)$ have been considered for instance in [74], [6], [7], see also the references in these papers.

Theorem 3.26 (**Representation of ϕ-volume**). *If $B \subseteq \Omega$ is a Borel set, then*

$$\mathcal{H}^n_\phi(B) = \int_B \mathrm{vol}_\phi \, dx. \tag{3.22}$$

The distance function d_ϕ is useful for various reasons; one of them is that it gives a natural extension of ν_{ϕ^o} out of ∂E.

3.4 Eikonal equation and extensions

Let $\phi \in \mathcal{M}_{\mathrm{reg}}(TM)$ and ∂E be compact and of class $\mathcal{C}^\infty$ (resp. of class $\mathcal{C}^2$). It is possible to prove that there exists a tubular neighbourhood of ∂E where the signed ϕ-distance d_ϕ in (3.19) is of class $\mathcal{C}^\infty$ (resp. $\mathcal{C}^2$), see also [105].
The proof of the following theorem can be found for instance in [48]. See [23], [62] for related results.

Theorem 3.27 (**Eikonal equation**). *Let ∂E be compact and let $U \subset M$ be a tubular neighbourhood of ∂E such that $d_\phi \in \mathcal{C}^\infty(U)$. Then d_ϕ satisfies the eikonal equation in U:*

$$(\phi^o(x, \nabla d_\phi(x)))^2 = 1, \qquad x \in U, \tag{3.23}$$

so that in particular

$$\nabla d_\phi = \nu_{\phi^o} \qquad \text{on } \partial E.$$

Definition 3.28 (**Extension of n_ϕ**). *Under the assumptions at the beginning of the section, we can extend the Cahn-Hoffman vector field n_ϕ on the whole of U as follows:*

$$N_\phi(x) := T_{\phi^o}(x, \nabla d_\phi(x)), \qquad x \in U. \tag{3.24}$$

Note that

$$\phi^o(x, N_\phi(x)) = 1, \qquad \langle \nabla d_\phi(x), N_\phi(x) \rangle = 1, \qquad x \in U.$$

3.5 Appendix: definitions of ∇_ϕ, div_ϕ, Δ_ϕ. ϕ-Distributional perimeter

Assume that $\phi^o \in \mathcal{M}_{\mathrm{reg}}(T^\star M)$. For completeness, we define here various operators([26]) naturally related to ϕ^o. If $u \in \mathcal{C}^2(M)$ we define the vector field

$$\nabla_\phi u(x) := T_{\phi^o}(x, \nabla u(x)), \qquad x \in M. \tag{3.25}$$

Note that if $\phi(x,\xi) = (\sum_{i,j=1}^n g_{ij}(x)\xi^i\xi^j)^{1/2}$ is a riemannian metric in M, then the i-th component of $\nabla_\phi u(x)$ equals $\sum_{j=1}^n g^{ij}(x)\nabla_j u(x)$ where (g^{ij}) is the inverse of (g_{ij}).
If $\eta \in \mathcal{C}^1(M;V)$ we set

$$\begin{aligned} \mathrm{div}_\phi \, \eta &:= \mathrm{div}\eta + \nabla\left(\log(\mathrm{vol}_\phi)\right) \cdot \eta, \\ \Delta_\phi u &:= \mathrm{div}_\phi \nabla_\phi u. \end{aligned} \tag{3.26}$$

With the above definitions we have the following Gauss-Green type formula.

[26]We give the definitions assuming the validity of (3.29) below, for the sake of simplicity. See also [35].

Proposition 3.29 (Divergence Theorem). *If $\Omega \subset M$ is a bounded open set of class $\mathcal{C}^1$, $u \in \mathcal{C}^1(\Omega)$ and $g \in \mathcal{C}^1(\Omega; \Lambda_1 V) \cap \mathcal{C}(\overline{\Omega}; \Lambda_1 V)$, then*

$$\int_\Omega u \mathrm{div}_\phi g \ d\mathcal{H}^n_\phi + \int_\Omega \nabla u \cdot g \ d\mathcal{H}^n_\phi = -\int_{\partial\Omega} u \ \nu^\Omega_{\phi^o} \cdot g \ \phi^o(x, \nu^\Omega) \mathrm{vol}_\phi \ d\mathcal{H}^{n-1}. \tag{3.27}$$

Proof. Definition (3.26) of div_ϕ implies

$$\begin{aligned} u \ \mathrm{div}_\phi g \ \mathrm{vol}_\phi &= u \ \mathrm{div} g \ \mathrm{vol}_\phi + u \ \nabla(\log(\mathrm{vol}_\phi)) \cdot g \ \mathrm{vol}_\phi \\ &= \mathrm{div}(u \ g \ \mathrm{vol}_\phi) - \nabla u \cdot g \ \mathrm{vol}_\phi. \end{aligned}$$

Hence, using the Gauss-Green theorem and recalling (3.22), we get

$$\begin{aligned} \int_\Omega u \ \mathrm{div}_\phi g \ d\mathcal{H}^n_\phi &= \int_{\partial\Omega} u \ \nu^\Omega \cdot g \ \frac{\omega_n}{\mathcal{L}^n(B_\phi)} \ d\mathcal{H}^{n-1} - \int_\Omega \nabla u \cdot g \ d\mathcal{H}^n_\phi \\ &= \int_{\partial\Omega} u \ {\nu_{\phi^o}}^\Omega \cdot g \ \phi^o(x, \nu^\Omega) \mathrm{vol}_\phi \ d\mathcal{H}^{n-1} - \int_\Omega \nabla u \cdot g \ d\mathcal{H}^n_\phi. \end{aligned}$$

□

In view of (3.27) it is natural to introduce the surface measure

$$d\mathcal{P}_\phi(B) := \int_{B \cap \partial E} \phi^o(x, \nu(x)) \mathrm{vol}_\phi(x) \ d\mathcal{H}^{n-1}(x), \qquad B \subseteq M. \tag{3.28}$$

We will make this choice in the next chapters.

Remark 3.30. A rather natural choice of the function a in (3.1) and (3.4) is

$$a = \mathrm{vol}_\phi. \tag{3.29}$$

With this choice we have that $\mathcal{F}_{\phi^o}$ equals the functional in (3.28) when $B = M$. This functional turns out to be the ϕ-perimeter, defined in the distributional sense [6], [48], and also the ϕ-Minkowski content [94], [14], [48], [27], defined as

$$\mathcal{M}^{n-1}_\phi(\partial E) := \lim_{\rho \to 0^+} \frac{\mathcal{H}^n_\phi(\{x \in M : \mathrm{dist}_\phi(x, \partial E) < \rho\})}{2\rho}, \tag{3.30}$$

but not([27]) the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}_\phi$ with respect to dist_ϕ, defined as

$$\mathcal{H}^{n-1}_\phi(S) := \frac{\omega_{n-1}}{2^{n-1}} \lim_{\rho \to 0^+} \ \inf \left\{ \sum_{i=1}^{+\infty} (\mathrm{diam}_{\mathrm{dist}_\phi}(S_i))^k : S \subseteq \bigcup_{i=1}^{+\infty} S_i, \ \mathrm{diam}_{\mathrm{dist}_\phi}(S_i) < \rho \right\}, \tag{3.31}$$

where $\omega_{n-1} := \mathcal{L}^{n-1}(\xi \in \mathbb{R}^{n-1} : |\xi| < 1\})$. It is interesting to observe that, adopting (3.28) as the definition of $(n-1)$-dimensional ϕ-measure, it turns out that B_ϕ satisfies the

[27] Even for a spatially homogeneous ϕ [48].

isoperimetric property [48]. Eventually, other geometric measures could be considered [146], [13], for instance the Benson area [49], [13].([28])

When ϕ^o is spatially homogeneous, the choice in (3.29) gives

$$a = \frac{\omega_n}{\mathcal{L}^n(B_\phi)} =: c_n. \tag{3.32}$$

[28]If $\xi^\star \in \Lambda^1 V$, it is possible to prove that $|\xi^\star| = \sup\{\det[\xi^\star, \nu_1, \dots, \nu_{n-1}]\}$, where the supremum is taken over all unit covectors $\nu_1, \dots, \nu_{n-1} \in \Lambda^1 V$, and $[\xi^\star, \nu_1, \dots, \nu_n]$ denotes the matrix having $\xi^\star, \nu_1, \dots, \nu_n$ as columns. Such an inequality can be checked using the Hadamard inequality $|\det A| \le \prod_{j=1}^n \left(\sum_{i=1}^n a_{ij}^2\right)^{1/2}$, where $A = (a_{ij})$ is an $(n \times n)$ matrix. The Benson area of ∂E is then defined as $\int_{\partial E} \mathfrak{b}(x, \nu)\, d\mathcal{H}^{n-1}$, where $\mathfrak{b}(x, \cdot) : \xi^\star \in \Lambda^1 V \to \max\left\{\det[\xi^\star, \nu_1, \dots, \nu_{n-1}] : \nu_i \in \Lambda^1 V, \phi^o(x, \nu_i) \le 1\right\}$. This surface measure is strictly related to the De Giorgi mass [80], [13], which turns out to be, for instance for a spatially homogeneous ϕ^o, the quantity $\int_{\partial E} \lambda(T)\, d\mathcal{H}^{n-1}$, where $\lambda(T) := \sup\{\mathcal{H}^{n-1}(\eta(B_\phi \cap T)) : \eta = (\eta_1, \dots, \eta_{n-1}) \in GL(T,T), \phi^o(\eta_i) \le 1\ \forall i \in \{1, \dots, n-1\}\}$ for any $T \in G$.

4 ϕ-regular sets

Assume in this section that ϕ^o is convex and spatially homogeneous([29]). The next definitions become interesting when T_{ϕ^o} is multivalued hence, roughly speaking, when B_{ϕ^o} has corners, edges, etc. In these notes, we will apply these definitions in the crystalline case.

Let ∂E be Lipschitz. In order to look for a solution of an anisotropic (and in particular crystalline) mean curvature flow starting from ∂E, it is necessary to devise a certain class of regularity for the flowing hypersurfaces.

We will give different definitions, depending on whether we want to consider a whole neighbourhood of ∂E or not. All definitions have advantages and disadvantages. One motivation for considering the neighbourhoods comes from phase transitions (in particular the reaction-diffusion approximation considered in Section 9), where the interface is diffuse.

Let us begin with the definitions using the neighbourhoods, and with the most stringent one. Recall that ∇d_ϕ naturally extends the covector field ν_{ϕ^o} out of ∂E, and N_ϕ extends the vector field n_ϕ.

Definition 4.1 (**Neighbourhood-Lipschitz ϕ-regular sets).** *We say that E is neighbourhood-Lipschitz ϕ-regular if there exists a tubular neighbourhood U of ∂E and a bounded vector field $\eta \in \mathrm{Lip}(U; \Lambda_1 V)$ such that $\eta(x) \in T_{\phi^o}(\nabla d_\phi(x))$ for almost every $x \in U$.*

If T_{ϕ^o} is single-valued then $T_{\phi^o}(\nabla d_\phi(x))$ is a singleton and it reduces to the vector field N_ϕ. Neirghbourhood-Lipschitz ϕ-regularity seems to be the strongest regularity one can require([30]). Nevertheless, a difficulty related to Definition 4.1 is that the divergence([31]) of η belongs just to $L^\infty(U)$, hence has not, a priori, a well defined trace on ∂E([32]). This difficulty remains in the following definition ([33]).

Definition 4.2 (**Neighbourhood-L^∞ ϕ-regular sets).** *We say that E is neighbourhood-L^∞ ϕ-regular if there exists a tubular neighbourhood U of ∂E and a bounded vector field η such that $\mathrm{div}\eta \in L^\infty(U)$ and $\eta(x) \in T_{\phi^o}(\nabla d_\phi(x))$ for almost every $x \in U$.*

Let us now pass to a (rather intrinsic) definition. Define

$$\mathrm{Nor}_\phi(\partial E; M) := \{N : \partial E \to M : N(x) \in T_{\phi^o}(\nu_\phi^E(x)) \text{ for } \mathcal{H}^{n-1} \text{ a.e. } x \in \partial E\}.$$

Definition 4.3 (**Lipschitz ϕ-regular sets).** *We say that E is Lipschitz ϕ-regular if there exists a vector field $\eta \in \mathrm{Nor}_\phi(\partial E; M) \cap \mathrm{Lip}(\partial E; M)$. We say that E is polyhedral Lipschitz ϕ-regular if E is Lipschitz ϕ-regular and it is polyhedral([34]).*

[29] Various definitions could be generalized for ϕ^o depending on x, at least when $\phi^o \in \mathcal{M}_{\mathrm{reg}}(TM)$.

[30] In the euclidean case $\phi(\cdot) = |\cdot|$ we have that E is neighbourhood-Lipschitz ϕ-regular if and only if ∂E is of class $\mathcal{C}^{1,1}$.

[31] One advantage: this divergence is taken in the ambient space M.

[32] See however [32, Remark (d3)].

[33] Definition 4.2 could be in turn relaxed by requiring $\mathrm{div}\eta \in L^2(U)$. Also in view of the L^∞-regularity result stated in Theorem 5.17, we will not use this relaxed definition in these notes.

[34] All polyhedral sets considered in these notes are assumed to have a finite number of facets. Facets of ∂E are defined as the closure of a connected component of the relative interior of $\partial E \cap T_x\partial E$ for some $x \in \partial E$ such that the tangent plane $T_x\partial E$ to ∂E at x exists.

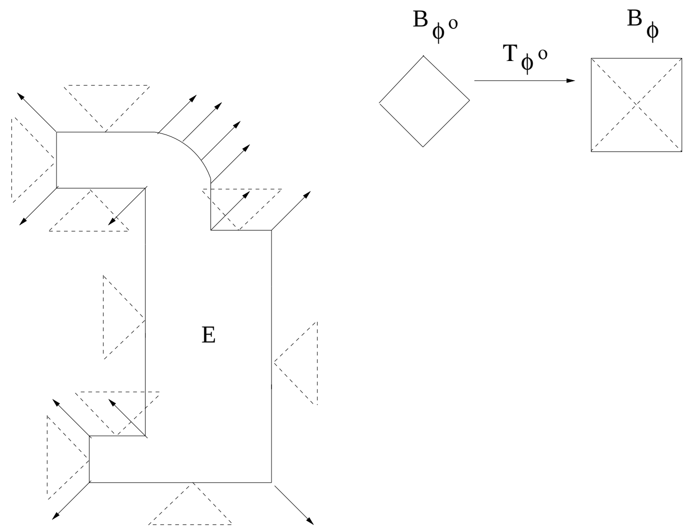

Figure 3: A Lipschitz ϕ-regular set E when B_ϕ is the square $[-1,1]^2$. Curved portions of ∂E may be present: we will see that there the crystalline curvature must vanish.

The difficulties related to constructing a vector field with Lipschitz regularity on ∂E in explicit examples are essentially the same as the ones in Definition 4.1; in addition, when talking about the divergence of η, we are forced now to speak about a tangential divergence. On facets, the tangential divergence we will consider will be the euclidean tangential divergence [136] div_τ. Again, one could relax the regularity of η in Definition 4.3, for instance by requiring η to be bounded with tangential divergence in $L^2(\partial E)$ or in $L^\infty(\partial E)$.

Definition 4.4 (L^∞**-**ϕ**-regular sets).** *We say that a polyhedral set E is L^∞-ϕ-regular if there exists a vector field $\eta \in \mathrm{Nor}_\phi(\partial E; M)$ having tangential divergence $\mathrm{div}_\tau \eta$ in $L^\infty(\partial E)$.*

Finally, we point out another notion that has been considered in [33].

Definition 4.5 (rB_ϕ**-condition).** *Let $r > 0$. We say that E satisfies the rB_ϕ-condition if, for any $x \in \partial E$, there exists $y \in M$ such that*

$$rB_\phi + y \subseteq E \qquad \text{and} \qquad x \in \partial\left(rB_\phi + y\right).$$

It turns out that if E is neighbourhood-Lipschitz ϕ-regular then there exists $r > 0$ such that E and $\overline{M \setminus E}$ satisfy the rB_ϕ-condition. Moreover, if E is convex then E is neighbourhood-L^∞ ϕ-regular if and only if E and $\overline{M \setminus E}$ satisfy the rB_ϕ-condition for some $r > 0$.

4.1 Examples

If $n = 2$, the structure of a Lipschitz ϕ-regular set E is, roughly speaking, the following: ∂E is a closed simple Lipschitz curve which is a sequence (with a precise order) of segments parallel to some edge of ∂B_ϕ and of segments or arcs corresponding to vertices of ∂B_ϕ.

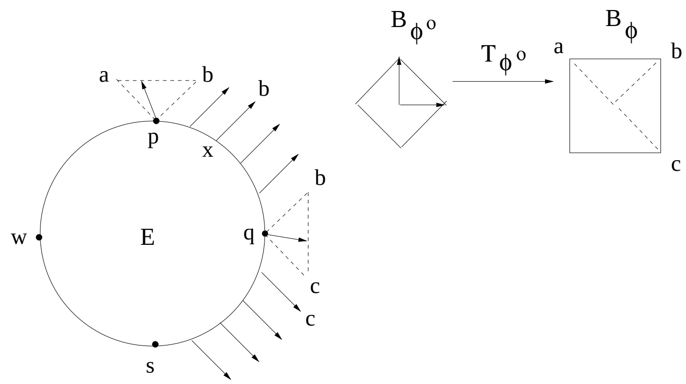

Figure 4: E is not L^∞-ϕ-regular. Any Cahn-Hoffman selection is forced to jump at the points p, q, s, w of ∂E.

Example 4.6 (A Lipschitz ϕ-regular curve). Let $\phi(\xi) := \max\{|\xi_1|, |\xi_2|\}$, so that $B_\phi = [-1,1]^2$, and let E be as in Figure 3. At the vertices of ∂E the vector ν_ϕ^E is not defined. However, let v be a vertex of ∂E, and let F_1 and F_2 be the two arcs or segments of ∂E having v as a vertex. For any x in the relative interior of F_i, the closed convex set $T_{\phi^o}(\nu_\phi^E(x))$ is either a segment or a singleton, independent of x and depending only on F_i. Let us denote it by K_i. What makes E Lipschitz ϕ-regular is the fact that $K_1 \cap K_2$ is a singleton. This produces a unique vector at each vertex of ∂E; then we can construct infinitely many vector fields $\eta \in \mathrm{Nor}_\phi(\partial E; \mathbb{R}^2) \cap \mathrm{Lip}(\partial E; \mathbb{R}^2)$ lying inside the dotted triangles with the assigned values at the vertices.

On the other hand, for the same ϕ as in Example 4.6, the euclidean unit ball is not Lipschitz ϕ-regular, and not even L^∞-ϕ-regular. Its regularity is analogous to the regularity of the square in the euclidean geometry.

Example 4.7 (The circle is not L^∞-ϕ-regular). Let $n = 2$ and ϕ be as in Example 4.6. Let $E := \{z \in \mathbb{R}^2 : |z| \le 1\}$, see Figure 4. Then E is not Lipschitz ϕ-regular. Indeed, $T_{\phi^o}(\nu_\phi^E(p))$ is the upper horizontal segment $[a,b]$ of ∂B_ϕ (we depict a corresponding dotted triangle at p). Similarly, $T_{\phi^o}(\nu_\phi^E(q))$ is the right vertical segment $[b,c]$ of ∂B_ϕ. On the other hand, any point x on ∂E lying in the (relatively) open arc A between p and q is such that $T_{\phi^o}(\nu_\phi^E(x)) = b$. We deduce that any vector field $\eta \in \mathrm{Nor}_\phi(\partial E; \mathbb{R}^2)$ must fulfill $\eta \equiv b$ on A, and $\eta \equiv c$ on the open arc on ∂E between q and ω. Hence, any vector we choose inside the dotted triangles (for instance, the triangle at q) will produce a discontinuity in the vector field η (at q). We conclude that E is not L^∞-ϕ-regular.

Example 4.8 (A Lipschitz ϕ-regular polyhedral surface). Let B_ϕ be as in Figure 1, and E as in Figure 5. If $x \in \mathrm{int}(Q)$ then $\nu_\phi^E(x)$ coincides with the top vertex of ∂B_{ϕ^o}, and $T_{\phi^o}(\nu_\phi^E(x))$ is the top facet $\widetilde{B}_\phi^Q$ of ∂B_ϕ. We depict $T_{\phi^o}(\nu_\phi^E(x))$ as a pyramid. Therefore $\eta(x)$ is constrained to lie in $\widetilde{B}_\phi^Q$. If $x \in \partial E$ is in the interior of an edge (say the edge l) of ∂Q, then ν_ϕ^E is not defined at x. However the intersection T_ϕ^l of $\widetilde{B}_\phi^Q$ with $\widetilde{B}_\phi^F$ is defined, and it is the top edge of the frontal facet of ∂B_ϕ. We have depicted this set as a triangle. Therefore

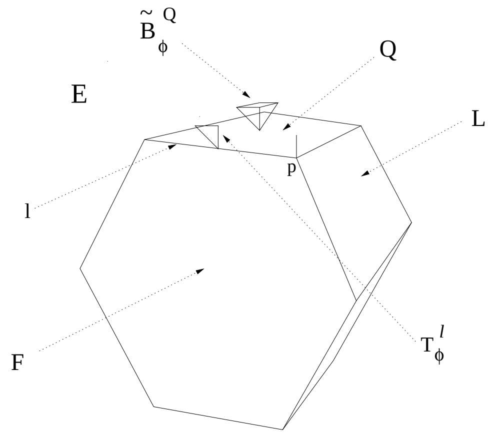

Figure 5: The set E. Here B_ϕ is as in Figure 1.

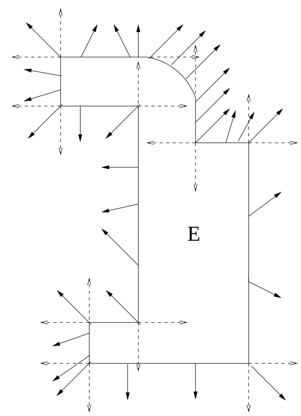

Figure 6: c_F is independent of the choice of η among all vector fields making E Lipschitz ϕ-regular.

$\eta(x)$ is constrained to lie in T_ϕ^l. If $x \in \partial E$ is a vertex (say the vertex p) of ∂Q, then ν_ϕ^E is not defined at p. What is defined is the intersection w of $\widetilde{B}_\phi^Q \cap \widetilde{B}_\phi^F \cap \widetilde{B}_\phi^L$, and we have depicted this point at p as a segment. Therefore $\eta(p)$ must coincide with w, see also Figure 1. A choice of a vector field $\eta \in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^3) \cap \mathrm{Lip}(\partial E;\mathbb{R}^3)$ can be made by hand.

4.2 Normal traces

We give here some notions that will be useful in the definition of the crystalline mean curvature. Recall the definition of $\widetilde{\nu}^F$ given in (2.2).

Definition 4.9 (The normal traces c_F). *Let E be a Lipschitz ϕ-regular set, let $\eta \in \mathrm{Nor}_\phi(\partial E; M) \cap \mathrm{Lip}(\partial E; M)$, and let $F \subset \partial E$ be a facet of ∂E. We define the normal trace function $c_F \in L^\infty(\partial F)$ as*

$$c_F := \widetilde{\nu}^F \cdot \eta. \tag{4.1}$$

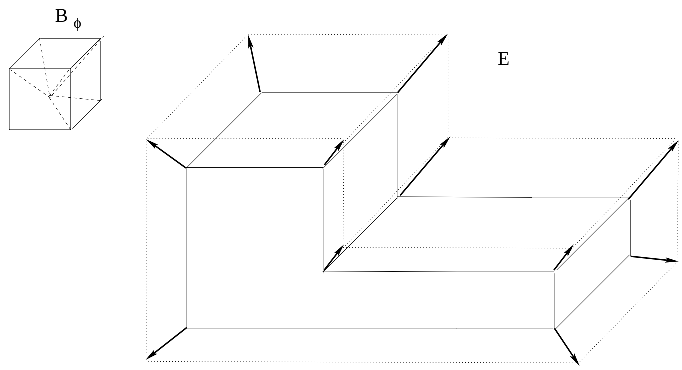

Figure 7: on the edges of ∂E the vector η must lye in the dotted regions

Example 4.10. Let $n = 2$, B_ϕ and E be as in Figure 3. In Figure 6 we depict a vector field η which makes ∂E Lipschitz ϕ-regular. The functions c_F do not depend on the particular choice of η. The dotted vectors at the vertices indicate the unit normals (in the line containing the facet F) pointing outward F (i.e., $\widetilde{\nu}^F$).

Example 4.11. Let $n = 3$, $B_\phi = [-1,1]^3$, and E be the set of Figure 8. E is a polyhedral Lipschitz ϕ-regular set, since it is possible to construct a vector field $\eta \in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^3) \cap \mathrm{Lip}(\partial E;\mathbb{R}^3)$. Indeed, first we identify η on the vertices of ∂E. If v is a vertex of ∂E, the intersection of $\widetilde{B}_\phi^Q$ over all facets Q of ∂E containing v is a singleton: we define this singleton to be the value of η at v (see the bold vectors in Figure 7). Next, on a facet $Q \subset \partial E$, it is enough to take suitable convex combinations of the values of η at the vertices of Q (possibly first subdividing Q into two or more rectangles if Q itself is not a rectangle) to obtain the required properties on η.
The bold vectors at the vertices of ∂E are the unique possible values for η. The vector field $\widetilde{\nu}^F$ points outside of F, and on $]p,q[$ points toward E. The pyramids with vertex on the relative interior of the two facets having $[p,q]$ in common represent the corresponding facets of ∂B_ϕ (for instance, $T_{\phi^o}(\nu_{\phi^o}(F))$ for the facet F), i.e. the range of admissibility of η. It follows that c_F is negative on $]p,q[$, while c_F is positive on the remaining relatively open edges of ∂F.

Given a Lipschitz ϕ-regular set E, in general it is possible to prove (see for instance [41], [42]) that c_F does not depend on the choice of η in $\mathrm{Nor}_\phi(\partial E;M) \cap \mathrm{Lip}(\partial E;M)$, and for $\mathcal{H}^{n-2}$-almost every $x \in \partial F$

$$c_F(x) = \begin{cases} \max\{\langle \widetilde{\nu}^F(x), \xi\rangle : \xi \in T_{\phi^o}(\nu_{\phi^o}(F))\} & \text{if } \widetilde{\nu}^F(x) \text{ points outside } E, \\ \\ \min\{\langle \widetilde{\nu}^F(x), \xi\rangle : \xi \in T_{\phi^o}(\nu_{\phi^o}(F))\} & \text{if } \widetilde{\nu}^F(x) \text{ points inside } E. \end{cases} \tag{4.2}$$

Remark 4.12. For a polyhedral Lipschitz ϕ-regular set, it is possible to extend the notion of normal trace also to vector fields $N \in \mathrm{Nor}_\phi(\partial E;M)$ with $\mathrm{div}_\tau N \in L^\infty(\partial E)$: such a normal trace turns out to coincide with the right hand side of (4.2).

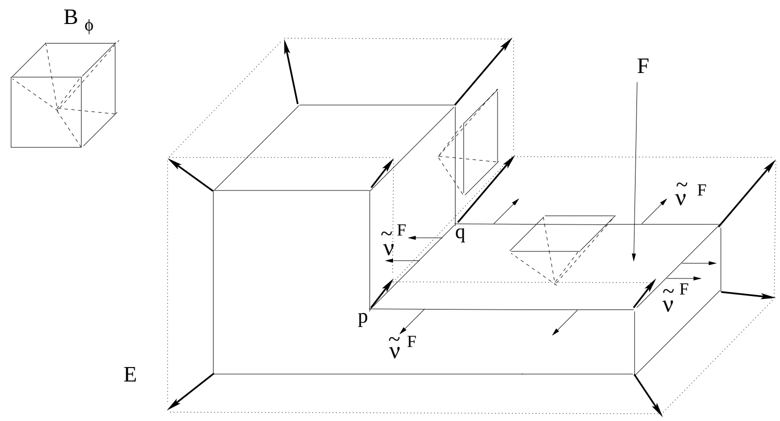

Figure 8: A Lipschitz ϕ-regular set E when B_ϕ is the cube. On the relative interior of $[p,q]$ the function c_F is negative (and constant).

5 First variations: functionals on boundaries

In this section we discuss the first variation of the functional $\mathcal{F}_{\phi^o}$, in order to devise a possible notion of ϕ-mean curvature. In the computations of this section it appears to be useful to have at our disposal quantities (such as the Cahn-Hoffman vector field) defined on a tubular neighbourhood of the interface ∂E.

5.1 Spatially homogeneous smooth ϕ^o

Let us assume that ϕ^o is spatially homogeneous and of class $\mathcal{C}^1(\Lambda^1 V \setminus \{0\})$. Let us also assume that ∂E is of class $\mathcal{C}^2$, and that there are no $x \in \partial E$ where $\phi^o(\nu^E(x)) = 0$ (this is in particular satisfied if ϕ^o is a metric on $\Lambda^1 V$, in view of (3.9)).
Let us introduce a class of admissible variations. Let $\Psi \in \mathcal{C}^\infty_c(M \times \mathbb{R}; M)$, and set $\Psi_\lambda(x) := \Psi(x,\lambda)$ for any $x \in M$ and $\lambda \in \mathbb{R}$. Assume that $\Psi_0 = \mathrm{Id}$, and $\Psi_\lambda - \mathrm{Id}$ has compact support in M for any $\lambda \in \mathbb{R}$. We can write

$$\Psi_\lambda(x) := x + \lambda X(x) + o(\lambda), \tag{5.1}$$

where $X := \frac{\partial \Psi_\lambda}{\partial \lambda}_{|\lambda=0}$. The vector field $X = (X^1, \ldots, X^n)$ can be considered as the initial velocity field of the deformation.
A direct computation shows that

$$\det(\nabla \Psi_\lambda) = 1 + \lambda \mathrm{tr}(\nabla X) + o(\lambda). \tag{5.2}$$

In particular

$$\frac{d}{d\lambda} |\det(\nabla \Psi_\lambda)|_{\lambda=0} = \mathrm{div} X. \tag{5.3}$$

Set

$$E_\lambda := \Psi_\lambda(E).$$

The next result was proved essentially in [47] (see also [37]), in the case of a convex regular metric ϕ^o. The proof that we present here does not require the convexity of ϕ^o, and it is slightly different. Recall the expression of the constant c_n in (3.32).

Theorem 5.1 (First variation, I). *We have*

$$\frac{d}{d\lambda}\mathcal{F}_{\phi^o}(E_\lambda)_{|\lambda=0} = c_n \int_{\partial E} \left(\mathrm{div}X - n^i_\phi \nu_{\phi^o}{}_j \nabla_i X^j\right) \phi^o(\nu)\ d\mathcal{H}^{n-1}. \tag{5.4}$$

Proof. Let $u \in \mathcal{C}^2(M)$ be such that $E = \{u \leq 0\}$, $\partial E = \{u = 0\}$, and $\nabla u \neq 0$ on ∂E. Then([35]) $\nu^E = \frac{\nabla u^T}{|\nabla u|}$ on ∂E. Define $v_\lambda : M \to \mathbb{R}$ as

$$v_\lambda(\Psi_\lambda(x)) := u(x), \qquad x \in M. \tag{5.5}$$

If $|\lambda|$ is small enough, we have

$$E_\lambda = \{\Psi_\lambda(x) : u(x) \leq 0\} = \{y : u(\Psi_\lambda^{-1}(y) \leq 0\} = \{v_\lambda \leq 0\},$$

$\partial E_\lambda = \{v_\lambda = 0\}$, $\nabla v_\lambda \neq 0$ on ∂E_λ, hence

$$\nu^{E_\lambda} = \frac{\nabla v_\lambda^T}{|\nabla v_\lambda|} \qquad \text{on } \partial E_\lambda.$$

In order to proceed in the proof, we recall the area and coarea([36]) formulas [94].

- The area formula: if $g : M \to \mathbb{R}$ is integrable, $f : M \to M$ is an injective Lipschitz map, and $\Omega \subseteq M$, then

$$\int_{f(\Omega)} g dy = \int_\Omega g(f)|\det(\nabla f)|\ dx.$$

- The coarea formula: if $w \in \mathrm{Lip}(M)$ satisfies $\mathrm{ess} - \inf |\nabla w| > 0$, $g : M \to \mathbb{R}$ is integrable, and $\mu \in \mathbb{R}$, then

$$\int_{\{w>\mu\}} g\ d\mathcal{H}^{n-1} = \int_\mu^{+\infty} \left(\int_{\{w=s\}} \frac{g}{|\nabla w|} d\mathcal{H}^{n-1}\right)\ ds. \tag{5.6}$$

It is now useful to make the following observation: for a given λ with $|\lambda|$ small enough, we have

$$\mathcal{H}^{n-1}(\partial E_\lambda) = \int_{\partial E} \frac{|\nabla v_\lambda(\Psi_\lambda)||\det(\nabla \Psi_\lambda)|}{|\nabla u|}\ d\mathcal{H}^{n-1}. \tag{5.7}$$

Indeed, if $\rho > 0$ is small enough, the area formula with the choice $f = \Psi_\lambda$, $\Omega = \{|u| < \rho\}$ (so that $f(\Omega) = \{|v_\lambda| < \rho\}$) and $g = |\nabla v_\lambda|$, gives

$$\int_{\{|v_\lambda|<\rho\}} |\nabla v_\lambda(y)|\ dy = \int_{\{|u|<\rho\}} |\nabla v_\lambda(\Psi_\lambda(x))||\det(\nabla \Psi_\lambda(x))|\ dx. \tag{5.8}$$

[35] $\nu^E = \nu$ is considered as a covector field (row), while ∇u as a vector field (column). Sometimes in the sequel of these notes we will omit the transposition symbol, identifying ν^E with $\frac{\nabla u}{|\nabla u|}$.

[36] An historical comment: the coarea formula appeared implicitely in [76].

Hence, by the coarea formula applied to the left hand side of (5.8) with the choice $w = v_\lambda$, and by the smoothness of ∂E_λ it follows

$$\begin{aligned} &\lim_{\rho \to 0^+} \frac{1}{2\rho} \int_{\{|u|<\rho\}} |\nabla v_\lambda(\Psi_\lambda)||\det(\nabla \Psi_\lambda)| \ dx \\ &= \lim_{\rho \to 0^+} \frac{1}{2\rho} \int_{-\rho}^{\rho} \mathcal{H}^{n-1}\left(\{v_\lambda = s\}\right) \ ds = \mathcal{H}^{n-1}(\partial E_\lambda). \end{aligned} \tag{5.9}$$

On the other hand, using again the coarea formula with the choice $w = u$ and the smoothness of u it follows

$$\lim_{\rho \to 0^+} \frac{1}{2\rho} \int_{\{|u|<\rho\}} |\nabla v_\lambda(\Psi_\lambda)||\det(\nabla \Psi_\lambda)| \ dx = \int_{\partial E} \frac{|\nabla v_\lambda(\Psi_\lambda)||\det(\nabla \Psi_\lambda)|}{|\nabla u|} \ d\mathcal{H}^{n-1}. \tag{5.10}$$

Then (5.7) follows from (5.9) and (5.10).

We now pass to the proof of (5.4). Using the area formula, and arguing as in the proof of (5.7), we have

$$\frac{1}{c_n}\mathcal{F}_{\phi^o}(E_\lambda) = \int_{\partial E_\lambda} \phi^o\left(\frac{\nabla v_\lambda}{|\nabla v_\lambda|}\right) \ d\mathcal{H}^{n-1} = \int_{\partial E} \phi^o\left(\frac{\nabla v_\lambda}{|\nabla v_\lambda|}(\Psi_\lambda)\right) |\det \nabla \Psi_\lambda| \frac{|\nabla v_\lambda(\Psi_\lambda)|}{|\nabla u|} \ d\mathcal{H}^{n-1}. \tag{5.11}$$

Differentiating (5.5) with respect to x^j and using (5.1) it follows $\frac{\partial u}{\partial x^j} = \frac{\partial v_\lambda}{\partial y^i}(\delta_{ij} + \lambda \frac{\partial X^i}{\partial y^j})$, hence if we set

$$J(x) := (\nabla X(x))^T, \qquad x \in U,$$

we have([37])

$$\nabla v_\lambda(\Psi_\lambda(x))^T = \nabla u(x)^T (\mathrm{Id} + \lambda J(x))^{-1}, \qquad x \in U. \tag{5.12}$$

In particular

$$\nabla v_\lambda(\Psi_\lambda) = \nabla u \quad \text{if } \lambda = 0, \quad \text{that is on } \partial E.$$

From (5.12) it follows

$$\frac{d}{d\lambda}\left(\nabla v_\lambda(\Psi_\lambda(x))^T\right)_{|\lambda=0} = -\nabla u(x)^T J(x), \qquad x \in U. \tag{5.13}$$

Using (5.13) and (5.3) it follows

$$\frac{d}{d\lambda}\left(|\det \nabla \Psi_\lambda| \frac{|\nabla v_\lambda(\Psi_\lambda)|}{|\nabla u|}\right)_{|\lambda=0} = \mathrm{tr}\left(\Big(\mathrm{Id} - \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|}\Big) \nabla X\right). \tag{5.14}$$

As a consequence of (5.12) and (5.13), at any point $x \in U$ we have

$$\frac{d}{d\lambda}\left[\frac{\nabla v_\lambda(\Psi_\lambda)^T}{|\nabla v_\lambda(\Psi_\lambda)|}\right]_{|\lambda=0} = -\frac{\nabla u^T}{|\nabla u|} J + \langle \frac{\nabla u^T}{|\nabla u|} J, \frac{\nabla u}{|\nabla u|}\rangle \ \frac{\nabla u^T}{|\nabla u|}. \tag{5.15}$$

[37] Recall that with our conventions the gradient vector field $\nabla u(x)$ is a column.

Using (3.5), (5.11), (5.15) and (5.14) we have

$$
\begin{aligned}
\frac{1}{c_n}\frac{d}{d\lambda}\mathcal{F}_{\phi^o}(E_\lambda)_{|\lambda=0} = &\int_{\partial E} \langle\ -\nu\nabla X^T + \langle \nu\nabla X^T, \nu^T\rangle\ \nu, n_\phi\ \rangle\ d\mathcal{H}^{n-1} \\
&+ \int_{\partial E} \left(\mathrm{div}X - \langle \nu^T\nabla X, \nu^T\rangle\right) \phi^o(\nu)\ d\mathcal{H}^{n-1}.
\end{aligned}
\tag{5.16}
$$

Recalling (3.13) we have that the second addendum $\langle\langle \nu^T\nabla X, \nu^T\rangle\nu, n_\phi\rangle$ under the integral on the right hand side of (5.16) can be written as

$$
\langle \nu^T\nabla X, \nu^T\rangle\langle \nu_{\phi^o}, n_\phi\rangle\phi^o(\nu) = \langle \nu^T\nabla X, \nu^T\rangle\phi^o(\nu)
$$

(where we have used (3.13)), and therefore cancels with the the fourth addendum. Then (5.4) follows. $\square$

Remark 5.2. We note once more that Theorem 5.1 is valid without assuming that ϕ^o is convex.

Remark 5.3. The previous computation holds also for a function σ (resp. ϕ^o) defined on a relatively open subset S of $\mathbb{S}^{n-1}$ (resp. on $\{\lambda\xi^\star : \xi^\star \in S, \lambda \in \mathbb{R}\}$), provided $\frac{\nabla v_\lambda}{|\nabla v_\lambda|}(x)$ still belongs to S.

Definition 5.4. *We set*

$$
\mathrm{div}_{\tau,\phi}X := \mathrm{tr}\left(\left(\mathrm{Id} - n_\phi \otimes \nu_{\phi^o}\right)\nabla X\right) = \mathrm{div}X - n_\phi^i \nu_{\phi^o_j}\nabla_i X^j.
$$

Notice that the matrix([38]) $\mathrm{Id} - n_\phi \otimes \nu_{\phi^o}$ is not symmetric.
Let $\phi^o \in \mathcal{M}_{\mathrm{reg}}(T^\star M)$ and let $N_\phi : U \to M$ be the extension of n_ϕ as defined in (3.24).

Definition 5.5 (**ϕ-mean curvature**)**.** *We define*([39])

$$
\kappa_\phi := \mathrm{div}_{\tau,\phi}N_\phi, \qquad \mathbf{H}_\phi := \kappa_\phi\nu_{\phi^o} \qquad \text{on } \partial E.
$$

The following result shows that we can equivalently define the ϕ-mean curvature using the divergence in the ambient space M.

Lemma 5.6. Let $\phi^o \in \mathcal{M}_{\mathrm{reg}}(T^\star M)$ and let $N_\phi : U \to M$ be the extension of n_ϕ as defined in (3.24). Then([40])

$$
\kappa_\phi = \mathrm{div}N_\phi = \phi^o_{\xi_i^\star\xi_j^\star}(\nabla d_\phi)\nabla^2_{ij}d_\phi = (\phi^{o2}/2)_{\xi_i^\star\xi_j^\star}(\nabla d_\phi)\nabla^2_{ij}d_\phi = \mathrm{div}_\tau N_\phi = \Delta_\phi d_\phi \qquad \text{on } \partial E.
\tag{5.17}
$$

Proof. For a fixed $x \in \partial E$ define $f(z) := \langle \nu^E(x), N_\phi(z)\rangle$ for any $z \in U$. Then f has a maximum at x (with value $\phi^o(\nu^E(x))$). Therefore $\nabla f(x) = 0$, i.e., $\nu^E_j(x)\nabla_i N^j_\phi(x) = 0$. Hence $\nu_{\phi^o_j}(x)\nabla_i N^j_\phi(x) = 0$, and therefore $\mathrm{div}_{\tau,\phi}N_\phi = \mathrm{div}N_\phi = \mathrm{div}_\tau N_\phi$. $\square$

[38] To be consistent with the indices, here Id has one lower index and one upper index, and $n_\phi \otimes \nu_{\phi^o}$ is a $(1,1)$ tensor.

[39] With this definition $\mathbf{H}_\phi$ is viewed as a covector. The corresponding vector is $\kappa_\phi N_\phi$.

[40] Observe that the matrix ∇N_ϕ is the product of two symmetric matrices, since $\nabla N_\phi = \nabla^2(\phi^{o2}/2)\nabla^2 d_\phi$, and $\nabla^2(\phi^{o2}/2)$ is positive definite. It is then possible to prove that ∇N_ϕ is diagonalizable and its eigenvalues are real.

Corollary 5.7 (First variation, II). *Let $\phi^o \in \mathcal{M}_{\rm reg}(T^\star M)$. We have*

$$\frac{d}{d\lambda}\mathcal{F}_{\phi^o}(E_\lambda)_{|\lambda=0} = c_n \int_{\partial E} \langle \mathbf{H}_\phi, X\rangle \ \phi^o(\nu) \ d\mathcal{H}^{n-1}. \tag{5.18}$$

Proof. Split X as

$$X = X_{\perp,\phi} + X_{\tau,\phi}, \qquad X_{\perp,\phi} := \langle \nabla d_\phi, X\rangle \ N_\phi =: \psi N_\phi, \qquad X_{\tau,\phi} := X - X_{\perp,\phi}.$$

Note that $\langle X_{\perp,\phi}, \nabla d_\phi\rangle = \langle X, \nabla d_\phi\rangle$, and that $\langle X_{\tau,\phi}, \nabla d_\phi\rangle = 0$, namely $X_{\tau,\phi}$ is a tangent vector field to ∂E. From (5.4) it follows that the function

$$X \to \frac{d}{d\lambda}\mathcal{F}_{\phi^o}(E_\lambda)_{|\lambda=0}$$

is linear with respect to X. Moreover, it is possible to show that the contribution of $X_{\tau,\phi}$ to $\mathcal{F}_{\phi^o}(E_\lambda)$ is of order $o(\lambda)$. Therefore we can neglect $X_{\tau,\phi}$ in the first variation, and consider only $X_{\perp,\phi}$. We have

$$\begin{aligned}
\frac{1}{c_n}\frac{d}{d\lambda}\mathcal{F}_{\phi^o}(E_\lambda)_{|\lambda=0} &= \int_{\partial E} \left({\rm div}(\psi N_\phi) - N^i_\phi {\nu_{\phi^o}}_j \nabla_i(\psi N^j_\phi)\right) \phi^o(\nu) \ d\mathcal{H}^{n-1} \\
&= \int_{\partial E} \left(\psi {\rm div} N_\phi - \psi N^i_\phi {\nu_{\phi^o}}_j \nabla_i N^j_\phi\right) \phi^o(\nu) \ d\mathcal{H}^{n-1},
\end{aligned}$$

where, recalling (3.13), we have used $N^i_\phi {\nu_{\phi^o}}_j \nabla_i(\psi N^j_\phi) = \langle \nabla\psi, N_\phi\rangle + \psi N^i_\phi {\nu_{\phi^o}}_j \nabla_i N^j_\phi$. Therefore

$$\frac{d}{d\lambda}\mathcal{F}_{\phi^o}(E_\lambda)_{|\lambda=0} = c_n \int_{\partial E} \psi \left({\rm div} N_\phi - N^i_\phi {\nu_{\phi^o}}_j \nabla_i N^j_\phi\right) \phi^o(\nu) \ d\mathcal{H}^{n-1}$$

which is (5.18). □

Corollary 5.8. *We have the integration by parts formula*

$$\int_{\partial E} {\rm div}_{\tau,\phi} X \ \phi^o(\nu) \ d\mathcal{H}^{n-1} = \int_{\partial E} \langle \mathbf{H}_\phi, X\rangle \phi^o(\nu) \ d\mathcal{H}^{n-1}, \qquad X \in \mathcal{C}^1_c(M; \Lambda_1 M).$$

B_ϕ has constant ϕ-mean curvature. More precisely we have the following

Example 5.9. Let $\phi^o \in \mathcal{M}_{\rm reg}(T^\star M)$ be spatially homogeneous. Then

$$\kappa_\phi = n-1 \qquad \text{on } \partial B_\phi. \tag{5.19}$$

Take $u(x) = \phi(x) - 1$ for $x \in M$; then $\nabla u(x) = \phi_\xi(x)$. Hence $\phi^o_\xi(\nabla u(x)) = x/\phi(x)$ on M. Consequently $\kappa_\phi = {\rm div}_\phi(\phi^o_\xi(\nabla u(x))) = {\rm div}(x/\phi(x))$. Then, as $x \cdot \phi_\xi(x) = \phi(x)$, we have

$${\rm div}\left(\frac{x}{\phi(x)}\right) = \frac{{\rm div} x}{\phi(x)} - \frac{x\cdot\phi_\xi(x)}{\phi^2(x)} = \frac{n}{\phi(x)} - \frac{1}{\phi(x)} = \frac{n-1}{\phi(x)}.$$

Example 5.10. Let $n = 2$, and assume that $\phi^o(\xi^\star) = \phi^o(\xi^\star) = \rho\gamma(\theta)$, where (ρ, θ) are polar coordinates in the $\xi^\star$-plane, i.e., $\xi^\star_1 = \rho\cos\theta$, $\xi^\star_2 = \rho\sin\theta$. Then the curvature κ_ϕ of a smooth curve ∂E is (see for instance [47] and the next section)

$$\kappa_\phi = \kappa(\gamma + \gamma_{\theta\theta}), \tag{5.20}$$

where $\gamma_{\theta\theta}$ denotes the second derivative of γ with respect to θ.

5.1.1 Curves: parametric computation

Let us compute the first variation of $\mathcal{F}_{\phi^o}$ in the special case $n = 2$, using a parametric approach. Write $\nu = \nu(\theta) = -(\cos\theta, \sin\theta) = \tau(\theta)^\perp$, where $^\perp$ denotes the counterclock-wise rotation of $\pi/2$, and define $\gamma : [0, 2\pi) \to \mathbb{R}$ as

$$\gamma(\theta) := \sigma(\nu).$$

Theorem 5.11 (First variation: curves). *Let $\alpha : [0,1] \to \mathbb{R}^2$ be a regular parametrization of ∂E. Let $\beta \in \mathcal{C}^2_c([0,1]; \mathbb{R}^2)$, $\lambda \in \mathbb{R}$, and $\alpha_\lambda := \alpha + \lambda\beta$. Then*

$$\frac{d}{d\lambda}\mathcal{F}_\sigma(E_\lambda)_{|\lambda=0} = \int_0^1 \langle (\gamma(\theta) + \gamma_{\theta\theta}(\theta))\kappa\nu, \beta\rangle \; dt, \tag{5.21}$$

where α_λ is a regular parametrization of ∂E_λ, and $\kappa = \frac{1}{|\alpha'|^2}(\alpha'' - \langle \alpha'', \frac{\alpha'}{|\alpha'|}\rangle \frac{\alpha'}{|\alpha'|})$ is the euclidean curvature of ∂E, where $'$ denotes the derivative with respect to $t \in [0,1]$.

Proof. Set $\tau_\lambda = \tau_\lambda(t) := \frac{\alpha'_\lambda(t)}{|\alpha'_\lambda(t)|} = (-\sin\theta_\lambda(t), \cos\theta_\lambda(t))$, and set $-\nu_\lambda := \tau_\lambda^\perp$. We have

$$\begin{aligned} \frac{d}{d\lambda}\mathcal{F}_\sigma(E_\lambda) = \frac{d}{d\lambda}\int_0^1 \gamma(\theta_\lambda(t))|\alpha'_\lambda(t)| \; dt \\ = \int_0^1 \gamma_\theta(\theta_\lambda)\frac{d\theta_\lambda}{d\lambda}|\alpha'_\lambda| \; dt + \int_0^1 \gamma(\theta_\lambda) \; \tau_\lambda \cdot \beta' \; dt =: I_\lambda + II_\lambda. \end{aligned}$$

We have, integrating by parts and using $\frac{d}{dt}\tau_{\lambda|\lambda=0} = -\kappa\nu$,

$$\begin{aligned} II_{|\lambda=0} &= -\int_0^1 \gamma_\theta(\theta)\frac{d\theta_\lambda}{dt}_{|\lambda=0}\tau \cdot \beta \; dt + \int_0^1 \gamma(\theta)\kappa\nu \cdot \beta \; dt \\ &= -\int_0^1 \gamma_\theta(\theta)\theta'\tau \cdot \beta \; dt + \int_0^1 \gamma(\theta)\kappa\nu \cdot \beta \; dt. \end{aligned} \tag{5.22}$$

To compute $\frac{d\theta_\lambda}{d\lambda}_{|\lambda=0}$ we differentiate $\alpha' + \lambda\beta' = |\alpha' + \lambda\beta'|(-\sin\theta_\lambda, \cos\theta_\lambda)$ with respect to λ. We have

$$\beta' = \tau \cdot \beta' \; \tau + |\alpha'|\nu\frac{d\theta_\lambda}{d\lambda}_{|\lambda=0}$$

which implies

$$\frac{d\theta_\lambda}{d\lambda}_{|\lambda=0} = \nu \cdot \frac{\beta'}{|\gamma'|}.$$

Substituting in (5.22), integrating by parts, using $\frac{d\nu}{d\theta} = -\tau$ and $\frac{d\theta}{dt} = \kappa$, gives

$$I_{|\lambda=0} = \int_0^1 \gamma_\theta(\theta)\nu \cdot \beta' \; dt = \int_0^1 \gamma_{\theta\theta}(\theta)\kappa\nu \cdot \beta \; dt + \int_0^1 \gamma_\theta(\theta)\theta'\tau \cdot \beta \; dt,$$

and (5.21) follows. □

5.2 Inhomogeneous ϕ^o

We define

$$\kappa_\phi := \mathrm{div}_\phi\ N_\phi = \mathrm{div} N_\phi + \nabla(\log(\mathrm{vol}_\phi)) \cdot N_\phi \qquad \text{on } \partial E, \tag{5.23}$$

and the vector mean curvature κ_ϕ to ∂E as $\mathbf{H}_\phi := \kappa_\phi \nu_{\phi^o}$.
The proof of the next theorem can be found in [47].

Theorem 5.12 **(First variation).** *Let $\phi^o \in \mathcal{M}(T^\star M)$. Adopting the same notation of Theorem 5.1, we have*

$$\frac{d}{d\lambda}\mathcal{F}_{\phi^o}(E_\lambda)_{|\lambda=0} = \int_{\partial E} \langle \mathbf{H}_\phi, X\rangle \phi^o(x,\nu)\mathrm{vol}_\phi\ d\mathcal{H}^{n-1}. \tag{5.24}$$

5.3 The crystalline case

The computation of the first variation of $\mathcal{F}_{\phi^o}$ is much more complicated in the crystalline case, because of the nondifferentiability of both the surface and the integrand. We report here some results from [41], [42], which indicate how to define the crystalline mean curvature. Let ϕ^o be crystalline. Let E be a polyhedral neighbourhood-Lipschitz ϕ-regular set, let $U \supset \partial E$ be an open set of M and $\eta \in \mathrm{Lip}(U;V)$ be such that $\eta \in T_{\phi^o}(\nabla d_\phi)$ almost everywhere in U, where $d_\phi = d_\phi^E$. Let $\Psi \in \mathrm{Lip}(U \times \mathbb{R}; M)$, with $\Psi(x,\lambda) := x + \lambda X(x)$, for a given initial velocity vector field $X \in \mathrm{Lip}(U;M)$. If we follow the reasoning of Section 5.1 for the computation of the first variation of $\mathcal{F}_{\phi^o}$, we now encounter some technical difficulties: for instance we have to be able to consider the divergence of X on ∂E. This is not immediately guaranteed from the regularity of X, since $\mathrm{div}X$ is only in $L^\infty(U)$, and ∂E has obviously zero Lebesgue measure. We therefore prefer to slightly change our point of view. Assume then E to be polyhedral Lipschitz ϕ-regular, and define

$$\mathcal{H}_{\mathrm{div}} := \big\{N \in \mathrm{Nor}_\phi(\partial E; M) : \mathrm{div}_\tau N \in L^2(\partial E)\big\}.$$

Let $X \in \mathrm{Lip}(\partial E; V)$. As in the smooth case, $\mathcal{F}_{\phi^o}$ does not change under infinitesimal tangential variations. Therefore we restrict ourselves to consider ϕ-normal vector fields, hence we assume that X can be written as $X = \psi\eta$, where $\psi \in \mathrm{Lip}(\partial E)$ and $\eta \in \mathrm{Nor}_\phi(\partial E; M) \cap \mathrm{Lip}(\partial E; M)$. In order to perform the computation, it appears to be useful to extend ψ and η in a suitable neighbourhood of ∂E. We choose the extension by lines along η itself. More precisely, one can show that there exist $\varepsilon > 0$ and an open set U containing ∂E such that the map $(x,\lambda) \in \partial E \times (-\varepsilon,\varepsilon) \to x + \lambda\eta(x) \in U$ is bilipschitz. We write $(\pi_\eta(\cdot), \lambda_\eta(\cdot)) \in \partial E \times\ (-\varepsilon,\varepsilon)$ on U the inverse of this map. Define $\psi^e \in \mathrm{Lip}(U)$, $\eta^e \in \mathrm{Lip}(U;M)$ as $\psi^e(z) := \psi(\pi_\eta(z))$ and $\eta^e(z) := \eta(\pi_\eta(z))$ for any $z \in U$. For $\lambda \in \mathbb{R}$ with $|\lambda|$ small enough and $z \in U$, define $\Psi(z,\lambda) := z + \lambda\psi^e(z)\eta^e(z)$, and let Ψ_λ and E_λ be as in Section 5.1.

Theorem 5.13. *We have*

$$\inf_{\psi \in \mathrm{Lip}(\partial E), c_n \int_{\partial E} \psi^2 \phi^o(\nu) d\mathcal{H}^{n-1} \le 1} \qquad \lim_{\lambda \to 0^+} \frac{\mathcal{F}_{\phi^o}(E_\lambda) - \mathcal{F}_{\phi^o}(E)}{\lambda} = -\min_{N \in \mathcal{H}_{\mathrm{div}}} (\mathcal{K}(N))^{\frac{1}{2}}, \tag{5.25}$$

where

$$\mathcal{K}(N) := c_n \int_{\partial E} (\mathrm{div}_\tau N)^2 \phi^o(\nu) d\mathcal{H}^{n-1}. \tag{5.26}$$

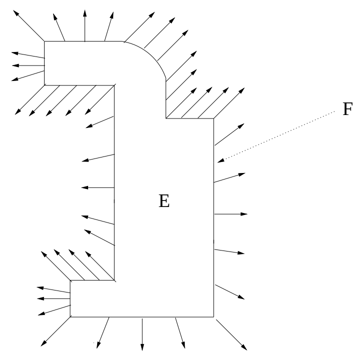

Figure 9: the vector field $N_{\min} : \partial E \to \mathbb{R}^2$ is, on facets and arcs of ∂E, the linear combination of the values of η at the vertices. In this example ∂E is not polygonal: indeed κ_ϕ^E can be defined for a generic Lipschitz p-regular set [41], [42].

The minimization problem in (5.25) in general may admit([41]) more than one solution, and two minimizers have the same divergence. In the following we denote by $N_{\min}^E = N_{\min} \in \mathcal{H}_{\mathrm{div}}$ a minimizer.

Definition 5.14 (**Crystalline mean curvature).** *We define the ϕ-mean curvature κ_ϕ^E of ∂E as $\kappa_\phi^E = \kappa_\phi := \mathrm{div}_\tau N_{\min} \in L^2(\partial E)$.*

It turns out([42]) that the ϕ-mean curvature of ∂B_ϕ is constantly equal to $n-1$.

Remark 5.15. When $\phi \in \mathcal{M}_{\mathrm{reg}}(TM)$ formula (5.25) reduces to

$$\inf_{\psi \in \mathrm{Lip}(\partial E), c_n \int_{\partial E} \psi^2 \phi^o(\nu) d\mathcal{H}^{n-1} \leq 1} \int_{\partial E} \langle \psi\eta, \nu \rangle \kappa_\phi \phi^o(\nu)\ d\mathcal{H}^{n-1} = -\int_{\partial E} (\kappa_\phi)^2 \phi^o(\nu)\ d\mathcal{H}^{n-1}.$$

Example 5.16 (**Polygonal curves).** Let $n = 2$. Let us compute explicitely the ϕ-curvature of a two-dimensional Lipschitz ϕ-regular set E, letting $\eta \in \mathrm{Nor}_\phi(\partial E; \mathbb{R}^2) \cap \mathrm{Lip}(\partial E; \mathbb{R}^2)$. Given a facet $F \subset \partial E$ (in this case F equals a segment $[z, w]$), the minimum problem (5.29) becomes

$$\inf \Big\{ \int_{]z,w[} (N'(s))^2 d\mathcal{H}^1(s) : N \in L^2(]z, w[; \Pi_{[z,w]}),\ N' \in L^2(]z, w[),$$

$$N(x) \in T_{\phi^o}(\nu_{\phi^o}(x)) \text{ for a.e. } x \in\ ]z, w[,\ N(z) = c_z, N(w) = c_w \Big\},$$

where c_z (resp. c_w) is the orthogonal projection of $\eta(z)$ (resp. of $\eta(w)$) on the line $\Pi_{[z,w]}$, with the correct sign, and $]z, w[$ is the relative interior of $[z, w]$.

[41]It would be interesting to find some argument for selecting one preferred minimizer.

[42]For instance using Remark 5.19 below.

We now observe that the above minimum problem has a unique solution $N^F_{\min}$, which is simply the linear function connecting c_z at z with c_w at w. Hence, when $n = 2$, not only the divergence of a minimizer is unique, but also the minimizer itself. If we now repeat this procedure for any facet, and on each facet we add to $N^F_{\min}$ the proper (constant) normal component to F, we end up with the vector field $N_{\min} : \partial E \to \mathbb{R}^2$ whose divergence is the ϕ-curvature of ∂E. An example of this vector field is depicted in Figure 9. Curved regions in ∂E have zero ϕ-curvature. On the other hand, if F is a facet of $\partial E \subset \mathbb{R}^2$ and $B_F \subset \partial B_\phi$ is the corresponding facet in ∂B_ϕ, κ^F_ϕ is constant in F and

$$\kappa^F_\phi = \delta_F \frac{|B_F|}{|F|} \qquad \text{in } \mathrm{int}(F), \tag{5.27}$$

where $\delta_F \in \{0, \pm 1\}$ is a convexity factor: $\delta_F = 1$ (resp. $\delta_F = -1$, $\delta_F = 0$) if E is locally convex (resp. if E is locally concave, E is neither locally convex nor locally concave) at F.

Lipschitz ϕ-regular sets have ϕ-curvature which is more regular than being only square integrable [42].

Theorem 5.17 (Regularity). *We have $\kappa_\phi \in L^\infty(\partial E)$. Moreover, κ_ϕ has bounded variation on all facets of ∂E corresponding to facets of ∂B_ϕ.*

Remark 5.18. Assume that ∂E is a polyhedral Lipschitz ϕ-regular set. We do not know under which further conditions on ∂E (if any) the functional $\mathcal{K}$ in (5.26) admits a minimizer in $\mathcal{H}_{\mathrm{div}} \cap \mathrm{Lip}(\partial E; M)$ or not. See also formula (8.5) (and Remark 5.19) below: in that case a discontinuous minimizing vector field with bounded divergence is constructed on the facet F.

5.3.1 A minimum problem on F: ϕ-mean curvature on F

In this section we assume for simplicity that $n = 3$ and that $c_n = 1$ (the case $n = 2$ is trivial), and that E is a polyhedral Lipschitz ϕ-regular set. We recall some notation that we have already occasionally used. The symbol F will always denote a (polyhedral) facet of ∂E such that $\widetilde{B}^F_\phi$ is a facet of B_ϕ. If $[p, q]$ is a closed edge of a polyhedral set, by $]p, q[$ we denote the relative interior of $[p, q]$.
Π_F is the affine plane spanned by the facet F. Whenever necessary, we identify Π_F with the plane parallel to Π_F and passing through the origin, and F with its orthogonal projection on this latter plane. We will assume for simplicity that $\widetilde{B}^F_\phi$ contains the origin of Π_F in its interior, and is symmetric with respect to the origin itself.
We let $\widetilde{\phi}_F : \Pi_F \to [0, +\infty[$ be the convex and one-homogeneous function on Π_F such that $\{\widetilde{\phi}_F \le 1\} = \widetilde{B}^F_\phi$. We denote by $\widetilde{\phi}^o_F$ the dual of $\widetilde{\phi}_F$ (recall the first item of Remark 3.18). If no confusion is possible, we omit the dependence on F of $\widetilde{\phi}_F$, thus writing $\widetilde{\phi}$ in place of $\widetilde{\phi}_F$. We indicate by $\kappa^B_{\widetilde{\phi}}$ the $\widetilde{\phi}$-curvature of the boundary of a Lipschitz $\widetilde{\phi}$-regular set $B \subset \Pi_F$. We also set

$$P_{\widetilde{\phi}}(F) := \int_{\partial F} \widetilde{\phi}^o(\widetilde{\nu}^F)\, d\mathcal{H}^1.$$

We want to recall another way to define the crystalline mean curvature κ^E_ϕ on a facet F of ∂E, using a localized minimum problem on F. Set

$$\mathrm{Nor}_\phi(F; \Pi_F) := \big\{ N \in L^\infty(F; \Pi_F) : \ N(x) \in T_{\phi^o}(\nu_{\phi^o}(F)) \text{ for } \mathcal{H}^2 \text{ a.e. } x \in F \big\}.$$

Any $N \in \mathrm{Nor}_\phi(F;\Pi_F)$ with $\mathrm{div}N \in L^2(\mathrm{int}(F))$ admits a normal trace $\langle \widetilde{\nu}^F, N\rangle$ on ∂F Set

$$\mathcal{H}_{\mathrm{div}}(F;\Pi_F) := \left\{N \in \mathrm{Nor}_\phi(F;\Pi_F) : \ \mathrm{div}N \in L^2(F), \ \langle \widetilde{\nu}^F, N\rangle = c_F \ \ \mathcal{H}^1 \text{ a.e. on } \partial F\right\}.$$

We define the functional $\mathcal{K}(\cdot, F) : \mathcal{H}_{\mathrm{div}}(F;\Pi_F) \to [0,+\infty)$ as

$$\mathcal{K}(N,F) := \int_F (\mathrm{div}N)^2 \ \phi^o(\nu^E) \ d\mathcal{H}^2 = \phi^o(\nu(F)) \int_F (\mathrm{div}N)^2 \ d\mathcal{H}^2. \tag{5.28}$$

The minimum problem

$$\inf \Big\{\mathcal{K}(N,F) : N \in \mathcal{H}_{\mathrm{div}}(F;\Pi_F)\Big\} \tag{5.29}$$

admits a solution, and two minimizers have the same divergence. Let us denote by $N^F_{\min}$ a solution of problem (5.29). It turns out that

$$\kappa^E_\phi = \mathrm{div}N^F_{\min} \qquad \mathcal{H}^2 \text{ a.e. in } F.$$

Notice once more that the above equality says that the crystalline ϕ-mean curvature of ∂E can be obtained, on the facet F, as the divergence of a vector field which minimizes a problem on F. However this minimum problem is nonlocal, in the sense that it depends on the shape of ∂E around F: indeed, we are assigning the normal trace of $N^F_{\min}$ on ∂F via the functions c_F.

The following remark is useful in concrete situations, and is a consequence of the strict convexity of the functional $\mathcal{K}(\cdot, F)$ in the divergence.

Remark 5.19 (Minimality criterion). Let $f = \mathrm{div}\overline{N}$ where $\overline{N}$ is a vector field belonging to $\mathcal{H}_{\mathrm{div}}(F;\Pi_F)$. Assume that f satisfies the Euler-Lagrange inequality

$$\int_F f \mathrm{div}(\overline{N} - N) \ d\mathcal{H}^2 \le 0, \qquad N \in \mathcal{H}_{\mathrm{div}}(F;\Pi_F). \tag{5.30}$$

Then $\overline{N}$ is a solution of (5.29).

As a corollary of this minimality criterion it follows that if f is constant in F then (5.30) is satisfied (with the equality in place of the inequality).

6 ϕ-calibrability

Theorem 5.17 makes possible to speak of the jump set of κ_ϕ on the facets of ∂E corresponding to facets of ∂B_ϕ. If $F \subset \partial E$ is such a facet, it may be of interest finding necessary and sufficient conditions on E and F ensuring that the jump set of κ_ϕ in F is empty: that is, to prove that κ_ϕ is continuous in F. Assume that this is the case: then for small times in the crystalline mean curvature flow, F is expected to translate parallely to itself if κ_ϕ is constant on F or to bend if κ_ϕ is continuous but not constant in F.

Definition 6.1 (Calibrability). *We say that F is ϕ-calibrable if κ_ϕ^E is constant in F.*

Recalling Definition 6.1, the Gauss-Green theorem (applied to $\int_F \mathrm{div} N \, d\mathcal{H}^{n-1}$) and the results of Section 5.3.1, we deduce, for instance in $n = 3$ dimensions, that a facet F is ϕ-calibrable if and only if there exists a vector field $N : F \to \mathbb{R}^3$ which is a solution to:

$$\begin{cases} N \in L^\infty(F; \mathbb{R}^3), \\ N(x) \in T_{\phi^o}(\nu_{\phi^o}(F)) \text{ for } \mathcal{H}^2 \text{ a.e. } x \in F, \\ \mathrm{div} N = \dfrac{1}{|F|} \displaystyle\int_{\partial F} c_F \, d\mathcal{H}^1, \\ \langle \widetilde{\nu}^F, N \rangle = c_F \quad \mathcal{H}^1 \text{ a.e. on } \partial F. \end{cases} \tag{6.1}$$

We note that what are important here are the two components of N on the plane Π_F containing F, since the third component (the orthogonal one) must be constant (and hence does not affect the computation of $\mathrm{div} N$).
The quantity

$$\frac{1}{|F|} \int_{\partial F} c_F \, d\mathcal{H}^1 =: v_F$$

can be interpreted as the mean velocity of F, and is sometimes called weighted mean curvature; in case of a convex F with E convex at F (see Definition 6.3 below) this velocity is positive. Hence

$$-v_F \, \nu_{\phi^o}(F)$$

represents the normal velocity vector of F.
To construct examples of facet-breaking in crystalline mean curvature flow, the first step consists exactly in finding facets which are not ϕ-calibrable. Therefore, we are led to look for criteria that allow to decide whether a facet is ϕ-calibrable or not [41], [42].
Given a finite perimeter set [14] B in the (hyper)plane Π_F containing F, we denote by $\partial^* B$ the reduced boundary of B. We also define the function $c_B : F \to \mathbb{R}$ as follows:

$$c_B(x) := \begin{cases} \max\left\{ \widetilde{\nu}^B(x) \cdot p : p \in \partial \widetilde{B}_\phi^F \right\} & \text{if } x \in \partial^* B \setminus \partial F, \\ c_F(x) & \text{otherwise.} \end{cases} \tag{6.2}$$

The following result ($n = 3$) is proved in [43].

Theorem 6.2 (**Characterization**)**.** *F is ϕ-calibrable if and only if for any $B \subseteq F$ of finite perimeter we have*([43])

$$v_B := \frac{1}{|B|}\int_{\partial^* B} c_B \; d\mathcal{H}^1 \geq \frac{1}{|F|}\int_{\partial F} c_F \; d\mathcal{H}^1 = v_F. \tag{6.3}$$

Namely, F is a solution of the minimum problem

$$\inf\left\{\frac{1}{|B|}\int_{\partial^* B} c_B \; d\mathcal{H}^1 : B \text{ of finite perimeter, } B \subseteq F\right\}.$$

Before sketching the proof of Theorem 6.2, let us recall [19], [20] that given a function u of bounded variation in F and a vector field $X \in L^\infty(F;\Pi_F)$ with bounded divergence([44]), the following generalized Gauss-Green formula holds:

$$\int_F u\mathrm{divX} \; dx + \int_F \theta(X, Du)|Du| = \int_{\partial F} \langle \widetilde{\nu}^F, X\rangle \; u \; d\mathcal{H}^1.$$

Here Du is the distributional derivative of u, which is a Radon measure; moreover, the density $\theta(X, Du)$ the total variation measure $|Du|$ [95], [14], and the normal trace $\langle \widetilde{\nu}^F, X\rangle$ of X on ∂F are suitably defined.
Sketch of proof of Theorem 6.2. The implication

$$F \quad \phi\text{–calibrable} \quad \Rightarrow \quad v_B \geq v_F \qquad \text{for } B \subseteq F$$

can be proved as follows. We know that $\mathrm{div}N = v_F$ in F. Therefore, integrating $\mathrm{div}N$ on F and using the Gauss-Green theorem we get

$$|B|\mathrm{div}N = \int_B \mathrm{div}N \; dx = \int_{\partial^* B} \widetilde{\nu}^B \cdot N \; d\mathcal{H}^1 \leq \int_{\partial^* B} c_B \; d\mathcal{H}^1,$$

where in the last equality we use also the definition (6.2) of c_B.
The implication

$$F \quad \phi\text{–calibrable} \quad \Leftarrow \quad v_B \geq v_F \qquad \text{for } B \subseteq F$$

can be proved as follows. Assume by contradiction that F is not ϕ-calibrable. Given any $\lambda \in \mathbb{R}$ define $\Omega_\lambda := \{x \in F : \mathrm{div}N_{\min}(x) < \lambda\}$. Using Theorem 5.17 it follows([45]) that

$$\text{there exists } \lambda < v_F \text{ such that } \Omega_\lambda \neq \emptyset \text{ has finite perimeter.}$$

We have, using the properties of functions of bounded variations [14] and the Gauss-Green theorem,

$$\begin{aligned}\int_{\Omega_\lambda} \mathrm{div}N_{\min} \; dx &= -\int_{\mathrm{int}(F)\cap\partial^*\Omega_\lambda} \theta(N_{\min}, D1_{\Omega_\lambda}) \; d\mathcal{H}^1 + \int_{\partial F} \langle \widetilde{\nu}^F, N_{\min}\rangle \; 1_{\Omega_\lambda} \; d\mathcal{H}^1 \\ &= -\int_{\mathrm{int}(F)\cap\partial^*\Omega_\lambda} \theta(N_{\min}, D1_{\Omega_\lambda}) \; d\mathcal{H}^1 + \int_{\partial F\cap\partial^*\Omega_\lambda} \langle \widetilde{\nu}^F, N_{\min}\rangle \; d\mathcal{H}^1.\end{aligned}$$

[43] Heuristically, subfacets of F would move faster than F, consistently with the comparison result for crystalline mean curvature flow.
[44] Divergence in $L^2(F)$ would be enough.
[45] Almost all sublevel sets of a BV function are of finite perimeter.

It is now possible to prove the following property:

$$-\theta(N_{\min}, D1_{\Omega_\lambda})(x) = \max\left\{\widetilde{\nu}^{\Omega_\lambda}(x)\cdot p : p \in \partial \widetilde{B}^F_\phi\right\} \qquad \text{for } \mathcal{H}^1 - \text{a.e. } x \in \mathrm{int}(F)\cap \partial^*\Omega_\lambda,$$

and also the property $\langle \widetilde{\nu}^F, N_{\min}\rangle = c_F = c_{\Omega_\lambda}$ on $\partial F \cap \partial^*\Omega_\lambda$. Therefore $-\theta(N_{\min}, D1_{\Omega_\lambda}) = c_{\Omega_\lambda}$ on $\mathrm{int}(F)\cap\partial^*\Omega_\lambda$, and hence

$$\int_{\Omega_\lambda} \mathrm{div} N_{\min}\ dx = \int_{\partial^*\Omega_\lambda} c_{\Omega_\lambda}\ d\mathcal{H}^1.$$

It follows

$$v_F > \lambda > \frac{1}{|\Omega_\lambda|}\int_{\Omega_\lambda} \mathrm{div} N_{\min}\ dx = \frac{1}{|\Omega_\lambda|}\int_{\partial^*\Omega_\lambda} c_{\Omega_\lambda}\ d\mathcal{H}^1 \geq v_F,$$

which is a contradiction. □

Heuristically, proving that a facet instantly breaks during the subsequent crystalline mean curvature flow means to find a subset $B \subset F$ such that $v_B < v_F$.

6.1 The case of convex facets

Definition 6.3 (Convexity at a facet). *We say that E is convex at F if E lies, locally around F, from one side of the hyperplane Π_F containing F.*

It is possible to prove that if the Lipschitz ϕ-regular set E is convex at F, then F is Lipschitz $\widetilde{\phi}_F$-regular (we set $\widetilde{\phi} = \widetilde{\phi}_F$). Under this convexity assumption([46]), we have([47]) that

$$v_F = \frac{1}{|F|}\int_{\partial F} \widetilde{\phi}^o(\widetilde{\nu}^F)\ d\mathcal{H}^1.$$

In addition κ_ϕ turns out to be convex in F.

The following result, useful in the applications, is proved in [43] ([48]).

Theorem 6.4 (Characterization for a convex E and a convex F). *Assume that E is convex at F and that F is convex. Then F is ϕ-calibrable if and only if*

$$\sup_{\partial F} \kappa^F_{\widetilde{\phi}} \ \leq\ \frac{1}{|F|}\int_{\partial F} \widetilde{\phi}^o(\widetilde{\nu}^F)\ d\mathcal{H}^1. \tag{6.4}$$

[46] We assume $B_{\widetilde{\phi}}$ to be centrally symmetric.

[47] We recall here that, in the euclidean case, $|\cdot|$-calibrability under suitable assumptions (essentially convexity of E at a facet) becomes the notion of Cheeger set. Indeed, a set $F \subset \mathbb{R}^2$ is called a Cheeger set if for any $B \subseteq F$ we have $\frac{P(B)}{|B|} \geq \frac{P(F)}{|F|}$, where $P(A) = P_{|\cdot|}(A)$ is the perimeter of $A \subseteq \mathbb{R}^2$. In the euclidean case it is known that if F is a Cheeger set then the curvature of its boundary is bounded above by $\frac{P(F)}{|F|}$, and the converse implication is true if F is convex (in accordance with Theorem 6.4). It is also known that the complement of a bounded convex set if $|\cdot|$-calibrable. On the other hand, the complement of two bounded convex sets is not necessarily $|\cdot|$-calibrable [2], [3], see also [31] for related results. We refer to [66], [67] and references therein for more, and also for some connections with the capillarity problem. In [66] it is possible to find relations of the above concepts with the prescribed curvature problem $\inf\{P(B) - \lambda|B| : B \subseteq F\}$. Relations with the isoperimetric problem $\inf\{P(B) : |B| = \mu\}$ can be considered as well. Finally, we mention the paper [65] for recent extensions.

[48] We do not know whether the convexity assumption on F in Theorem 6.4 can be relaxed, in order to obtain the same thesis.

The sup in (6.4) is the essential supremum, since $\kappa^F_{\widetilde{\phi}}$ is a function in $L^\infty(\partial F)$. Recall that $\kappa^F_{\widetilde{\phi}}$ is the $\widetilde{\phi}_F$-curvature of ∂F (as a subset of Π_F).
Hence, under the assumptions of Theorem 6.4, problem (6.1) is solvable if and only if the $\widetilde{\phi}$-curvature of ∂F is bounded above by the constant on the right hand side of (6.4); this means, roughly speaking, that the edges of ∂F cannot be too "short".

7 Anisotropic mean curvature flow

In this section we quickly recall the definition of anisotropic mean curvature flow, and we present the main example of evolution. We will not consider the case of unbounded hypersurfaces (such as graphs on the whole of $\mathbb{R}^{n-1}$, for instance).

7.1 Regular case

We assume in this subsection that $\phi \in \mathcal{M}_{\mathrm{reg}}(TM)$.

Definition 7.1 (**ϕ-mean curvature flow**). *Let $T > 0$ and, for any $t \in [0,T]$, let $E(t) \subset M$ be a set with compact boundary. We say that $(E(t))_{t\in[0,T]}$ is a smooth ϕ-mean curvature flow in $[0,T]$ starting from $E = E(0)$ if:*

(i) there exists an open set $A \subset M \times [0,+\infty)$ such that $\cup_{t\in[0,T]}(\partial E(t) \times \{t\}) \subset A$ and, if we define

$$d_\phi(z,t) := \mathrm{dist}_\phi(z, E(t)) - \mathrm{dist}_\phi(z, M \setminus E(t)), \qquad z \in M,\ t \in [0,T],$$

we have $d_\phi \in \mathcal{C}^\infty(A)$;

(ii) the following equation holds([49]):

$$\frac{\partial}{\partial t} d_\phi(x,t) = \Delta_\phi d_\phi(x,t), \qquad x \in \partial E(t),\ t \in [0,T]. \tag{7.1}$$

Observe that $\frac{\partial}{\partial t} d_\phi$ is positive for an expanding set.

Example 7.2. Given $R_0 > 0$, let us show that $\{\xi \in M : \phi(\xi) < R_0\}$ has an evolution shrinking self-similarly under the flow (7.1). Looking for a solution of the form $\{\xi \in M : \phi(\xi) < R(t)\}$, we have $d_\phi(z,t) = \phi(z) - R(t)$, and (7.1) becomes $\dot{R} = -\frac{n-1}{R}$ (recall Example 5.9). Hence $R(t) = \sqrt{R_0^2 - 2(n-1)t}$ for $t \in [0, \frac{R_0^2}{2(n-1)})$, which disappears for times larger than $\frac{R_0^2}{2(n-1)}$.

The evolution law (7.1) is the gradient flow of $\mathcal{F}_{\phi^o}$, see [5], [78], [9]. We refer for instance to the papers [47] and references therein for more.

7.2 Crystalline case

Assume now that ϕ is crystalline. Unless $n = 2$, the definition of crystalline mean curvature flow is much more involved. Let us begin with the two-dimensional case ($M = \mathbb{R}^2$).

[49] Recall that from the last equality in (5.17) it follows that $\Delta_\phi d_\phi(x,t)$ is the ϕ-mean curvature $\kappa_\phi = \mathrm{div} N_\phi^{E(t)}$ of $\partial E(t)$ at $x \in \partial E(t)$. Moreover $-\frac{\partial d_\phi}{\partial t} N_\phi^{E(t)}$ is the normal velocity of the flow.

7.2.1 Curves

Let $\partial E \subset \mathbb{R}^2$ be a closed simple polygonal curve, $S_j \subset \partial E$ be an edge of length $L_j > 0$ and ν_j the exterior euclidean unit normal to S_j. We define δ_{S_j} to be 1 (resp. -1) if S_j and its two adjacent edges form a convex (resp. concave) curve, and 0 otherwise. Let $L_{B_\phi}(\nu_j)$ be the length of the edge of ∂B_ϕ having ν_j as exterior normal: we will restrict here to consider polygonal curves ∂E (and $\partial E(t)$) which consist of a sequence of segments having the same ordered set of normal orientations as ∂B_ϕ. Such a ∂E is neighbourhood-Lipschitz ϕ-regular. Recall that the crystalline curvature of S_j equals

$$\kappa_\phi^j := \delta_{S_j} \frac{L_{B_\phi}(\nu_j)}{L_j},$$

see (5.27).
Given two parallel segments S_1, S_2, we call the distance vector of S_2 from S_1 the vector having norm $\mathrm{dist}(S_1, S_2)$ pointing from S_1 to S_2.
Let us define the local in time crystalline curvature flow of a polygonal Lipschitz ϕ-regular curve (with a finite number of sides), supposing that no side disappears.

Definition 7.3. *Let $\partial E(t)$ be a family of time-parametrized polygonal Lipschitz ϕ-regular curves. We say that $\partial E(t)$ moves by crystalline curvature in $[0,T]$, $T > 0$, if each side either translates parallel to itself or stays still (and does not disappear) for any j, the distance vector $h_j(t)$ between the edge $S_j(t)$ and $S_j(0)$ is of class $\mathcal{C}^1([0,T])$, and*([50])

$$\frac{\dot{h}_j(t)}{\phi^o(\nu_j)} = -\kappa_\phi^j(t)\nu_j, \qquad t \in [0,T].$$

The left hand side of the above equation is the time derivative of the signed ϕ-distance function from $S_j(t)$.
Convex portions of the curve contract in the direction of their inner normal, while concave portions expand in the direction of the outer normal (see Figure 10). See [141], [143], [121], [98], [101], [107], [108], [102], [103], [121], [137], [138], [148] for various qualitative properties.

7.2.2 Hypersurfaces

In this section we recall a definition of crystalline mean curvature flow (for which the set B_ϕ shrinks self-similarly([51])). Recall that $\partial E(t)$ is always assumed to be compact and Lipschitz. The next definition has been used in [39] (see also [36] in two dimensions) to prove a comparison principle for crystalline mean curvature flow.

[50] See also Remark 10.13 below.

[51] This self-similar evolution is not expected to be stable (for $n = 3$) for all choices of ϕ [129]. It seems to be reasonable to believe also that, for $n = 3$, there exist $\phi \in \mathcal{M}_{\mathrm{reg}}(TM)$ for which the evolution of B_ϕ is not stable. We recall that if $n = 2$ and ϕ is crystalline the evolution of B_ϕ is stable; but there exist crystalline ϕs (still for $n = 2$) not even for which B_ϕ is not stable. See also [128].

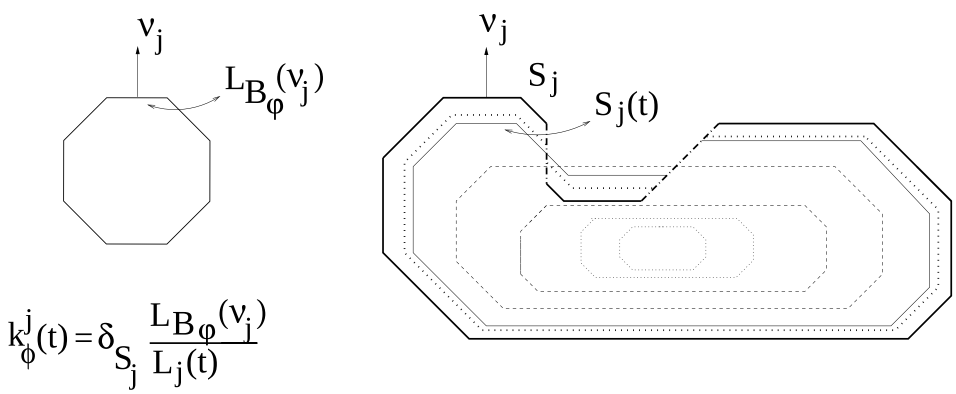

Figure 10: The motion by crystalline curvature starting from the curve in bold when B_ϕ is an octagone. We depict the motion also after the disappearing of some edge.

Definition 7.4 (Neighbourhood-L^∞ ϕ-regular flow). *Let $T > 0$. A neighbourhood-L^∞ ϕ-regular flow on $[0, T]$ is a map $t \in [0, T] \to E(t) \subset M$ satisfying the following properties:*

(i) there exists an open set $A \subset M \times [0, +\infty)$ such that $\cup_{t\in[0,T]}(\partial E(t) \times \{t\}) \subset A$ and $d_\phi(z, t) := \mathrm{dist}_\phi(z, E(t)) - \mathrm{dist}_\phi(z, M \setminus E(t))$ is Lipschitz in A;

(ii) there exists a bounded vector field $n : A \to \Lambda_1 V$ such that $n \in T_{\phi^o}(\nabla d_\phi)$ almost everywhere in A, $\mathrm{div} n \in L^\infty(A)$, and there exists $\lambda > 0$ such that

$$\left| \frac{\partial d_\phi}{\partial t}(z, t) - \mathrm{div} n(z, t) \right| \leq \lambda |d_\phi(z, t)| \qquad \text{for a.e. } (z, t) \in A. \tag{7.2}$$

In (7.2) the divergence of n is taken in M, hence we avoid to restrict it on a specific boundary. Notice also that the left hand side of (7.2) tends to zero as $d_\phi(z, t)$ tends to zero([52]).
Other definitions could be given by imposing for instance in addition that $E(t)$ and $M \setminus E(t)$ satisfy the rB_ϕ-ball condition (see [68], [33])([53]), or by imposing the evolution law only on the flowing manifold, possibly using([54]) the vector field $N_{\min}^{E(t)}$. We do not give any detail here. We refer to the already mentioned papers, to [26] and to Section 8.

Remark 7.5. We are not aware of a local existence result of a neighbourhood-L^∞ ϕ-regular flow even starting from a Lipschitz ϕ-regular polyhedral set E in $n \geq 3$ dimensions, one of

[52] For euclidean motion by mean curvature ($\phi(\cdot) = |\cdot|$), the left hand side of (7.2) can be controlled by $|d_{|\ |}|$ times the L^∞-norm squared of the length of the second fundamental form of the flowing boundaries.

[53] We recall that if $\phi \in \mathcal{M}_{\mathrm{reg}}(TM)$, then ∂E satisfies an interior and exterior ball condition if and only if ∂E is neighbourhood Lipschitz ϕ-regular. If ϕ is crystalline and ∂E is neighbourhood Lipschitz ϕ-regular, then ∂E satisfies and interior and exterior ball condition. See [32] for more.

[54] We have not used a minimizer $N_{\min}^{E(t)}$ (which is a vector field not extended out of $\partial E(t)$) in the definition of the flow. Instead, we have given a definition using a vector field n defined in a neighbourhood of $\partial E(t)$, one reason being that, for such a class of flows, a comparison principle is available (see Corollary 9.5 and its application in Remark 8.4). The hope is that, for almost each $t \in [0, T]$, the divergence of n can be restricted to $\partial E(t)$ and it solves the minimum problem in (5.26) (with E replaced by $E(t)$), so that it coincides with the divergence of $N_{\min}^{E(t)}$ on $\partial E(t)$.

the reasons probably being the presence of the facet-breaking phenomenon [148], [40], [43], [28]. A short time existence and uniqueness result (as well as the existence of a global weak solution) has been proved in [32], provided E is convex. In view of the poor knowledge on existence and uniqueness of crystalline mean curvature flow in three dimensions, it is not yet completely clear that Definition 7.4 is the most natural one for this kind of geometric flows.

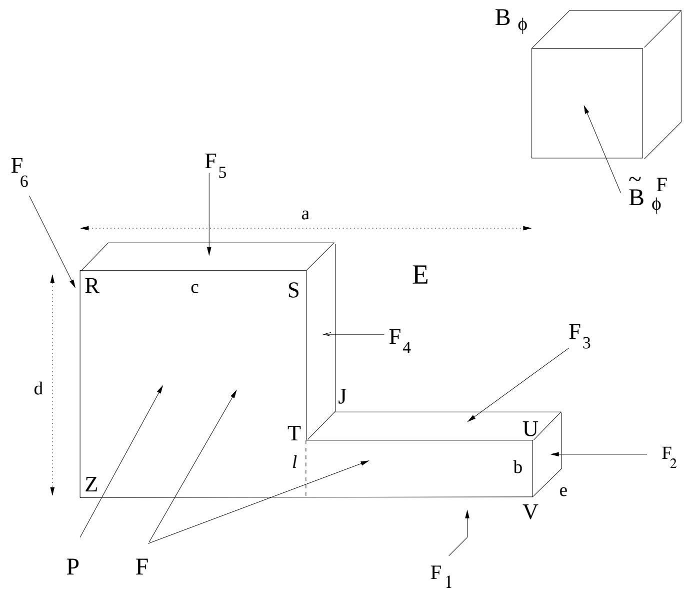

Figure 11: the initial set $E = E(0)$ and $B_\phi = [-1,1]^3$.

8 Facet-breaking in crystalline mean curvature flow in three dimensions

In this section we assume $n = 3$ (so that $M = \mathbb{R}^3$), we fix $B_\phi := [-1,1]^3$ and we let $E = E(0)$ be the set depicted in Figure 11. We want to construct a short-time crystalline mean curvature flow $E(t)$ starting from E (under proper choices of a, b, c, d, e) in the sense of Definition 7.4, in the case when the frontal facet F (and its opposite one) splits (at time zero) into two facets, while all the other facets (some of which do not remain rectangular for small positive times) do not split.
In what follows, we often use the same symbol to indicate an edge and its length.

8.0.3 On ϕ-calibrability of F

It is clear that E is convex at F, but F is not convex, hence Theorem 6.4 cannot be applied. Nevertheless, the following proposition holds [40].

Proposition 8.1. *F is ϕ-calibrable if and only if*

$$b \geq \frac{cd}{c+d}, \qquad c \geq \frac{ab}{a+b}. \tag{8.1}$$

Remark 8.2. The implication F ϕ-calibrable $\Rightarrow$ (8.1) is proved in [40] applying Theorem 6.2, taking B as the rectangle with sides c and d, and next as the rectangle with sides a and b. The opposite implication is proved using Remark 8.3 below to three subrectangles of F partitioning F.

We notice however that inequalities (8.1) express exactly condition (6.4). Indeed $\kappa^F_{\widetilde{\phi}} = 0$ on the edges $]S,T[$ and $]T,U[$. Moreover if $|L|$ stands for the length of one of the four remaining edges $d = [R,Z]$, $a = [Z,V]$, $b = [U,V]$ and $c = [R,S]$, which we generically denote by L, we have, recalling (5.27),

$$\kappa^F_{\widetilde{\phi}} = \frac{2}{|L|} \qquad \text{on int}(L).$$

Therefore the supremum of $\kappa^F_{\widetilde{\phi}}$ is reached either at the edge b or at the edge c. We distinguish two cases. The first case is when $b \le c$ (as in Fig. 11), so that the supremum of $\kappa^F_{\widetilde{\phi}}$ is reached at the edge b. Then the second inequality in (8.1) is automatically satisfied, since $a/(a+b) < 1$. A direct computation gives $c_F = 1$ on ∂F (see (4.1) and Example 4.11; therefore

$$\int_{\partial F} c_F \; d\mathcal{H}^1 = 2(a+d).$$

Since $|F| = cd + b(a-c)$, the inequality (6.4) reads as $\frac{2}{b} \le \frac{2(a+d)}{cd+ab-bc}$, which is equivalent to $b \ge \frac{cd}{c+d}$, and gives the first inequality in (8.1).
If $b \ge c$, the supremum of $\kappa^F_{\widetilde{\phi}}$ is reached at the edge c. Then the first inequality in (8.1) is automatically satisfied. Moreover inequality (6.4) reads as $\frac{2}{c} \le \frac{2(a+d)}{cd+ab-bc}$, which is equivalent to $c \ge \frac{ab}{a+b}$.

8.0.4 On ϕ-calibrability of the other facets of ∂E

All facets of ∂E different from F and its opposite one are ϕ-calibrable, since they are rectangles. For rectangular facets F such that E is convex at F this is a direct consequence of Theorem 6.4. For instance, consider the right lateral facet F_2 of E: the edges of F_2 are b and e. Assume $b \le e$. Theorem 6.4 reads as $\frac{2}{b} \le \frac{2(b+e)}{be}$, which is always satisfied (with the strict inequality).
However there are rectangular facets $Q \subset \partial E$ such that E is not convex at Q. The ϕ-calibrability of those facets([55]) follows from the following result.

Remark 8.3. Let $R \subset \mathbb{R}^2$ be a rectangle with edges $l_1, \dots, l_4$ parallel to the coordinate axes, let $\widetilde{\nu}_i$ be the exterior unit normal to R at int(l_i) and let $|l_i|$ be the length of l_i. Let l_1 and l_3 be the edges parallel to the x-axis, l_1 the lower one, and l_2 be the right edge. Fix for simplicity the origin at the intersection between l_4 and l_1. Let $\alpha_i \in [-1,1]$ for $i = 1, \dots, 4$. Consider the vector field $n := (n_1, n_2)$ defined, for $(x,y) \in R$, as

$$n_1(x,y) := \frac{\alpha_2 x}{|l_1|} - \alpha_4 \left(1 - \frac{x}{|l_1|}\right) = n_1(x), \qquad n_2(x,y) := \frac{\alpha_3 y}{|l_4|} - \alpha_1 \left(1 - \frac{y}{|l_4|}\right) = n_2(y).$$

Notice that n_1 (resp. n_2) depends only on x (resp. on y). Then

$$\text{div} n = \frac{\alpha_2 + \alpha_4}{|l_1|} + \frac{\alpha_3 + \alpha_1}{|l_4|} = \frac{|l_2|(\alpha_2 + \alpha_4)}{|R|} + \frac{|l_3|(\alpha_3 + \alpha_1)}{|R|} = |R|^{-1} \sum_{i=1}^{4} \alpha_i |l_i|.$$

Moreover, $\langle \widetilde{\nu}_i, n \rangle = \alpha_i$ for $i = 1, \dots, 4$. Indeed, for instance on l_3 (resp. on l_4) we have $\langle \widetilde{\nu}_3, n(x,y) \rangle = n_2(x, |l_4|) = \alpha_3$ (resp. $\langle \widetilde{\nu}_4, n(x,y) \rangle = -n_1(0,y) = \alpha_4$).

[55] After adding the proper constant third component on each Q to the vector field (n_1, n_2) defined in Remark 8.3.

The vector field n satisfies also $|n_1|, |n_2| \le 1$. Summarizing

$$\begin{cases} \max\{|n_1|, |n_2|\} \le 1 & \text{in int}(R), \\ \text{div } n = |R|^{-1} \sum_{i=1}^{4} \alpha_i |l_i| & \text{in int}(R), \\ \langle n, \widetilde{\nu}_i \rangle = \alpha_i & \text{in int}(l_i). \end{cases} \tag{8.2}$$

Fix now

$$a = 2,\ b = 1/4,\ c = 1,\ d = 1,\ e = 1/2. \tag{8.3}$$

With these choices,

$$\text{the facet } F \text{ is not } \phi-\text{calibrable},$$

in view of Proposition 8.1.

8.0.5 On ϕ-calibrability of facets of $\partial E(t)$, $t > 0$

Let us consider a set $E(t)$ of the form depicted in Figure 13, for a fixed $t > 0$ small enough, hence in particular the edge $[\alpha(t), \beta(t)]$ is short enough.

Let us consider facets $F_2(t)$, $F_5(t)$, $F_6(t)$, $P(t)$ and its opposite one: these are rectangular facets where the set $E(t)$ is convex. These facets are ϕ-calibrable as a consequence of Theorem 6.4.

Concerning facets $F_3(t)$, $(F \setminus P)(t)$ and its opposite one: these are rectangular facets, and they are ϕ-calibrable thanks to Remark 8.3.

Facets $F_1(t)$ and $F_4(t)$: these are non rectangular facets. Notice that $F_1 = F_1(0)$ satisfies $\sup_{\partial F_1} \kappa_{\widetilde{\phi}}^{F_1} < P_{\widetilde{\phi}}(F_1)/|F_1|$, which implies that $\sup_{\partial F_1(t)} \kappa_{\widetilde{\phi}}^{F_1(t)} < P_{\widetilde{\phi}}(F_1(t))/|F_1(t)|$ for short times. ϕ-calibrability of $F_1(t)$ follows from Theorem 6.2.

The most delicate analysis concerns the facet $F_4(t)$, since $E(t)$ is neither convex nor concave at $F_4(t)$. It is possible to prove that $F_4(t)$ is ϕ-calibrable under the assumptions (8.3), for $t > 0$ small enough (this is due to the specific form of the normal traces to $\partial F_4(t)$).

8.0.6 Construction of the flow

We now want to show that $E(t)$ is a crystalline mean curvature flow, in the sense of Definition 7.4. Each set $E(t)$ is polyhedral Lipschitz ϕ-regular, since a vector field in $\text{Nor}_\phi(\partial E(t); \mathbb{R}^3) \cap \text{Lip}(\partial E(t); \mathbb{R}^3)$ can be constructed by hand.

Step 1. Construction of the velocity field div$N(\cdot, 0)$ *on* ∂E.

We construct a vector field $N(\cdot, 0) \in \mathcal{H}_{\text{div}}$ at time 0 as follows. Let Q be a facet of ∂E, consider $N_{\min}^Q$, and define, for $\mathcal{H}^2$-almost every $x \in \text{int}(Q)$, the two components of $N(x, 0)$ lying in the plane Π_Q as $N_{\min}^Q(x)$. Add the proper constant third component on $\text{int}(Q)$ in such a way that the three-dimensional vector field (still denoted by $N(\cdot, 0)$) belongs to $\mathcal{H}_{\text{div}}$. The initial normal velocity of ∂E is then div$N(\cdot, 0)$ on $\text{int}(Q)$. For the moment, this definition of velocity is not explicit.

Step 2. Identification of div$N(\cdot, 0)$ *on facets different from* F *and its opposite one.*

Each facet Q of ∂E different from F and its opposite is ϕ-calibrable. It follows that, on Q, the initial normal velocity equals the constant appearing on the right hand side of the partial

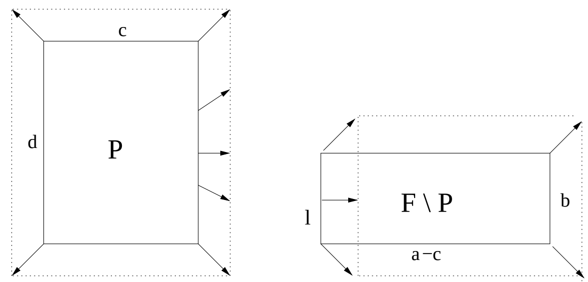

Figure 12: P and $F \setminus P$ is a partition of the frontal facet F of ∂E. We depict the vector field $\overline{N}$ on ∂P and on $\partial(F \setminus P)$. The construction is based on Remark 8.3.

differential equation in (8.2) expressing the divergence of the vector field, namely v_Q. This is a consequence of Remark 5.19 and the sentence after it.
To determine this constant we have to find the values of α_i, i.e. the value of c_Q on each facet Q. We have

(i) $c_{F_6} = 1$ on ∂F_6, $c_{F_1} = 1$ on ∂F_1, $c_{F_2} = 1$ on ∂F_2, $c_{F_5} = 1$ on ∂F_5. Hence

$$\begin{aligned}
\mathrm{div} N^E_{\min} &= \tfrac{2(d+e)}{de} \text{ in } \mathrm{int}(F_6)\\
\mathrm{div} N^E_{\min} &= \tfrac{2(a+e)}{ae} \text{ in } \mathrm{int}(F_1),\\
\mathrm{div} N^E_{\min} &= \tfrac{2(b+e)}{be} \text{ in } \mathrm{int}(F_2),\\
\mathrm{div} N^E_{\min} &= \tfrac{2(c+e)}{ce} \text{ in } \mathrm{int}(F_5).
\end{aligned}$$

(ii) $c_{F_3} = 1$ on ∂F_3 and $c_{F_4} = 1$ on ∂F_4 except that on $]T, J[$, where $c_{F_3} = c_{F_4} = -1$, see Figures 11 and 7. Hence $\mathrm{div} N^E_{\min} = \frac{2}{e}$ in $\mathrm{int}(F_3)$ and $\mathrm{div} N^E_{\min} = \frac{2}{e}$ in $\mathrm{int}(F_4)$.

Step 3. Identification of $\mathrm{div} N(\cdot, 0)$ *on* F *and on its opposite facet.*
Let us consider the facet F (the arguments for the facet opposite to F are the same). We have $c_F = 1$ on ∂F. We know that there does not exist a vector field defined on $\mathrm{int}(F)$ having constant divergence, whose normal trace on ∂F is one and lying in $T_{\phi^o}(\nu_{\phi^o}(F))$.
Let us subdivide F into two rectangles P and $F \setminus P$ as in Figure 11; in Figure 12 the two rectangles are depicted disjoint. We use the explicit construction of Remark 8.3 separately on P and $F \setminus P$, taking the constants α_i as follows.

- On ∂P all α_i are equal to one;
- on $\partial(F \setminus P)$ the α_i are equal to one except that on the dotted segment l, where the corresponding α_j is equal to -1.

Remark 8.3 provides two explicit vector fields

$$M_P : P \to \mathbb{R}^2, \qquad M_{F \setminus P} : F \setminus P \to \mathbb{R}^2,$$

with the following properties:

(a) $M_P \in \mathcal{H}_{\mathrm{div}}(P; \Pi_F)$, $\mathrm{div} M_P \equiv \frac{2(d+c)}{dc}$ on $\mathrm{int}(P)$;

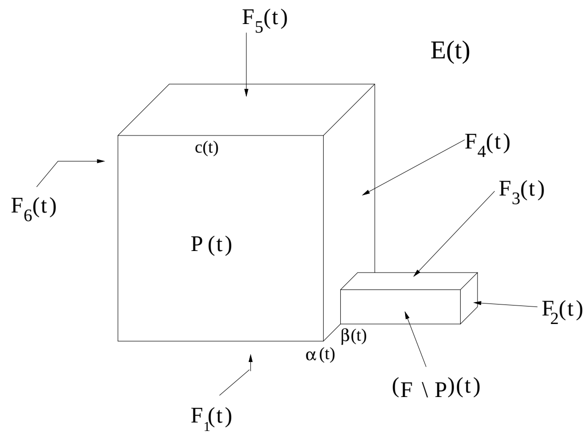

Figure 13: Picture of the set $E(t)$ for small positive t

(b) $M_{F\setminus P} \in \mathcal{H}_{\rm div}(F \setminus P; \Pi_F)$, ${\rm div} M_{F\setminus P} \equiv \frac{2(a-c)}{b(a-c)} = \frac{2}{b}$.

We let

$$\overline{N} := \begin{cases} M_P & \text{in } {\rm int}(P), \\ M_{F\setminus P} & \text{in } {\rm int}(F\setminus P). \end{cases}$$

The vector field $\overline{N}$ is explicit, since the construction in Remark 8.3 is explicit.
Observe that

$$\frac{1}{2} = {\rm div} M_P < {\rm div} M_{F\setminus P} = 1. \tag{8.4}$$

It is interesting to observe that the component of $\overline{N}$ in Π_F orthogonal to l is continuous along l, see Figure 12. On the other hand, the component of $\overline{N}$ in Π_F tangent to l is discontinuous along l. It follows that

$$\overline{N} \text{ is discontinuous on } {\rm int}(F),$$

and

$${\rm div}\overline{N} \in L^\infty(F).$$

In particular, $\overline{N} \in \mathcal{H}_{\rm div}(F; \Pi_F)$.
Let us now check that $\overline{N}$ satisfies the Euler-Lagrange inequality (5.30). The divergence of $\overline{N}$ is constant in the interior of P and $F \setminus P$, and therefore, to check that (5.30) holds, we have to prove that

$$\frac{2(d+c)}{dc} \int_{{\rm int}(P)} {\rm div}(\overline{N} - N)\ d\mathcal{H}^2 + \frac{2}{b} \int_{{\rm int}(F\setminus P)} {\rm div}(\overline{N} - N)\ d\mathcal{H}^2 \le 0 \qquad \forall N \in \mathcal{H}_{\rm div}(F; \Pi_F). \tag{8.5}$$

We have

$$\begin{aligned} &\frac{2(d+c)}{dc} \int_{{\rm int}(P)} {\rm div}(\overline{N} - N)\ d\mathcal{H}^2 + \frac{2}{b} \int_{{\rm int}(F\setminus P)} {\rm div}(\overline{N} - N)\ d\mathcal{H}^2 \\ =& \frac{2(d+c)}{dc} \int_{\partial P} \langle \widetilde{\nu}^P, \overline{N} - N\rangle\ d\mathcal{H}^1 + \frac{2}{b} \int_{\partial(F\setminus P)} \langle \widetilde{\nu}^{F\setminus P}, \overline{N} - N\rangle\ d\mathcal{H}^1 \\ =& \left(\frac{2(d+c)}{dc} - \frac{2}{b}\right) \int_l \langle \widetilde{\nu}^P, \overline{N} - N\rangle\ d\mathcal{H}^1 \le 0, \end{aligned}$$

since $\frac{2(d+c)}{dc} - \frac{2}{b} < 0$ by (8.4), and since, by construction, the normal trace of $\overline{N}$ on $\mathrm{int}(l)$ is maximal (in the direction of $\widetilde{\nu}^P$) among all vector fields satisfying the same constraints (see Figure 12), so that $\langle \widetilde{\nu}^P(x), \overline{N}(x) - N(x) \rangle \geq 0$ for $\mathcal{H}^1$-almost every $x \in l$.
Using Remark 5.19, we conclude that $\overline{N}$ is a solution of (5.29), and therefore $\mathrm{div}\overline{N}$ is the ϕ-mean curvature of ∂E on $\mathrm{int}(F)$, and $\mathrm{div}\overline{N} = \mathrm{div}N(\cdot, 0)$ on $\mathrm{int}(F)$.

Step 4. Construction of the normal velocity of $\partial E(t)$.
Let us now consider the set $E(t)$ for small positive times, constructed by flowing (shrinking) a generic facet $Q(t)$ of $\partial E(t)$ with constant normal velocity equals to $\frac{1}{|Q(t)|}\int_{\partial Q(t)} c_{Q(t)}\, d\mathcal{H}^1$. Observe that all facets of $\partial E(t)$ are ϕ-calibrable, so they do not further subdivide. In addition, on each $\mathrm{int}(Q(t))$ the normal velocity equals the divergence of a solution of (5.29) (where F is replaced by $Q(t)$)([56])
Through *steps 1-4* we have constructed a flow starting from E. Actually, this is the unique crystalline mean curvature flow of E in a reasonably large class of flows, as explained in the following observation.

Remark 8.4. The vector field N previously defined admits an extension (by lines) in $U \times [0, T]$, where U is a suitable open set containing $\partial E(t)$, $t \in [0, T]$, and $T > 0$ is small enough. More precisely, let $y \in U$ and let $x \in \partial E(t)$ be the unique point with the property that y belongs to the straight line $\{x + sN(x,t)\}_{s \in \mathbb{R}}$ (this property is fulfilled if U is sufficiently thin, and for those points x where $N(x,t)$ is continuous, hence if $t \in (0, T]$ for all points, while if $t = 0$ excluding points on the segment l). Then we define $N(y,t) := N(x,t)$ (extension of N by lines). With this definition the evolution that we have constructed is in the sense of Definition 7.4. It turns out that this evolution is unique in that class (see the next section).

Before concluding this section, we remark that various qualitative properties of the jump set of $\mathrm{div}N^E_{\min}$ on F (such as the facts that the jump set must reach the boundary of ∂F, and that it must be contained in the boundary of an homothetic of $\widetilde{B}^F_\phi$) can be found in [42]. Moreover, in [28] it is discussed with some detail an example of facet bending([58]), previously described in [43], see also [148].

Remark 8.5. It interesting to point out here a connection between anisotropic mean curvature flow and the total variation flow [30] (see also [130]). Let us consider in $\mathbb{R}^3$ the cylindrical anisotropy $B_\phi := \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1^2 + \xi_2^2 \leq 1\} \times [-1, 1]$. Let $u : \mathbb{R}^2 \times [0, +\infty) \to \mathbb{R}$ be a solution to the total variation flow [16] starting from the characteristic function of a bounded

[56]This step requires solving a system of ordinary differential equations for([57]) $a(t), b(t), c(t), d(t)$ and $e(t)$. For instance, the equation for $\dot{a}$, using the evolution of the facets F_6 and F_2 is given by

$$\frac{\dot{a}(t)}{\phi^o(\nu_{F_2})} = -2\frac{d(t) + e(t) + (\beta(t) - \alpha(t))}{d(t)[e(t) + (\beta(t) - \alpha(t))]} - 2\frac{b(t) + e(t)}{b(t)e(t)}.$$

Similarly, the evolution for $\dot{d}$ is given by

$$\frac{\dot{d}(t)}{\phi^o(\nu_{F_5})} = -2\frac{c(t) + e(t) + (\beta(t) - \alpha(t))}{c(t)[e(t) + (\beta(t) - \alpha(t))]} - 2\frac{a(t) + e(t) + (\beta(t) - \alpha(t))}{a(t)[e(t) + (\beta(t) - \alpha(t))]},$$

and so on.

[58]This example shows that the class of polyhedral Lipschitz ϕ-regular sets is a too restricted class when looking for a solution of crystalline mean curvature flow (starting from a polyhedral Lipschitz set). Curved portions on $\partial E(t)$ may appear, and the constraint of being polyhedral must be abandoned.

set $\Omega \subset \mathbb{R}^2$. The graph of the BV-function $u(\cdot, t)$ (i.e., the boundary of the subgraph $E(t)$ of $u(\cdot, t)$) has in general flat portions and curved portions and vertical walls. If Ω is $|\cdot|$-calibrable ($|\cdot|$ as usual is the euclidean norm) the solution $u(\cdot, t)$ remains a characteristic function (of the same initial set E), and typically it moves in vertical direction with velocity $P(\Omega)/|\Omega|$. If Ω is not $|\cdot|$-calibrable then curved portions in the graph of $u(\cdot, t)$ appear([59]) in correspondence of points of $\partial\Omega$ of large curvature.

[59]The evolution of the curved smooth regions of $\partial E(t)$ and of the vertical walls is not the one given by ϕ-mean curvature flow.

9 The reaction-diffusion approximation

Motion by mean curvature can be approximated by the zero-level sets of solutions of a singularly perturbed parabolic equation of Ginzburg-Landau type [83], [71]. This approximation result can be generalized to anisotropic and crystalline mean curvature flow, and several authors contributed to the final results (for instance [93], [126], [87]), which are sometimes valid even efter the onset of singularities (excluding fattening). A partial list of references can be found for instance in the papers [46], [35]. In this section we briefly recall the main statement in the crystalline case, and one of its consequences, namely the comparison principle, which implies a uniqueness result.

Assume that ϕ^o is crystalline. Let us introduce the relaxed evolution law. Let $\Omega \subset M$ be a smooth bounded open set. For $s \in [-1,1]$ let $W(s) := (1-s^2)^2$ and $\psi := W'/2$. We denote by γ the unique smooth strictly increasing function([60]) exponentially asymptotic, at $\pm\infty$, to the two stable zeroes ± 1 of ψ, satisfying

$$-\gamma'' + \psi(\gamma) = 0, \qquad \gamma(0) = 0. \tag{9.1}$$

Let $\delta \geq 3$ be a fixed natural number such that, if for any $\varepsilon \in (0,1]$ we let $\xi_\varepsilon := \delta|\log \varepsilon|$, then $\gamma(\pm\xi_\varepsilon) = \pm 1 + O(\varepsilon^{2\delta})$, $\gamma'(\pm\xi_\varepsilon) = O(\varepsilon^{2\delta})$. Denote by γ_ε a smooth increasing function which coincides with γ on $[-\xi_\varepsilon, \xi_\varepsilon]$ and assumes the corresponding asymptotic values ± 1 outside the interval $(-2\xi_\varepsilon, 2\xi_\varepsilon)$.

Let Ω be a smooth bounded open st, let $\varepsilon \in (0,1]$, $T > 0$ and let u_0 belong to the Sobolev space $H^1(\Omega)$, and suppose also that

$$\mathcal{E}_\phi(u_0) := \int_\Omega \phi^o(\nabla u_0)^2 + W(u_0)\ dx < +\infty.$$

Let us consider the problem

$$\begin{cases} \varepsilon u_t - \varepsilon \mathrm{div}(T_{\phi^o}(\nabla u)) + \dfrac{1}{\varepsilon}\psi(u) \ni 0 & \text{in } \Omega \times (0,T), \\[2ex] u(\cdot,0) = u_0(\cdot) & \text{in } \Omega, \\[2ex] T_{\phi^o}(\nabla u) \cdot \nu^\Omega = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{9.2}$$

Let us define what is a solution to (9.2). For the definitions of parabolic spaces, we refer for instance to [88]. For an introduction to parabolic partial differential equations we refer for instance to [122] and [54].

[60] An hyperbolic tangent.

Definition 9.1 (Sub/super solutions). *A pair (u, ζ) is a subsolution of (9.2) if, for any $T > 0$, the following properties hold:*

(i) $u \in L^\infty(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ *and* $\zeta \in (L^2(\Omega \times (0, T)))^n$*;*

(ii) for any $\varphi \in H^1(\Omega; [0, +\infty))$ *and a.e.* $t \in (0, T)$

$$\int_\Omega \left(\varepsilon u_t \varphi + \varepsilon \zeta \cdot \nabla \varphi + \frac{1}{\varepsilon} \psi(u) \varphi \right) dx \leq 0; \tag{9.3}$$

(iii) $u(x, 0) \leq u_0(x)$ *for a.e.* $x \in \Omega$*;*

(iv) for a.e. $(x, t) \in \Omega \times (0, T)$

$$\zeta(x, t) \in T_{\phi^o}(\nabla u(x, t)). \tag{9.4}$$

The pair (u, ζ) is a supersolution of (9.2) if (i) and (iv) hold, and conditions (ii) and (iii) hold with $\geq$ in place of $\leq$. The couple (u, ζ) is a solution of (9.2) if it is both a subsolution and a supersolution.

By (i), (iv) and the one-homogeneity of T_{ϕ^o}, it follows that $\zeta \in L^\infty\big(0, T; (L^2(\Omega))^n\big)$.
The following results hold.

Lemma 9.2 (Comparison). *Let (u^-, ζ^-) and (u^+, ζ^+) be respectively a subsolution and a supersolution of (9.2). Then $u^- \leq u^+$ a.e. in $\Omega \times (0, T)$.*

Theorem 9.3 (Existence and uniqueness). *Problem (9.2) admits a solution (u, ζ). Moreover, if (u_1, ζ_1) and (u_2, ζ_2) are two solutions of (9.2), then $u_1 = u_2$ a.e. in $\Omega \times (0, T)$.*

9.1 Approximation and comparison principle

Following [36] and [39] we recall the convergence and comparison results.

Theorem 9.4 (Convergence). *Let $E(t)$ be a neighbourhood-L^∞ $\phi-$regular flow on $[0, T]$. For any $\varepsilon > 0$ let u_ε be the solution of problem (9.2) with the ε-dependent initial datum*

$$u_\varepsilon(x, 0) = u_\varepsilon^0(x) := \gamma_\varepsilon \left(\frac{d_\phi(x, 0)}{\varepsilon} \right), \tag{9.5}$$

where as usual $d_\phi(x, 0) := \mathrm{dist}_\phi(x, E(0)) - \mathrm{dist}_\phi(x, M \setminus E(0))$. Let $\Sigma_\varepsilon(t)$ denote the zero level set of $u_\varepsilon(\cdot, t)$([61]). Then there exist $\varepsilon_0 \in (0, 1]$ and a constant C depending on $(E(t))_{t \in [0,T]}$, and independent of $\varepsilon \in]0, \varepsilon_0]$, such that for any $\varepsilon \in (0, \varepsilon_0]$

$$\begin{aligned} \Sigma_\varepsilon(t) &\subset \{x \in \Omega : \mathrm{dist}(x, \partial E(t)) \leq C\varepsilon |\log \varepsilon|^2\}, \\ \partial E(t) &\subset \{x \in \Omega : \mathrm{dist}(x, \Sigma_\varepsilon(t)) \leq C\varepsilon |\log \varepsilon|^2\}, \end{aligned} \qquad t \in [0, T]. \tag{9.6}$$

Using Lemma 9.2 and Theorem 9.4 it is possible to deduce the following result.

[61] Since $u_\varepsilon(\cdot, t)$ is not a priori a continuous function, this zero level set must be properly defined.

Corollary 9.5 (Uniqueness). *Let $E_1(t)$ and $E_2(t)$ be two neighbourhood-L^∞ ϕ-regular flows on $[0,T]$. Then*

$$E_1(0) \subseteq E_2(0) \Rightarrow E_1(t) \subseteq E_2(t), \qquad t \in [0,T].$$

Hence $E_1(0) = E_2(0) \Rightarrow E_1(t) = E_2(t)$ for any $t \in [0,T]$.

As a consequence, a ϕ-regular flow depends only on $E(0)$, hence it does not depend on the choice of the vector field which makes it neighbourhood-L^∞ ϕ-regular.

Remark 9.6. We are not aware of a direct proof of the comparison principle for crystalline mean curvature flow in $n \geq 3$ dimensions, without using the reaction-diffusion approximation, or without using a heat-type approximation [68], [69]([62]).

Remark 9.7. The comparison result of Corollary 9.5 should allow to implement the barriers method, and produce a global notion of weak solution to crystalline mean curvature flow. Namely, one should define what is for instance a neighbourhood L^∞ ϕ-regular subsolution in the spirit of Definition 7.4, and then should take the family of all these (local in time) subsolutions as the test family for constructing the minimal barrier and its regularized versions. See [45] and [38] for more on the barriers method. Another notion of weak solution in the convex case has been used in [32], and previously in [4] in two dimensions and without convexity assumptions.

[62]The uniqueness result in [68] is obtained among flows in the sense of Definition 7.4, and satisfying a further interior/exterior ball condition.

10 Anisotropic functionals on partitions and crystalline flow of planar triods

Functionals defined on boundaries have a rather natural extension as functionals defined on bounded variation functions taking a finite number of values (sometimes called functionals on partitions)([63]). As in the two-phases case, we will not study such functionals in full generality([64]), and we will confine ourselves to the following particular situation: only one anisotropy ϕ^o will be used, that will be assumed convex and spatially homogeneous. Only special partitions will be considered, consisting of a finite number of Lipschitz phases.

Let $\phi \in \mathcal{M}(TM)$ be spatially homogeneous. By a Lipschitz hypersurface with Lipschitz boundary we mean a $(n-1)$-dimensional set $\Sigma \subset M$ which can be written locally as the graph of a Lipschitz function defined on an open subset of $\mathbb{R}^{n-1}$, and such that each point of its relative boundary $\partial\Sigma$ can be written locally as the graph of a Lipschitz function defined on an open Lipschitz subset of $\mathbb{R}^{n-2}$. If $x \in \Sigma$ (resp. $x \in \partial\Sigma$) we denote by $T_x(\Sigma)$ (resp. $T_x(\partial\Sigma)$) the tangent space to Σ (resp. to $\partial\Sigma$) at x. We also denote by $\Pi_{T_x(\Sigma)}$ (resp. $\Pi_{T_x(\partial\Sigma)}$) the orthogonal projection on $T_x(\Sigma)$ (resp. on $T_x(\partial\Sigma)$). Any Lipschitz function or vector field defined on Σ will be considered as defined up to $\partial\Sigma$.

Given a Lipschitz hypersurface $\Sigma \subset M$ with boundary, we define

$$\mathcal{M}_\phi(\Sigma) := \int_\Sigma \phi^o(\nu)\, d\mathcal{H}^{n-1}, \tag{10.1}$$

where $\nu(x)$ is a euclidean unit normal vector to Σ at $\mathcal{H}^{n-1}$-almost every $x \in \Sigma$.

Definition 10.1 (Lipschitz partitions). *A Lipschitz (resp. smooth) partition of M is a finite family $\{E_i\}_i$ of subsets([65]) of M such that $\cup_i \overline{E}_i = \mathbb{R}^n$, $E_i \cap E_j = \emptyset$ for $i \neq j$, and $\partial E_i \cap \partial E_j$, when it is nonempty, is a Lipschitz (resp. smooth) hypersurface with Lipschitz (resp. smooth) boundary, called interface. If $n = 2$, by a m-multiple junction of $\{E_i\}$ ($m \geq 3$ a natural number) we mean a point q belonging to m distinct interfaces. If in addition $m = 3$ we say that q is a triple junction of $\{E_i\}$.*

Given a Lipschitz partition $\{E_i\}$ of M, we set

$$\Sigma_{ij} := \partial E_i \cap \partial E_j,\ i \neq j, \qquad \Gamma := \bigcup_{i,j} \Sigma_{ij}, \qquad J := \bigcup_{i,j} \partial\Sigma_{ij}, \tag{10.2}$$

and

$$\mathcal{M}_\phi(\Gamma) := \sum_{i,j} \mathcal{M}_\phi(\Sigma_{ij}). \tag{10.3}$$

We denote by ν^{ij} a $\mathcal{H}^{n-1}$-a.e. defined euclidean unit normal to Σ_{ij} and we set ${\nu_{\phi^o}}^{ij} := \nu^{ij}/\phi^o(\nu^{ij})$. For notational simplicity, when $n = 2$ the sets $\partial E_i \cap \partial E_j$ are also denoted by Σ_k, using one index only, and ${\nu_{\phi^o}}^{ij}$ will be denoted by ${\nu_{\phi^o}}^k$. When $n = 2$ the set Γ is sometimes called network.

[63] These latter functionals are in turn generalized by functionals defined on special functions of bounded variation, such as the Mumford-Shah functional [14].

[64] See [14], [63], [64], [70] and references therein.

[65] Called phases.

When $\{E_1, E_2, E_3\}$ is a partition of $\mathbb{R}^2$ into three sets having only one triple junction (denoted by q) the set Γ defined in (10.2) will be called triod, and denoted by Π. If the partition is Lipschitz ϕ-regular in the sense of Definition 10.8 below, the triod is said to be Lipschitz ϕ-regular. We call angles of Π the three angles at q between Σ_1, Σ_2, Σ_3.

10.1 First variation

Assume $\phi \in \mathcal{M}_{\rm reg}(TM)$ is spatially homogeneous. We assume that Σ is a $(n-1)$-dimensional smooth bounded embedded orientable manifold with (smooth) boundary. ν is a smooth euclidean unit normal vector field to Σ, smoothly defined up to $\partial\Sigma$. We define, at each point of $\overline{\Sigma}$, $\nu_{\phi^o} := \nu/\phi^o(\nu)$, $n_\phi := T_{\phi^o}(\nu_{\phi^o})$, and on Σ the ϕ-mean curvature κ_ϕ of Σ as $\kappa_\phi := {\rm div}_\tau n_\phi$.

Definition 10.2 (**ϕ-conormal vector**). *We denote by $n_\phi^{\partial\Sigma} : \partial\Sigma \to M$ the vector field defined as follows: if $x \in \partial\Sigma$ then*

(i) $n_\phi^{\partial\Sigma}(x) \in \Big\{{\rm span}\Big(T_x(\partial\Sigma), n_\phi(x)\Big)\Big\}^\perp$;

(ii) $|n_\phi^{\partial\Sigma}(x)| = |n_\phi(x) - \Pi_{T_x(\partial\Sigma)} n_\phi(x)|$;

(iii) $n_\phi^{\partial\Sigma}(x)$ *points out of* Σ.

Observe that $\dim \Big\{{\rm span}\Big(T_x(\partial\Sigma), n_\phi(x)\Big)^\perp\Big\} = 1$, since $n_\phi(x)$ and $T_x(\partial\Sigma)$ are linearly independent, as a consequence of $\langle \nu_{\phi^o}(x), n_\phi(x)\rangle = 1$.
If $\phi(\xi) = |\xi|$, then $n_\phi^{\partial\Sigma}$ is the usual conormal unit euclidean vector pointing out of Σ. Note also that in $n = 2$ dimensions condition (i) reduces to $n_\phi^{\partial\Sigma}(x) \cdot n_\phi(x) = 0$, and condition (ii) reduces to $|n_\phi^{\partial\Sigma}(x)| = |n_\phi(x)|$.

10.1.1 The smooth 2-dimensional case

In this subsection we assume $n = 2$ (hence $M = \mathbb{R}^2$) and we compute the first variation of $\mathcal{M}_\phi$ using a parametric approach [44], for $\phi \in \mathcal{M}_{\rm reg}(TM)$.

Theorem 10.3 (**Curves with boundary**). *Let $\Sigma \subset \mathbb{R}^2$ be a smooth simple curve with boundary $\partial\Sigma = \{p, q\}$, $p \neq q$. Let $\alpha : [0,1] \to \mathbb{R}^2$ be a regular parametrization of Σ with $\alpha(0) = p$ and $\alpha(1) = q$. Let $\beta \in \mathcal{C}^2([0,1];\mathbb{R}^2)$, $\lambda \in \mathbb{R}$, and let Σ_λ be the curve parametrized by $\alpha + \lambda\beta$. Then*

$$\frac{d}{d\lambda}\mathcal{M}_\phi(\Sigma_\lambda)_{|\lambda=0} = \int_\Sigma \kappa_\phi \nu_{\phi^o} \cdot \beta \ \phi^o(\nu) d\mathcal{H}^1 + n_\phi^{\partial\Sigma}(q) \cdot \beta(1) + n_\phi^{\partial\Sigma}(p) \cdot \beta(0). \tag{10.4}$$

Proof. Set $\tau := \frac{\alpha'}{|\alpha'|}$ and $\nu := \tau^\perp$, where $^\perp$ is the counterclockwise rotation of $\pi/2$. Recalling (10.1) we have

$$\begin{aligned}\frac{d}{d\lambda}\mathcal{M}_\phi(\Sigma_\lambda)_{|\lambda=0} &= \frac{d}{d\lambda}\int_0^1 \phi^\circ\left(\left(\alpha' + \lambda\beta'\right)^\perp\right)\ dt_{|\lambda=0} \\ &= \int_0^1 \phi^o_\xi(\nu)\cdot(\beta^\perp)'\ dt = -\int_0^1 \frac{d}{dt}(\phi^o_\xi(\nu))\cdot\beta^\perp\ dt - \phi^o_\xi(\nu(q))^\perp\cdot\beta(1) + \phi^o_\xi(\nu(p))^\perp\cdot\beta(0).\end{aligned} \tag{10.5}$$

We now observe that $\beta^\perp = -\beta\cdot\nu\tau + \beta\cdot\tau\nu$. Moreover, $\phi^o_\xi(\nu) = n_\phi$ by definition, and from [47] we have $\phi^o_{\xi\xi}(\nu)\tau\cdot\nu = 0$ and $\kappa_\phi = \kappa\phi^o_{\xi\xi}(\nu)\tau\cdot\tau$, where κ is the euclidean curvature. Therefore, using $\frac{d\nu}{dt} = \frac{d\nu}{ds}\frac{ds}{dt} = \kappa\nu|\alpha'|$ where s is the arclength parameter, we have

$$\int_0^1 \frac{d}{dt}(\phi^o_\xi(\nu))\cdot\beta^\perp\, dt = -\int_0^1 \kappa\phi^o_{\xi\xi}(\nu)\tau\cdot\tau\ \nu\cdot\beta|\alpha'|\, dt = -\int_\Sigma \kappa_\phi\nu_{\phi^o}\cdot\beta\ \phi^o(\nu)d\mathcal{H}^1. \tag{10.6}$$

Then (10.4) follows from (10.5) and (10.6). $\square$

Corollary 10.4 (Networks). *Let $\{E_i\}$ be a smooth partition of $\mathbb{R}^2$ and let q be a m-multiple junction of $\{E_i\}$, $m\geq 3$. Let $\Sigma_1,\dots,\Sigma_m$ be the m arcs of the partitions meeting at q. Let $\alpha_i : [0,1]\to\mathbb{R}^2$ be a regular parametrization of Σ_i such that $\alpha_i(1) = q$ for any $i = 1,\dots,m$. Let $\beta_i \in \mathcal{C}^2([0,1];\mathbb{R}^2)$ be such that $\beta_i(0) = 0$ and $\beta_i(1) = \beta_j(1) =: \beta(1)$ for every $i,j\in\{1,\dots,m\}$, let $\lambda\in\mathbb{R}$ and Σ^i_λ be the curve parametrized by $\alpha_i+\lambda\beta_i$ and $\Gamma_\lambda := \bigcup_{i=1}^m \Sigma^i_\lambda$. Then*

$$\frac{d}{d\lambda}\mathcal{M}_\phi(\Gamma_\lambda)_{|\lambda=0} = \int_\Gamma \kappa_\phi\nu_\phi\cdot\beta\ \phi^o(\nu)d\mathcal{H}^1 + \beta(1)\cdot\sum_{i=1}^m n_\phi^{\partial\Sigma_i}(q). \tag{10.7}$$

In particular, if for any β_i as above we have $\frac{d}{d\lambda}\mathcal{M}_\phi(\Gamma_\lambda)_{|\lambda=0} = 0$, then each Σ_i has zero ϕ-mean curvature, and

$$\sum_{i=1}^m n_\phi^{\partial\Sigma_i}(q) = 0. \tag{10.8}$$

10.1.2 The smooth n-dimensional case

In this subsection we assume $n\geq 2$ and we state the first variation of $\mathcal{M}_\phi$ [44]. Let Ψ_λ and X be as in Section 5.1.

Theorem 10.5 (First variation: manifolds with boundary). *Let $\Sigma\subset M$ be a smooth hypersurface with boundary. Set $\Sigma_\lambda := \Psi_\lambda(\Sigma)$. Then*

$$\frac{d}{d\lambda}\mathcal{M}_\phi(\Sigma_\lambda)_{|\lambda=0} = \int_\Sigma \kappa_\phi\nu_{\phi^o}\cdot X\ \phi^o(\nu)d\mathcal{H}^{n-1} + \int_{\partial\Sigma} n_\phi^{\partial\Sigma}\cdot X\ d\mathcal{H}^{n-2}. \tag{10.9}$$

Remark 10.6. If $n = 2$, the right hand side of (10.9) reduces to right hand side of (10.4).

Corollary 10.7 (Partitions). *Let $\{E_i\}$ be a smooth partition of M. Set $\Sigma^{ij}_\lambda := \Psi_\lambda(\Sigma_{ij})$ and $\Gamma_\lambda := \bigcup_{i=1}^m \Sigma^{ij}_\lambda$. Then*

$$\frac{d}{d\lambda}\mathcal{M}_\phi(\Gamma_\lambda)_{|\lambda=0} = \int_\Gamma \kappa_\phi\nu_\phi\cdot X\ \phi^o(\nu)d\mathcal{H}^{n-1} + \int_\Gamma\left(\sum_{i,j} n_\phi^{\partial\Sigma_{ij}}\right)\cdot X\ d\mathcal{H}^{n-2}. \tag{10.10}$$

In particular, if $\frac{d}{d\lambda}\mathcal{M}_\phi(\Gamma_\lambda)_{|\lambda=0} = 0$, then each Σ_{ij} has zero ϕ-mean curvature and the balance condition holds:

$$\sum_{i,j} n_\phi^{\partial\Sigma_{ij}} = 0 \qquad \text{on } \Gamma. \tag{10.11}$$

From now on, up to the end of the notes, we will assume $n = 2$ (so that $M = \mathbb{R}^2$) and ϕ crystalline.

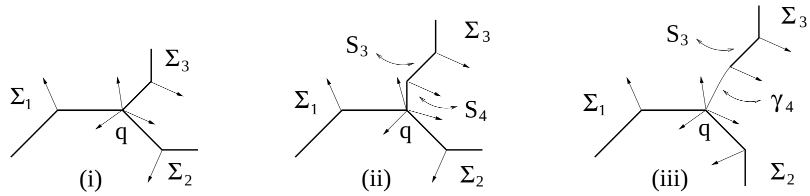

Figure 14: (i) Elementary, (ii) quasi-elementary, (iii) non-polygonal triod ($B_\phi = P_8$). Note that $\kappa_\phi = 0$ on S_3 in (i) and (ii), $\kappa_\phi < 0$ on S_4 in (ii), and $\kappa_\phi = 0$ on γ_4 in (iii).

10.1.3 The crystalline case in $n = 2$ dimensions

We denote by $\mathrm{Lip}_{\nu,\varphi}(\Gamma; \mathbb{R}^2)$ the space of vector fields $N : \Gamma \to \mathbb{R}^2$ such that $N_{|\Sigma_{ij}} \in \mathrm{Lip}(\Sigma_{ij}; \mathbb{R}^2)$ and $N_{|\Sigma_{ij}}(x) \in T_{\phi^o}({\nu_{\phi^o}}^{ij}(x))$ for $\mathcal{H}^1$-almost every $x \in \Sigma_{ij}$. Set

$$\mathcal{N} := \left\{ N \in \mathrm{Lip}_{\nu,\varphi}(\Gamma, \mathbb{R}^2) \, : \, \sum_{i,j} (N_{|\Sigma_{ij}})^{\partial \Sigma_{ij}} = 0 \text{ on } J \right\}. \tag{10.12}$$

See the appendix for more on the balance condition.

Definition 10.8 (**Lipschitz ϕ-regular partitions**). *If $\mathcal{N} \neq \emptyset$, the partition $\{E_i\}$ is said to be Lipschitz ϕ-regular.*

We now want to define the ϕ-curvature of Γ. if $\{E_i\}_i$ is a Lipschitz ϕ-regular partition of M then the minimum problem

$$\min \left\{ \int_\Gamma (\mathrm{div}_\tau N)^2 \, \phi^o(\nu) \, d\mathcal{H}^1 \, : \, N \in \mathcal{N} \right\} \tag{10.13}$$

admits a unique([66]) solution which identifies the direction along which the functional (10.3) decreases most quickly. Let $N_{\min} : \Gamma \to \mathbb{R}^2$ be the solution of problem (10.13).

Definition 10.9 (**Crystalline curvature of a network**). *Let $\{E_i\}$ be a Lipschitz ϕ-regular partition. We define the ϕ-curvature κ_ϕ of Γ as*

$$\kappa_\phi := \mathrm{div}_\tau N_{\min}, \qquad \textit{a.e. on } \Gamma.$$

10.2 Triods

In this section we report some results on triods from [34]. We denote by $\mathfrak{n}$ a positive integer and we assume that $B_\phi = P_\mathfrak{n}$, where $P_\mathfrak{n}$ denotes the regular polygon of $\mathfrak{n}$ ($\mathfrak{n}$ even) sides of length L inscribed in the unit circle centered at the origin of $\mathbb{R}^2$, having two horizontal sides and oriented in clockwise sense.

[66]Remember that we are considering partitions in the plane.

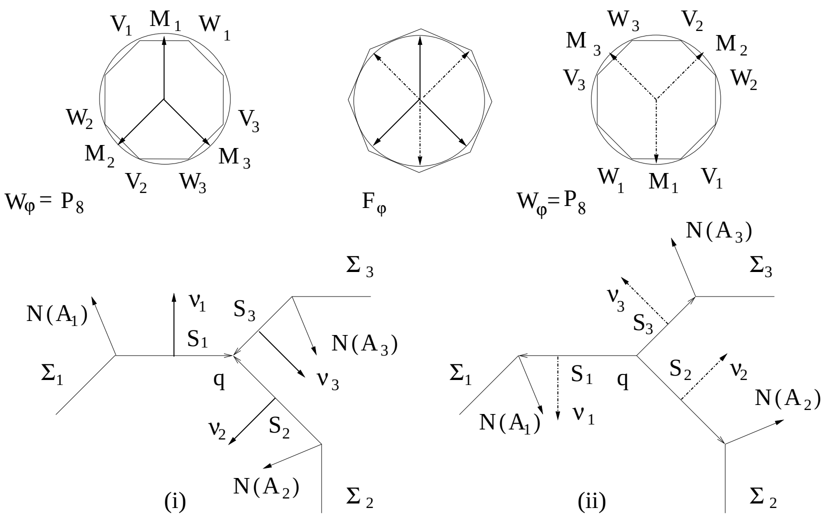

Figure 15: These triods have the same evolution according to system (10.15). Our convention is to take the orientation as in (i).

Definition 10.10 (**Elementary, quasi-elementary and non-polygonal triods).** *Let $\Pi = \cup_{j=1}^3 \Sigma_j$ be a Lipschitz ϕ-regular triod. We say that Π is elementary if*

($\mathcal{E}$) each interface Σ_j is the union of a segment S_j of finite length $L_j > 0$ and a half-line R_j such that S_j and R_j reproduce two consecutive sides of B_ϕ, see Figure 14 (i).

We say that Π is degenerate if two interfaces satisfy ($\mathcal{E}$) and the remaining one is a half-line. We say that Π is quasi-elementary if two interfaces satisfy ($\mathcal{E}$) and the remaining one Σ_k is the union of two segments S_4 and S_k of finite lengths, $L_4 > 0$ and $L_k > 0$ respectively, and a half-line R_k such that S_4 and S_k, and S_k and R_k, reproduce two consecutive sides of B_ϕ, see Figure 14 (ii).
We say that Π is non-polygonal if two interfaces satisfy ($\mathcal{E}$) and the remaining one Σ_k is the union of a curve γ_4, a segment S_k of finite length $L_k > 0$ and a half-line R_k such that S_k and R_k reproduce two consecutive sides of B_ϕ, see Figure 14 (iii).

Given a triod Π and $N \in \mathcal{N}$, we set $A_j := \overline{S_j} \cap \overline{R_j}$ for any $j = 1, 2, 3$ such that $R_j \neq \emptyset$, $A_4 := \overline{S_4} \cap \overline{S_k}$ if Π is quasi-elementary, and $A_4 := \overline{\gamma_4} \cap \overline{S_k}$ if Π is non-polygonal.
Conventions: let ν be the $\mathcal{H}^1$-almost everywhere defined euclidean unit normal to Π oriented in such a way that $\nu_{|\mathrm{int}\,(S_j)} \cdot N(A_j) > 0$. We set $\nu_j := \nu_{|\mathrm{int}\,(S_j)}$, $\tau_j := -\nu_j^\perp$ and $l_j := L_j \tau_j$, for any $j = 1, 2, 3$, and also $j = 4$ if Π is quasi-elementary. Thus $\{\tau_j, \nu_j\}$ is a positively oriented basis of $\mathbb{R}^2$ and, without loss of generality, we assume that each l_j points towards q. We denote by $\kappa_\phi(l_j)$ the ϕ-curvature of S_j.
For an elementary triod, we assume that S_1 is horizontal and Σ_2 and Σ_3 are given in counterclockwise sense as in Figure 15. We denote by V_j, W_j the vertices of the side of $P_\mathfrak{n}$ (in clockwise sense) having ν_j as outer normal and by M_j the middle point of the segment $[V_j, W_j]$. Note that

$$\tau_1 \cdot \nu_3 = -\tau_1 \cdot \nu_2, \quad \nu_1 \cdot \tau_3 = -\nu_1 \cdot \tau_2, \quad \tau_1 \cdot \nu_3 = -\nu_1 \cdot \tau_3. \tag{10.14}$$

We recall the notion of stability [44].

Definition 10.11 (**Stable triods).** *Let Π be a ϕ-regular triod. We say that Π is stable if $(N_{\min})_{|\Sigma_j}(q)$ is not a vertex of B_ϕ for any $j = 1, 2, 3$. We say that Π is unstable if it is not stable.*

Non-polygonal triods are always unstable, while elementary, degenerate and quasi-elementary triods can be either stable or unstable.

10.3 Crystalline flows of triods

As usual, given two parallel (possibly infinite) segments S_1, S_2, we call the distance vector of S_2 from S_1 the vector having norm $\mathrm{dist}(S_1, S_2)$ pointing from S_1 to S_2.

Definition 10.12. *Let $T > 0$ and Π be an elementary triod (resp. degenerate). For any $t \in [0, T]$, let $\Pi(t)$ be a Lipschitz ϕ-regular triod and $q(t)$ its triple junction. We say that $t \in [0, T] \mapsto \Pi(t)$ is a ϕ-curvature flow starting from $\Pi = \Pi(0)$ if for any $t \in (0, T)$*

(i) $\Pi(t)$ *is either elementary or quasi-elementary or non-polygonal (resp. degenerate);*

(ii) for any $j = 1, 2, 3$, each $R_j(t)$ has zero normal velocity and each $S_j(t)$ is parallel to $S_j(0) = S_j$;

(iii) for each $j = 1, 2, 3$, and also $j = 4$ if $\Pi(t)$ is quasi-elementary, denoting by $h_j(t)$ the distance vector of the segment $S_j(t)$ from $S_j(0) = S_j$, then $h_j \in C^1([0, T]; \nu_j \mathbb{R})$ and

$$\begin{cases} \dfrac{\dot{h}_j(t)}{\phi^o(\nu_j)} = -\kappa_\phi(l_j(t))\ \nu_j = -\dfrac{1}{L_j(t)}\Big[\ N_{\min}\big|_{\Sigma_j(t)}(\,q(t)\,) - N_{\min}(\,A_j(t)\,)\ \Big] \cdot\ \tau_j\ \nu_j. \\ h_j(0) = 0. \end{cases} \tag{10.15}$$

Remark 10.13. $S_j(t)$ moves in the same direction of ν_j if and only if $\kappa_\phi(l_j(t)) < 0$. Furthermore, system (10.15) is invariant under the change of the orientation of $\Pi(t)$ (see Figure 15).

Finally, it is possible to prove the following short time existence and uniqueness theorem for the ϕ-curvature flow of a triod.

Theorem 10.14. *Let Π be elementary and stable. Then there exist $T > 0$ and a unique stable ϕ-curvature flow $t \in [0, T) \mapsto \Pi(t)$ starting from Π for any $t \in [0, T]$.*

10.4 Appendix

The angles of an elementary triod are given by the angles between the vectors ν_j's and are determined by the balance condition at q (see (10.12)) that, in turn, is related to the existence of admissible triplets.

Definition 10.15 (**Admissible triplets).** *We call admissible triplet any triplet of vectors $(X, Y, Z) \in (\partial B_\phi)^3$ satisfying*

$$X + Y + Z = 0. \tag{10.16}$$

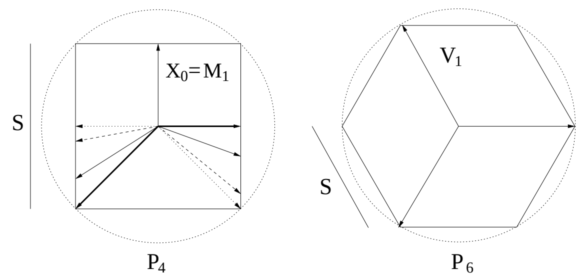

Figure 16: P_4 admits infinitely many unordered pairs $\{Y, Z\}$ satisfying $X_0 + Y + Z = 0$ in correspondence of $X_0 = M_1$. P_6 has a unique pair in correspondence of all $X \in \partial P_6$.

It is possible to prove the following result.

Lemma 10.16 (Geometry of admissible triplets). *Let $\psi : \mathbb{R}^2 \to [0, +\infty)$ be a convex norm on $\mathbb{R}^2$. Let $X \in \partial B_\psi$. Then there exist two distinct vectors Y, Z in ∂B_ψ such that (X, Y, Z) is an admissible triplet. Moreover, if either B_ψ is strictly convex or for any segment $S \subset \partial B_\psi$ parallel to $X \in \partial B_\psi$ we have $|S| \leq |X|$, then the unordered pair $\{ Y, Z \}$ is unique. Finally, if there exist $X_0 \in \partial B_\psi$ and a segment $S \subset \partial B_\psi$ parallel to X_0 with $|S| > |X_0|$, then there are infinitely many unordered pairs $\{ Y, Z \}$ of distinct vectors in ∂B_ψ such that (X_0, Y, Z) is an admissible triplet.*

Example 10.17. If $B_\psi = P_4$ and $X_0 = M_1$ (see Figure 16), then $|S| = 2|X_0|$; hence there are infinitely many pairs $\{Y, Z\}$ of distinct vectors in ∂P_4 satisfying $X_0 + Y + Z = 0$. Moreover, any elementary triod has always two angles of $\pi/2$. If $B_\psi = P_6$ and $X = V_1$ (see Figure 16), then $|S| = |V_1|$; hence for any $X \in B_\psi$ there exists a unique unordered pair $\{Y, Z\}$ satisfying (10.16).

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