BV-entropy solutions to nonlinear strongly degenerate parabolic equations

Hiroshi Watanabe

Department of Mathematics, Chuo University, Tokyo

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Initial boundary value problem.

The initial boundary value problem for a strongly degenerate parabolic equation of the form

$$(\mathrm{P})\begin{cases} u_t + \nabla\cdot A(x,t,u) + B(x,t,u) = \Delta\beta(u), \quad (x,t)\in\Omega\times(0,T), \\ \dfrac{\partial\beta(u)}{\partial\mathbf{n}}(x,t) = 0, \quad (x,t)\in\partial\Omega\times(0,T), \\ u(x,0) = u_0(x), \quad u_0\in L^\infty(\Omega)\cap BV(\Omega). \end{cases}$$

$\Omega\subset\mathbb{R}^N$: bounded Lipschitz domain.

$A(x,t,\xi) = (A^1,\dots,A^N)(x,t,\xi)$: $\mathbb{R}^N$-valued differentiable function defined on $\overline{\Omega}\times[0,T]\times\mathbb{R}$.

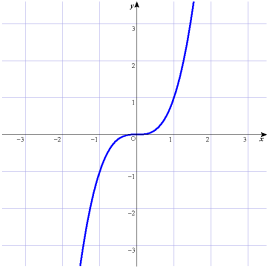
$B(x,t,\xi)$: $\mathbb{R}$-valued differentiable function on $\overline{\Omega}\times[0,T]\times\mathbb{R}$.

$\beta(\xi)$: monotone nondecreasing and locally Lipschitz continuous on $\mathbb{R}$.

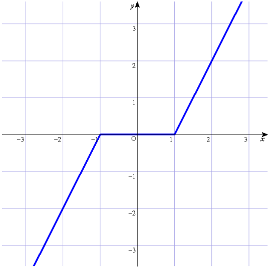
Back ground.

This type of equation can be applied to the sedimentation-consolidation processes of particulate suspensions , filtration problems, Stefan problems and many others,

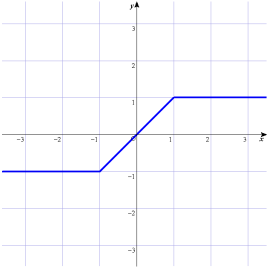
Porous medium type



Stefan type



Sedimentation-consolidation



Due to assumptions on β, this equation has the following properties :

- β is strictly increasing $\Rightarrow$ "parabolicity $>$ hyperbolicity".
- β is monotone nondecreasing $\Rightarrow$ "parabolicity $\lesssim$ hyperbolicity".

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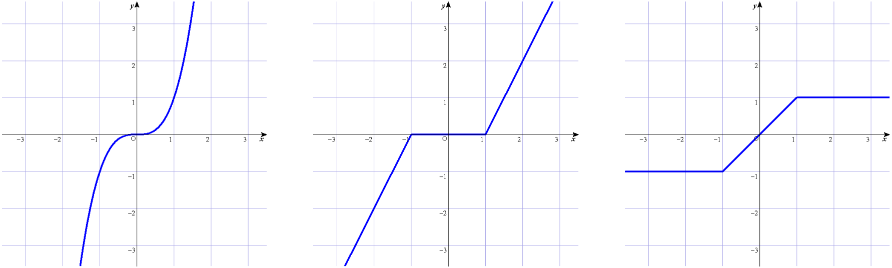
Porous medium type Stefan type Sedimentation-consolidation



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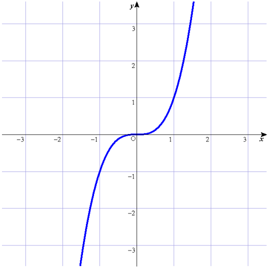

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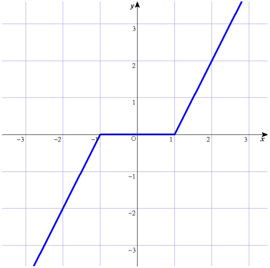

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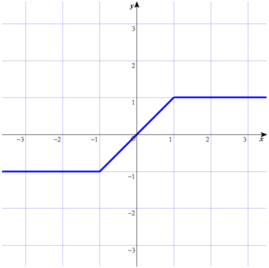

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Keywords of my research

Methods of my reseach

1. Degenerate parabolic equation → entropy solutions
2. The space BV → bounday trace theorem, compactness theorem
3. Nonlinear semigroup theory → difference approximation

Previous works

1. J. Carrillo [Arch. Rational Mech. Anal., 147 (1999)]
   Strongly degenerate, the boundary condition $\beta(u) = 0$
   unique existence of entropy solutions
2. C. Mascia, A. Porretta and A. Terracina [Arch. Rational Mech. Anal., 163 (2002)]
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Our work (degenerate parabolic equations).

H. Watanabe (Adv. Math. Sci. Appl. 19 (2009))

has proved the unique existence of BV solutions of (P) below under the assumption that β is strictly increasing.

Definition

Let $u_0 \in L^\infty(\Omega) \cap BV(\Omega)$. A function $u \in L^\infty(\Omega \times (0,T)) \cap BV(\Omega \times (0,T))$ is called a *BV solution* of the problem (P), if it satisfies the two conditions below:

(1) u belongs to $C([0,T]; L^1(\Omega))$ and $L^1\text{-}\lim_{t\downarrow 0} u(\cdot,t) = u_0$;

(2) $\nabla\beta(u) \in L^2(0,T; L^2(\Omega)^N)$ and for $\varphi \in C_0^\infty(\mathbb{R}^N \times (0,T))$,

$$\int_0^T \int_\Omega (u\varphi_t + A(x,t,u)\cdot\nabla\varphi - B(x,t,u)\varphi - \nabla\beta(u)\cdot\nabla\varphi)dxdt$$

$$- \int_0^T \int_{\partial\Omega} A(x,t,T_r u)\cdot \mathbf{n}(x)\ \varphi\ d\mathcal{H}^{N-1}dt = 0,$$

where $T_r : BV(\Omega) \to L^1(\partial\Omega; \mathcal{H}^{N-1})$ is the trace operator.

BV-entropy solutions.

Definition

Let $u_0 \in L^\infty(\Omega) \cap BV(\Omega)$. A function $u \in L^\infty(\Omega \times (0,T)) \cap BV(\Omega \times (0,T))$ is called a *BV-entropy solution* of the problem (P), if it satisfies:

(1) u belongs to $C([0,T]; L^1(\Omega))$ and $L^1\text{-}\lim_{t\downarrow 0} u(\cdot,t) = u_0$;

(2) $\nabla\beta(u) \in L^2(0,T; L^2(\Omega)^N)$ and for $\varphi \in C_0^\infty(\mathbb{R}^N \times (0,T))^+$ and $k \in \mathbb{R}$,

$$\begin{aligned}\int_0^T \int_\Omega |u-k|\varphi_t dxdt \geq \int_0^T \int_\Omega &\mathrm{sgn}(u-k)(\nabla\beta(u)\cdot\nabla\varphi \\ &-[A(x,t,u)-A(x,t,k)]\cdot\nabla\varphi + [B(x,t,u)+\nabla\cdot A(x,t,k)]\varphi)dxdt \\ &+\int_0^T \int_{\partial\Omega} \mathrm{sgn}(T_r u-k)[A(x,t,T_r u)-A(x,t,k)]\cdot \mathbf{n}(x)\varphi d\mathcal{H}^{N-1}dt.\end{aligned}$$

The relationship between the concept of BV-entropy solution and that of BV solution is important.

In the case of $\Omega = \mathbb{R}^N$, a BV-entropy solution is a BV solution.

Inflow-Outflow condition.

We consider the problem in the case that β is monotone nondecreasing. We then concentrate on the framework such that a BV-entropy solution is a BV solution. To investigate this, we assume the following condition :

Inflow-Outflow condition

Let $v \in L^\infty(\Omega) \cap BV(\Omega)$. For any $k \in \mathbb{R}$,

$$\mathrm{sgn}(T_r v - k)[A(x,t,T_r v) - A(x,t,k)] \cdot \mathbf{n}(x) \geq 0. \tag{1}$$

holds for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$ and $\mathcal{L}^1$-a.e. $t \in [0,T]$.

Under this Inflow-Outflow condition, BV-entropy solutions are BV solutions.

In C. Bardos, A. Y. Leroux and J. C. Nedelec (1979), a similar condition for the Dirichlet boundary condition are introduced in the first order quasilinear equations.

Neumann boundary condition.

We employ a natural way for approximating outward normal on the boundary. Let a sequence $\{\zeta_\delta\}$ of $C^2(\Omega) \cap C^0(\overline{\Omega})$ functions be such that

$$0 \leq \zeta_\delta \leq 1 \text{ in } \overline{\Omega}, \quad \zeta_\delta = 0 \text{ on } \partial\Omega, \quad \lim_{\delta \to 0} \zeta_\delta = 1 \text{ in } \Omega.$$

Then we see that the $-\nabla\zeta_\delta$ converges to the outward normal $\mathbf{n}$ of the boundary as $\delta \downarrow 0$ in the following sense :

$$\lim_{\delta \downarrow 0} \int_\Omega \varphi \cdot \nabla\zeta_\delta dx = -\lim_{\delta \downarrow 0} \int_\Omega \mathrm{div}(\varphi)\zeta_\delta dx = -\int_\Omega \mathrm{div}(\varphi) dx = -\int_{\partial\Omega} \varphi \cdot \mathbf{n} \, d\mathcal{H}^{N-1},$$

for $\varphi \in [C^\infty(\Omega)]^N$. In C. Mascia, A. Porretta and A. Terracina (2002), such a sequence $\{\zeta_\delta\}$ is called a boundary-layer sequence.
Here, we substitute $\varphi = (1 - \zeta_\delta)\xi$, for $\xi \in C_0^\infty(\mathbb{R}^N)^+$ in the definition of BV-entropy solutions. To assert that for $\xi \in C_0^\infty(\mathbb{R}^N)^+$,

$$\lim_{\delta \downarrow 0} \int_\Omega \nabla\beta(u) \cdot \nabla\zeta_\delta \xi dx = 0. \qquad (2)$$

Hence, we may interpret the Neumann boundary condition in the sense of (2).

Abstract Cauchy Problems.

The problem (P) is converted to a time-dependent abstract Cauchy problems in $L^1(\Omega)$. To see this, we define the following differential operator $\mathcal{A}(t)$.

Definition

Let $t \in [0,T]$. We say that $v \in \mathscr{D}(\mathcal{A}(t))$ and $w = \mathcal{A}(t)v$ iff $v, w \in L^\infty(\Omega) \cap BV(\Omega)$ and

$$\langle \operatorname{sgn}(v-k)w, \varphi \rangle \le -\langle \operatorname{sgn}(v-k)\nabla\beta(v), \nabla\varphi \rangle$$

$$+\langle \operatorname{sgn}(v-k)[A(\cdot,t,v) - A(\cdot,t,k)], \nabla\varphi \rangle - \langle \operatorname{sgn}(v-k)[B(\cdot,t,v) + \nabla \cdot A(\cdot,t,k)], \varphi \rangle$$

$$-\langle \operatorname{sgn}(T_r v - k)[A(\cdot,t,T_r v) - A(\cdot,t,k)] \cdot \mathbf{n}, \varphi \rangle_{\partial\Omega}$$

for $\varphi \in C_0^\infty(\mathbb{R}^N)^+$, $k \in \mathbb{R}$.

$\langle \cdot, \cdot \rangle$: the duality pairing between $L^p(\Omega)$ and $L^q(\Omega)$ with $1/p + 1/q = 1$.

$$\text{(ACP)} \quad \begin{cases} (d/dt)u(t) = \mathcal{A}(t)u(t) & \text{for } t \in (0,T), \\ u(0) = v \in L^\infty(\Omega) \cap BV(\Omega), \end{cases}$$

where $(d/dt)u(t)$ is understood to be the derivative of $u(\cdot)$ in a generalized sense.

Basic hypotheses.

(H.1) Let $a^i = (\partial/\partial\xi)A^i$ for $i = 1, \ldots, N$, and $b = (\partial/\partial\xi)B$. For each $r > 0$, the functions A^i, $\partial_i A^i$, a^i, $\partial_{ij} A^i$, $\partial_i a^i$, B, $\partial_i B$, $i, j = 1, \ldots, N$ are all bounded, continuous on $Q_r = \overline{\Omega} \times [0, T] \times [-r, r]$.

(H.2) There exist constants $\alpha, \alpha^{'}$ such that

$$-\nabla \cdot a(x,t,\xi) - b(x,t,\xi) \le \alpha \ \text{ and } \quad - b(x,t,\xi) \le \alpha'$$

for all $(x,t,\xi) \in \overline{\Omega} \times [0,T] \times \mathbb{R}$.

(H.3) For each $r > 0$ and $\tau > 0$,

$$\rho(\tau; r) = \max \left\{ \begin{array}{l} \sup |\partial_i A^i(x,t,\xi) - \partial_i A^i(x,s,\xi)|, \\ \sup |a^i(x,t,\xi) - a^i(x,s,\xi)|, \\ \sup |B(x,t,\xi) - B(x,s,\xi)| \end{array} \right\}$$

over $x \in \overline{\Omega}$, $s, t \in [0,T]$, $\xi \in \mathbb{R}$ with $|s - t| \le \tau$, $|\xi| \le r$, and $i = 1, \ldots, N$. Then

$$\rho(\tau; r) \to 0 \ \text{ as } \tau \downarrow 0 \text{ for each } r > 0.$$

Unique-existence of BV-entropy solutions.

Theorem (Generation of nonlinear evolution operators [1])

The one-parameter family $\{\mathcal{A}(t); t \in (0,T)\}$ is defined above. Then, there exists an evolution operator $\{\mathcal{U}(t,s); 0 \le s \le t \le T\}$ on $L^\infty(\Omega)$ such that for each $v \in L^\infty(\Omega) \cap BV(\Omega)$

$$\mathcal{U}(t,s)v = L^1(\Omega)\text{ - }\lim_{\lambda \downarrow 0} \prod_{i=1}^{[(t-s)/\lambda]} (I - \lambda \mathcal{A}(s+i\lambda))^{-1} v. \tag{3}$$

In addition, for $v \in \widehat{D} \cap BV(\Omega)$, the function $u(x,t) = \mathcal{U}(t,0)v(x)$ gives a BV-entropy solution of (P). Here

$$\widehat{D} \equiv \{v \in L^\infty(\Omega)\ ;\ \liminf_{\lambda \downarrow 0} \lambda^{-1} ||(I - \lambda \mathcal{A}(t))^{-1} v - v||_1 < \infty\}.$$

Theorem (Uniqueness of BV-entropy solutions [1])

Let u, v be BV-entropy solutions of (P) with initial data u_0, v_0, respectively. Then,

$$||u - v||_{L^1(\Omega)} \le e^{\alpha' t} ||u_0 - v_0||_{L^1(\Omega)}.$$

Here, α' is a positive constant such that $-b(x,t,\xi) \le \alpha'$.

[1] H. Watanabe, S. Oharu, "BV-entropy solutions to strongly degenerate parabolic equations". Adv. Differential Equations 15 (2010), no. 7-8, 757–800.

Main results.

1. Generation of nonlinear evolution operators which give BV-entropy solutions,
2. Trotter type approximation theorem and Chernoff type convergence theorem,
3. Time global existence of BV-entropy solutions,
4. Uniqueness of BV-entropy solutions,
5. Continuous dependence of BV-entropy solutions,
6. Comparison principle for BV-entropy solutions,
7. Applications (to degenerate parabolic systems with nonlocal couplings).

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