# Phase Transition Models Based on Linear-growth Interfacial Energies

Shirakawa, Ken (Kobe Univ., Japan)

“Singular Diffusion and Evolving Interfaces”, Hokkaido Univ., August 5 (2010)

Tutorial lectures and international workshop in:
“A Minisemester on Evolution of Interfaces”, Sapporo July 12–August 13 (2010)

# 1. Phase transition models of Fix-Caginalp type

**Keypoint.** Coupled system of heat exchange and phase field dynamics

$\Omega \subset \mathbb{R}^N$ $(1 \leq N \leq 3)$: b.d.d. domain, $\Gamma := \partial\Omega$: smooth

**Basic formulation [Fix](1983)-[Caginalp](1986)**

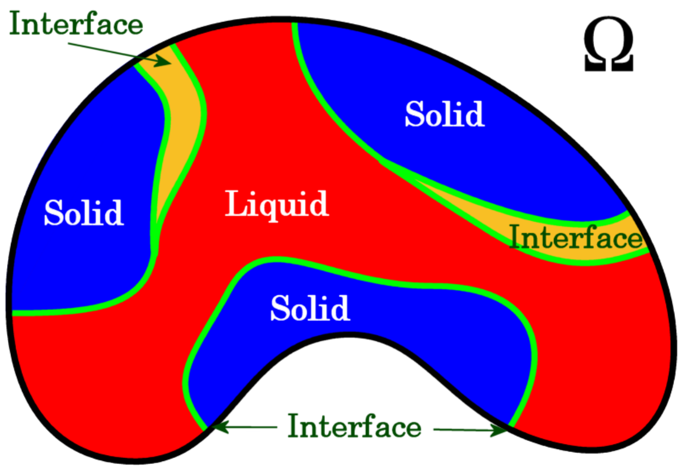
$$\begin{cases} \left(c_0\theta + \frac{L}{2}w\right)_t - \kappa_0 \Delta\theta = h(t,x), \ (t,x) \in Q := (0,+\infty) \times \Omega, \\ -w_t = \nabla_w \mathscr{F}_\theta(w) \quad \text{in } Q, \\ \text{(B.C.) + (I.C.)} \end{cases}$$

- $\theta = \theta(t,x)$: relative temperature,

  $\theta = 0$: critical temperature;

- $w = w(t,x)$: order parameter,

$$\begin{cases} w = 1, & \text{on liquid phase,} \\ w = -1, & \text{on solid phase,} \\ -1 < w < 1, & \text{on interface.} \end{cases}$$



# 1. Phase transition models of Fix-Caginalp type

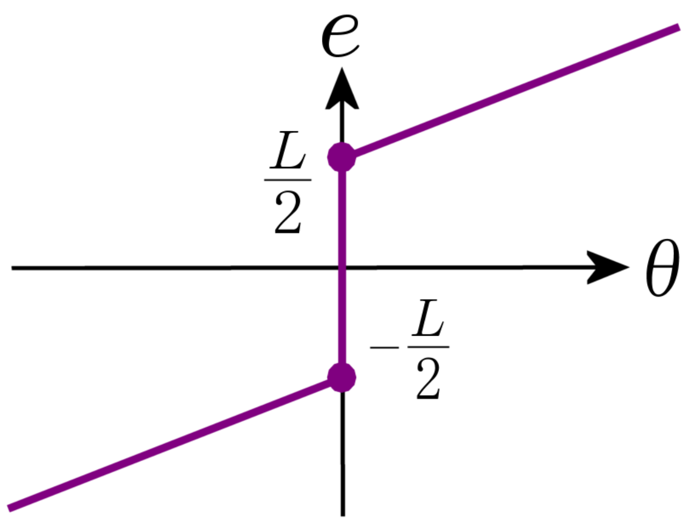
**Keypoint.** Coupled system of heat exchange and phase field dynamics

$\Omega \subset \mathbb{R}^N$ $(1 \leq N \leq 3)$: b.d.d. domain, $\Gamma := \partial\Omega$: smooth

**Basic formulation [Fix](1983)-[Caginalp](1986)**

$$\begin{cases} \left(c_0\theta + \frac{L}{2}w\right)_t - \kappa_0\Delta\theta = h(t,x), \ (t,x) \in Q := (0,+\infty) \times \Omega, \\ -w_t = \nabla_w \mathscr{F}_\theta(w) \quad \text{in } Q, \\ \text{(B.C.) + (I.C.)} \end{cases}$$

- $e = c_0\theta + \frac{L}{2}w$: enthalpy density, a kind of heat quantity;
- $L > 0$: latent heat,
- $c_0 > 0$: specific heat,
- $\kappa_0 > 0$: heat conduction,
- $h = h(t,x)$: heat source.



## 1. Phase transition models of Fix-Caginalp type

**Keypoint.** Coupled system of heat exchange and phase field dynamics

$\Omega \subset \mathbb{R}^N$ $(1 \leq N \leq 3)$: b.d.d. domain, $\Gamma := \partial\Omega$: smooth

**Basic formulation [Fix](1983)-[Caginalp](1986)**

$$\begin{cases} \left( c_0\theta + \frac{L}{2} w \right)_t - \kappa_0 \Delta\theta = h(t,x), \ (t,x) \in Q := (0,+\infty) \times \Omega, \\ -w_t = \nabla_w \mathscr{F}_\theta(w) \quad \text{in } Q, \\ \text{(B.C.) + (I.C.)} \end{cases}$$

- $\mathscr{F}_\theta(w)$: free-energy, where $\nabla_w$: derivative w.r.t. $w$

**Keypoint:** $\kappa > 0$: given constant

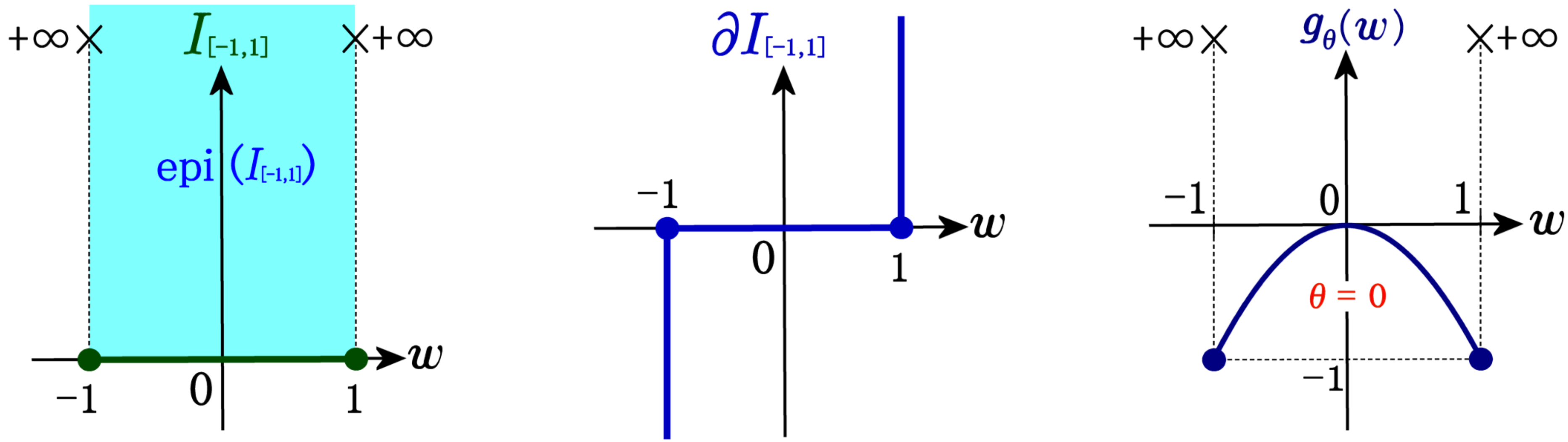
$$w \in L^2(\Omega) \mapsto \mathscr{F}_\theta(w) := \kappa \, \mathscr{I}(Dw) \quad + \int_\Omega g_\theta(w)\, dx$$

Interfacial energy

Double-well potential

## ◇ **Possible choice of free-energy** [Visintin](1996)

- $\mathscr{I}(Dw) := \int_\Omega |Dw| := \sup\left\{ \int_\Omega w \,\mathrm{div}\, \varphi \, dx \;\middle|\; \begin{array}{l} \varphi \in C^1_c(\Omega)^N, \\ |\varphi| \le 1 \text{ on } \Omega \end{array} \right\}$ (total variation)
- $g_\theta(w) := I_{[-1,1]}(w) - \frac{1}{2}w^2 - \theta w$ ($I_{[-1,1]}$: indicator function on $[-1,1]$)




$$\nabla_w \mathscr{F}_\theta(w) = -\kappa \,\mathrm{div}\left(\frac{Dw}{|Dw|}\right) + \partial I_{[-1,1]}(w) - w - \theta$$

Singular diffusion | Set-valued map by subdifferential

subject to homogeneous Neumann type B.C.

†. In this talk, we adopt anisotropic version of the above (anisotropic total variation flow)

## 1.2. Statement of the mathematical model

$\Omega \subset \mathbb{R}^2$: 2D-bounded domain, $W \subset \mathbb{R}^2$: origin symmetric, compact, convex, $\mathrm{int}\,(W) \ni 0$

**System $(S)_W$:**

$$\begin{cases} (\theta + w)_t - \Delta(\theta + \mu\theta_t) = 0 \ \text{ in } Q, \\ \theta = 0 \ \text{ on } \Sigma := (0, +\infty) \times \Gamma, \quad \theta(0, x) = \theta_0(x), \quad x \in \Omega; \end{cases} \qquad \text{(1st.eq.)}$$

$$\begin{cases} w_t - \kappa\,\mathrm{div}\,(\partial f_W^\circ(Dw)) + \partial I_{[-1,1]}(w) \ni w + \theta \text{ in } Q, \\ \boxed{\text{anisotropic total variation flow}} \\ \text{(homogeneous Neumann type B.C.)}, \quad w(0, x) = w_0(x), \quad x \in \Omega. \end{cases} \qquad \text{(2nd.eq.)}$$

- $c_0 = \kappa_0 = 1$, $L = 2$, $h \equiv 0$;
- $\mu\theta_t$ : time-relaxation ($\mu > 0$ : small const.)
- $W$ : structural unit of crystal (Wulff shape).
- $f_W$ : 2D-norm (Finsler norm), s.t. $W = \{ \ \xi \in \mathbb{R}^2 \mid f_W(\xi) \leq 1 \ \}$,
  $f_W^\circ$ : dual norm of $f_W$.
- $\partial f_W^\circ$ : subdifferential of $f_W^\circ$ $\left( W = \mathbb{D}^2 := \overline{\mathrm{conv}}(\mathbb{S}^1) \implies \partial f_W^\circ(Dw) \approx \dfrac{Dw}{|Dw|} \right)$

†. Hereafter, let us denote by $\mathscr{F}_\theta(w)_W$ the free-energy, in each case of Wulff shape $W$.

## 1.3. Mathematical treatement of total variation flow $W$: Wulff shape

- $\mathscr{I}(Dw) := \int_\Omega f_W^\circ(Dw) := \sup \left\{ \int_\Omega w \operatorname{div} \varphi \, dx \,\middle|\, \begin{array}{l} \varphi \in C_c^1(\Omega)^N, \\ f_W(\varphi) \le 1 \text{ on } \Omega \end{array} \right\}$
- $V_W(w) := \mathscr{I}(Dw) + \int_\Omega I_{[-1,1]}(w)\, dx, \quad \forall w \in L^2(\Omega).$

**Definition by evolution equation:** $\forall W \subset \mathbb{R}^2$: origin-symmetric, compact, convex, $\forall \theta \in L^2_{\mathrm{loc}}([0,+\infty); L^2(\Omega))$, $w = w(t,x)$ is called a solution of (2nd.eq.), iff:

$w \in W^{1,2}_{\mathrm{loc}}([0,+\infty); L^2(\Omega)) \cap L^\infty_{\mathrm{loc}}([0,+\infty); BV(\Omega))$, s.t.

$$w_t(t) + \kappa\, \partial V_W(w(t)) \ni w(t) + \theta(t) \text{ in } L^2(\Omega), \text{ a.e. } t > 0,$$

equivalently,

$$\big(w(t) + \theta(t) - w_t(t), z - w(t)\big)_{L^2(\Omega)} \le \kappa V_W(z) - \kappa V_W(w(t)),$$
$$\forall z \in L^2(\Omega), \quad \forall t > 0;$$

subject to

$$w(0) = w_0 \in D(V_W) = D_V := \{ \ z \in BV(\Omega) \mid |z| \le 1, \text{ a.e. in } \Omega \ \},$$

where $\partial V_W$ is the subdifferential of $V_W$ in $L^2(\Omega)$.

## 1.3. Mathematical treatement of total variation flow

- $\mathscr{I}(Dw) := \int_\Omega f_W^\circ(Dw) := \sup\left\{ \int_\Omega w \operatorname{div}\varphi\, dx \;\middle|\; \begin{array}{l} \varphi \in C_c^1(\Omega)^N, \\ f_W(\varphi) \le 1 \text{ on } \Omega \end{array} \right\}$
- $V_W(w) := \mathscr{I}(Dw) + \int_\Omega I_{[-1,1]}(w)\, dx, \quad \forall w \in L^2(\Omega).$

**Well-posedness:** [Brézis](1973), [Mazón](2001), [Kenmochi-Mizuta-Nagai](1980)

Total variation flow (2nd.eq.) admits a unique solution, and in particular:

$$|(w_1 - w_2)^+(t)|_{L^2(\Omega)}^2 \le e^{3t}\left( |(w_{0,1} - w_{0,2})^+|_{L^2(\Omega)}^2 + |(\theta_1 - \theta_2)^+|_{L^2(0,t;L^2(\Omega))}^2 \right),$$

for all $t > 0$, all two solutions $w_i$ with initial values $w_{0,i} \in D_V$ and source terms $\theta_i \in L^2_{\mathrm{loc}}([0,+\infty); L^2(\Omega))$, $i = 1, 2$.

**Keypoint:** [Kenmochi-Mizuta-Nagai](1980) $T$-monotonicity

$$(z_1^* - z_2^*, (z_1 - z_2)^+)_{L^2(\Omega)} \ge 0, \quad \forall [z_i, z_i^*] \in \partial V_W, \quad i = 1, 2.$$

**The purpose in this talk:**

to see some definite association between geometric characterisitics of Wulff shapes and interfaces represented by system $(S)_W$

## 2. Geometric characteristics of Wulff shapes

(Case 1) (Isotropic setting) $W := \mathbb{D}^2 := \overline{\mathrm{conv}}(\mathbb{S}^1)$.

(Case 2) (Crystalline type setting) $W \in \mathscr{P}$, where:

$$\mathscr{P} := \{ P \subset \mathbb{R}^2 \mid P\text{: origin symmetric compact polygon, circumscribed to } \mathbb{S}^1 \}.$$

**Lemma 2.1. (Limiting situation)** $\{P_n\} \subset \mathscr{P}$,

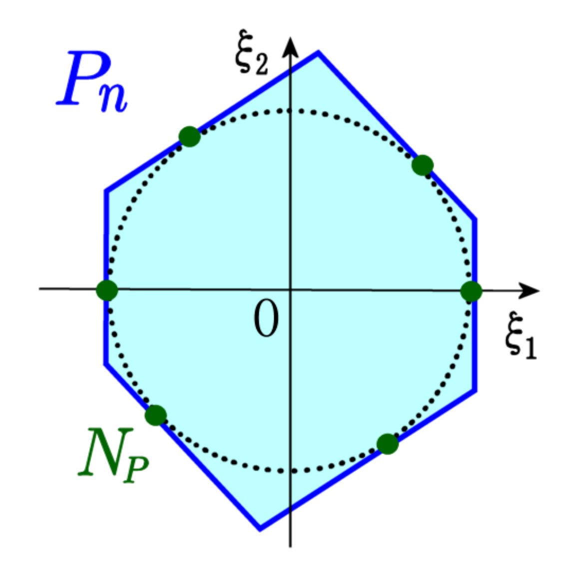
$$(\ell)_* \quad d_*(\partial P_n, \mathbb{S}^1) := \max\{ \sup_{v \in \partial P_n} \mathrm{dist}(v, \mathbb{S}^1), \sup_{v \in \mathbb{S}^1} \mathrm{dist}(v, \partial P_n)\} \to 0 \text{ as } n \to +\infty$$

$\Longrightarrow$ $V_{P_n} \to V_{\mathbb{D}^2}$, on $L^2(\Omega)$, as $n \to +\infty$,
in the sense of [Mosco] (1969):

(M1) $\varliminf_{n\to\infty} V_{P_n}(z_n) \geq V_{\mathbb{D}^2}(z)$, if
$z_n \to z$ weakly in $L^2(\Omega)$ as $n \to +\infty$.

(M2) $\forall z \in D_V$, $\exists \{z_n\} \subset D_V$, s.t.

$$\begin{cases} z_n \to z \text{ in } L^2(\Omega), \\ V_{P_n}(z_n) \to V_{\mathbb{D}^2}(z), \end{cases} \quad \text{as } n \to +\infty.$$




†. This implies the continuous dependence for (2nd.eq.) w.r.t. Wulff shapes, under $(\ell)_*$.

## 3. Key-properties of system $(S)_W$

**Proposition 3.1 (Solvability and boundedness)** $\forall W \subset \mathbb{R}^2$: Wulff shape (origin-symmetric, compact, convex), $\forall[\theta_0, w_0] \in (H^2(\Omega) \cap H_0^1(\Omega)) \times D_V$,

(i) $\exists!\,[\theta, w] \in W^{1,2}(0,+\infty; H^2(\Omega)) \times (W^{1,2}_{\mathrm{loc}}([0,+\infty); L^2(\Omega)) \cap L^\infty_{\mathrm{loc}}([0,+\infty); BV(\Omega)))$, which solves $(\mathrm{S})_W$;

(ii) $\exists N_{0,W} = N_{0,W}(1/\mu) > 0$ s.t. :

$$|\theta|^2_{W^{1,2}(0,+\infty;H^2(\Omega))} + |w_t|^2_{L^2(0,+\infty;L^2(\Omega))} + \kappa |V_W(w)|_{L^\infty(0,+\infty)} \leq N_{0,W}\left(1 + |\theta_0|^2_{H^2(\Omega)} + \kappa |w_0|_{BV(\Omega)}\right).$$

**Keypoint:** (equivalent reformulation) [Kenmochi-Niezgódka](1994)

Property of energy-dissipation. The following function is nonincreasing:

$$t \in [0,+\infty) \mapsto J(t) := \frac{1}{2}|\theta(t)|^2_{L^2(\Omega)} + \frac{\mu}{2}|\theta(t)|^2_{H_0^1(\Omega)} + \mathscr{F}_0(w(t))_W$$

†. Under the situation: $(\ell)_*$ $\{P_n\} \subset \mathscr{P}$, $d_*(\partial P_n, \mathbb{S}^1) \to 0$ as $n \to +\infty$, the constant $N_{0,W}$ can be taken independently of $W$, because $V_{P_n} \sim V_{\mathbb{D}^2}(w)$

**Proposition 2.2 (Large-time behavior)** $W \subset \mathbb{R}^2$: Wulff shape,
$[\theta, w]$: sol. of $(S)_W$ with initial value $[\theta_0, w_0] \in (H^2(\Omega) \cap H_0^1(\Omega)) \times D_V$.

(I) (Convergence of $\theta$)

$$\theta(t) \to 0 \text{ in } H^2(\Omega) \ (\text{in } C(\overline{\Omega})) \text{ as } t \to +\infty;$$

(II) ($\omega$-limit points of $w$) Let us set:

$$\omega_\infty(w) := \left\{ w_\infty \in L^2(\Omega) \;\middle|\; \begin{array}{l} \exists \{t_n\} \subset (0, +\infty) \text{ s.t. } t_n \nearrow +\infty \text{ and} \\ w(t_n) \to w_\infty \text{ in } L^2(\Omega) \text{ as } n \to +\infty \end{array} \right\}.$$

Then:

(i) $\omega_\infty(w) \neq \emptyset$, connected and compact in $L^2(\Omega)$;

(ii) (steady-state problem) $\forall w_\infty \in \omega_\infty(w)$ satisfies that:

$$(S_\infty)_W \quad \kappa \partial V_W(w_\infty) \ni w_\infty \text{ in } L^2(\Omega)$$

**Main focus: interfacial pattern, represented by steady-state solutions $[0, w_\infty]$ (steady-state pattern)**

## 4. Structural observation in (Case 1) – isotropic case –

**Key-Lemma 1. ($\partial V_{\mathbb{D}^2}$ for piecewise constant functions)**
[Andreu-Ballester-Caselles-Mazón](2001), [S.-Kimura](2005).
$D \subset \Omega$: domain, $D^{\mathrm{ex}} := \Omega \setminus \overline{D}$, $\Gamma_D := \partial D \cap \Omega$: Lipschitz,

$$w_D(x) := \chi_D(x) - \chi_{D^{\mathrm{ex}}}(x), \quad \forall x \in \Omega, \text{ and } w_D^* \in L^2(\Omega).$$

Then, $[w_D, w_D^*] \in \partial V_{\mathbb{D}^2}$ in $L^2(\Omega) \times L^2(\Omega)$, if $\exists \nu_D^* \in C^{0,1}(\Omega)^2$, s.t.

(a) $|\nu_D^*| \leq 1$, a.e. in $\Omega$;

(b) $\nu_D^* = -n_{\Gamma_D}$, $\mathscr{H}^1$-a.e. on $\Gamma_D$,
where $n_{\Gamma_D}$ is the unit outer normal on $\Gamma_D$;

$$\nu_D^* \approx \frac{Dw_D}{|Dw_D|} \quad \left( \nu_D^* \in \partial f_{\mathbb{D}^2}^{\circ}\left(\frac{Dw_D}{|Dw_D|}\right) \right)$$

(c) $\nu_D^* \equiv 0$, $\mathscr{H}^1$-a.e. on $\Gamma$.

(d) $$-\mathrm{div}\, \nu_D^*(x) \begin{cases} \leq w_D^*(x), & \text{if } w_D(x) = 1, \\ = w_D^*(x), & \text{if } -1 < w_D(x) < 1, \\ \geq w_D^*(x), & \text{if } w_D(x) = -1, \text{ a.e. } x \in \Omega. \end{cases}$$

$$w_D^* + \mathrm{div}\, \nu_D^*(x) \in \partial I_{[-1,1]}(w_D)$$

## 4.1. Solutions of the isotropic steady-state problem $(S_\infty)_{\mathbb{D}^2}$

**Main Theorem 1.** [S.](2004), [Bellettini-Caselles-Novaga] (2002) Let $\mathcal{X}_0$ be the solution class of $(S_\infty)_{\mathbb{D}^2}$. Then:

$$\mathcal{S}(\mathbb{D}^2) := \{\ w_D := \chi_D - \chi_{D^{\mathrm{ex}}}\ \big|\ D \subset \Omega : \text{domain, fulfilling } (\mathrm{A1})_0\text{-}(\mathrm{A4})_0\ \} \subset \mathcal{X}_0.$$

$(\mathrm{A1})_0$ $\Gamma_D := \partial D \cap \Omega$: Jordan curve

$(\mathrm{A2})_0$ $\exists r > 2\kappa$, s.t.

$$D = \bigcup_{x \in D, W_0(x;r) \subset D} W_0(x;r),$$

$$D^{\mathrm{ex}} = \bigcup_{x \in D^{\mathrm{ex}}, W_0(x;r) \subset D^{\mathrm{ex}}} W_0(x;r),$$

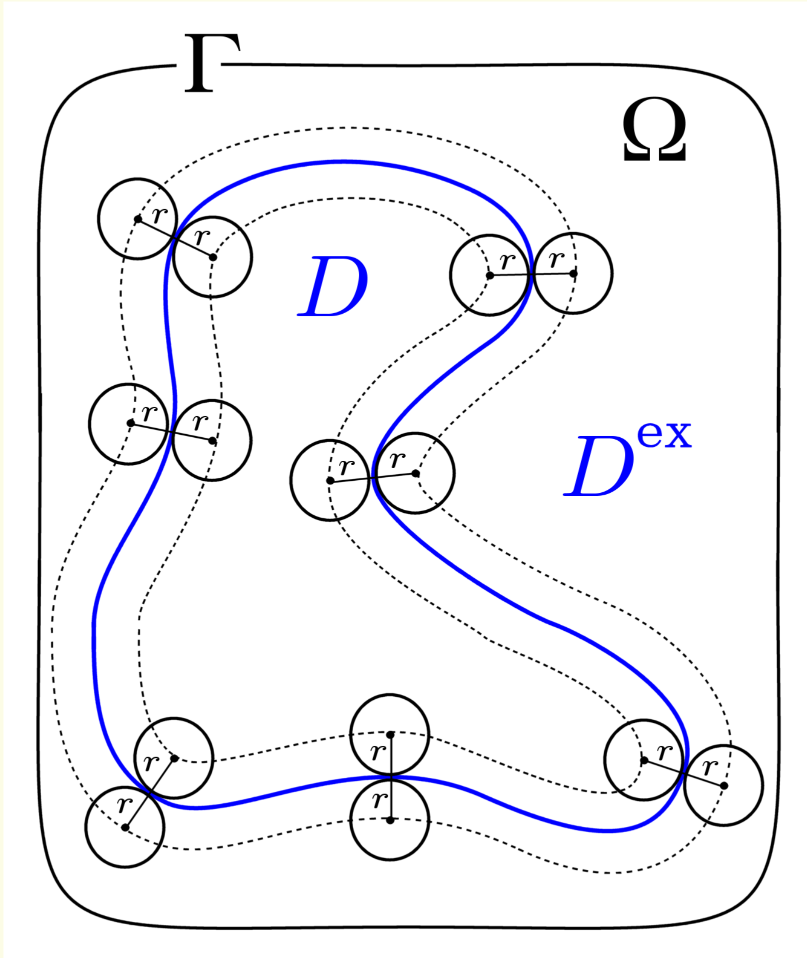
where $W_0(x;r) := (x + rB_1) \cap \Omega$.

$(\mathrm{A3})_0$ $0 < \rho < r$, the neighborhood:

$$\Gamma_D(\rho) := \{\ x \in \Omega\ \big|\ \mathrm{dist}(x, \Gamma_D) \le \rho\ \};$$

is $C^2$-diffeomorphic with $[0,1] \times [-1,1]$.

$(\mathrm{A4})_0$ (condition for stability) The number of inflection points of $\Gamma_D$ is finite.



◇ **Expectation:** (A3)$_0$-(A4)$_0$ are unnecessary

**Why necessary?** $\Longleftarrow$ Technical reason in Geometry.

$\forall w_D = \chi_D - \chi_{D^{\mathrm{ex}}} \in \mathcal{S}(\mathbb{D}^2)$,

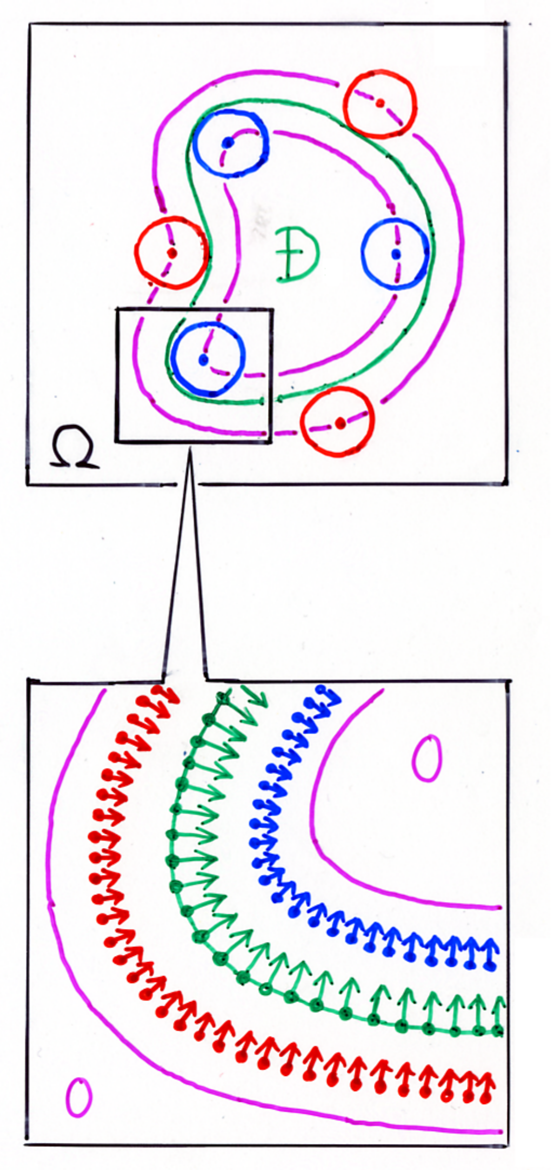
$$\nu_D^* = \begin{cases} \lambda_r(d_D(x))\nabla d_D(x), & \text{if } x \notin \Gamma_D, \\ -n_{\Gamma_D}, & \text{if } x \in \Gamma_D, \end{cases}$$

where

- $\lambda_r$: piecewise linear function, s.t.

$$\mathrm{spt}\lambda_r = [-r, r], \ \lambda_r(0) = |\lambda_r|_{C(\mathbb{R})} = 1;$$

- $$d_D(x) = \begin{cases} \mathrm{dist}(x, \Gamma_D), & \text{if } x \in D, \\ -\mathrm{dist}(x, \Gamma_D), & \text{if } x \in D^{\mathrm{C}}. \end{cases}$$




† $\nu_D^*$ filfills (a)- (d) as in Key-Lemma, because:

$$\nu_D^* \in C^{0,1}(\Omega)^2, \ \nu_D^* \equiv 0 \text{ on } \Gamma, \ \mathrm{div}\, \nu_D^* \approx \Delta d_D \approx (\text{curvature on } \Gamma_D)$$

## 5. Structural observation in (Case 2) – crystalline case –

**Key-Lemma 2. ($\partial V_P$ for piecewise constant functions)**

[Caselles-Chambolle-Moll-Novaga](2008), [S.](2007). $W$: Wulff shape

$D \subset \Omega$: domain, $D^{\mathrm{ex}} := \Omega \setminus \overline{D}$, $\Gamma_D := \partial D \cap \Omega$: Lipschitz,

$$w_D(x) := \chi_D(x) - \chi_{D^{\mathrm{ex}}}(x), \quad \forall x \in \Omega, \text{ and } w_D^* \in L^2(\Omega).$$

Then, $[w_D, w_D^*] \in \partial V_P$ in $L^2(\Omega) \times L^2(\Omega)$, if $\exists \nu_D^* \in C^{0,1}(\Omega)^2$, s.t.

(a) $f_W(\nu_D^*) \leq 1$, a.e. in $\Omega$;

(b) $f_W(\nu_D^*) = 1$ and $\nu_D^* \cdot (-n_{\Gamma_D}) = f_W^\circ(n_{\Gamma_D})$, $\mathscr{H}^1$-a.e. on $\Gamma_D$;

(a), (b) $\Rightarrow$ $\nu_D^* \in \partial f_W^\circ\left(\frac{Dw_D}{|Dw_D|}\right)$

(c) $\nu_D^* \equiv 0$, $\mathscr{H}^1$-a.e. on $\Gamma$.

(d) $$-\mathrm{div}\, \nu_D^*(x) \begin{cases} \leq w_D^*(x), & \text{if } w_D(x) = 1, \\ = w_D^*(x), & \text{if } -1 < w_D(x) < 1, \\ \geq w_D^*(x), & \text{if } w_D(x) = -1, \text{ a.e. } x \in \Omega. \end{cases}$$

$w_D^* + \mathrm{div}\, \nu_D^*(x) \in \partial I_{[-1,1]}(w_D)$

## 4.1. Solutions of anisotropic steady-state problem $(S_\infty)_P$ $\quad\forall P \in \mathscr{P}$: fixed

**Main Theorem 2.** [S.](2007) Let $\mathcal{X}_P$ be the solution class of $(S_\infty)_P$. Then:

$$\mathcal{S}(P) := \left\{ w_D := \chi_D - \chi_{D^{\mathrm{ex}}} \,\middle|\, D \subset \Omega : \text{domain, fulfilling } (\mathrm{A1})_P\text{-}(\mathrm{A3})_P \right\} \subset \mathcal{X}_P.$$

$(\mathrm{A1})_P$ $\Gamma_D$: piecewise linear Jordan curve,

$$\exists n_D \in \mathbb{N}, \text{ s.t. } \Gamma_D := \partial D \cap \Omega = \bigcup_{k=1}^{n_D} L_k$$

$(\ L_k$: compact segment, $1 \le k \le n_D\ )$

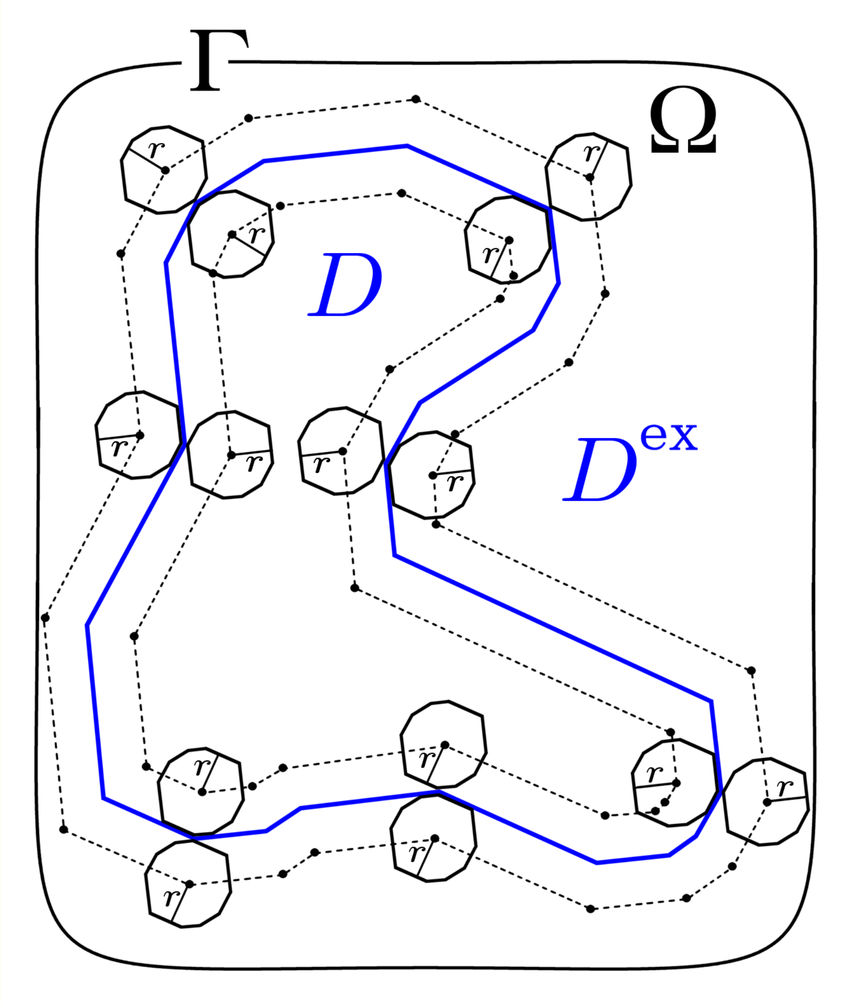
$(\mathrm{A2})_P$ $\exists r > 2\kappa$, s.t.

$$D = \bigcup_{x \in D,\, W_P(x;r) \subset D} W_P(x;r)$$

$$D^{\mathrm{ex}} = \bigcup_{x \in D^{\mathrm{ex}},\, W_P(x;r) \subset D^{\mathrm{ex}}} W_P(x;r),$$

where

$$W_P(x;r) := (x + rP) \cap \Omega.$$



$(\mathrm{A3})_P \quad 0 \leq \forall \rho < r,$

$$C_D(\rho)_P := \left\{ x \;\middle|\; \inf_{y \in \Gamma_D} f_P(y - x) = \rho \right\}.$$

Then:

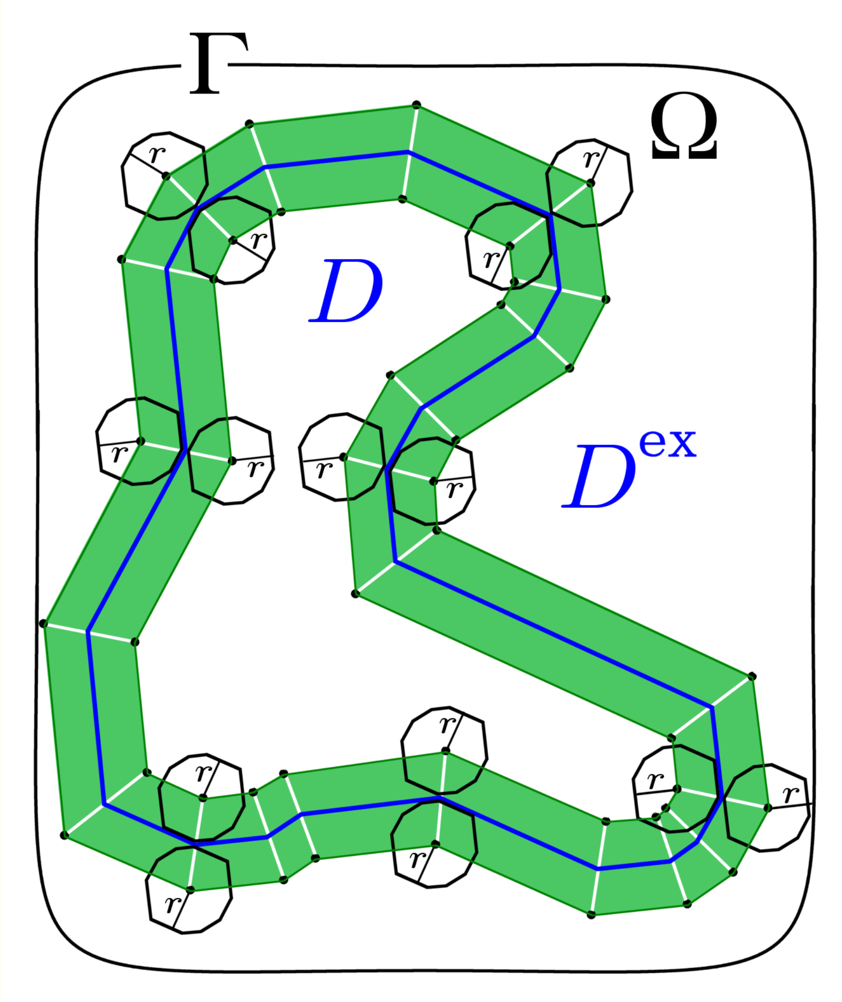
$$\Gamma_\rho := C_D(\rho)_P \cap D, \; \Gamma_\rho^{\mathrm{ex}} := C_D(\rho)_P \cap D^{\mathrm{ex}}$$

are all Jordan curves. In other words,

$$\Gamma_D(\rho)_P := \bigcup_{0 \leq \tau \leq \rho} C_D(\tau)_P = \bigcup_{k=1}^{n_D} L_k(\rho);$$

where, $L_k(\rho)$ $(k = 1, \cdots, n_D)$ are trapezoids, s.t. :

- $\mathrm{int}(L_{k_1}(\rho)) \cap \mathrm{int}(L_{k_2}(\rho)) = \emptyset$, if $1 \leq k_1 \neq k_2 \leq n_D$.




† Geometric situation is quite easier than that as in the isotropic case $(S_\infty)_{\mathbb{D}^2}$.

‡ Limiting observation, based on Mosco convergence, could cover the geometric difficulty as in the limiting problem $(S_\infty)_{\mathbb{D}^2}$.

## 5. Stability analysis

$[0, w_D]$: steady-state solution, where:

$w_D = \chi_D - \chi_{D^{\rm ex}} \in \mathcal{S}(W)$ with Wulff shape $W$ in either (Case 1) or (Case 2).

**Definition 4.1 (Range of oscillations)** $0 < \delta, \rho < 2\kappa$,

$$U_W\left(\begin{bmatrix} 0 \\ w_D \end{bmatrix}; \delta, \rho\right) := \left\{ \begin{bmatrix} \tilde{\theta} \\ \tilde{w} \end{bmatrix} \in \begin{matrix} H^2(\Omega) \cap H^1_0(\Omega) \\ \times \\ D_V \end{matrix} \;\middle|\; \begin{matrix} |\tilde{\theta}|_{H^2(\Omega)} \le \delta \\ |\tilde{w} - w_D| \le \delta \ \text{ a.e. in } \Omega \setminus \Gamma_D(\rho)_W \\ V_W(\tilde{w}) \le V_W(w_D) + \delta \end{matrix} \right\},$$

where $\Gamma_D(\rho)_W := \left\{ \ x \in \Omega \ \middle| \ \inf_{y \in \Gamma_D} f_W(y - x) \le \rho \ \right\}.$

**Main Theorem 3. (Stability analysis)**
The steady-state solution $[0, w_D]$ shows stability in the following sense:

$0 < \exists\, \delta_* < 2\kappa$ (small) which satisfies the following $(*)$.

$(*)$ $0 \le \forall \delta, \forall \rho < \delta_*$, $\exists\, t_* = t_*(\delta, \rho) > 0$ s.t.

$$w(t) = w_D \ \text{ a.e. in } \Omega \setminus \Gamma_D(\rho)_W;$$

$\forall t \ge t_*$, $\forall [\theta, w]$: solution of $(S)_W$ subject to $\begin{bmatrix} \theta_0 \\ w_0 \end{bmatrix} \in U_W\left(\begin{bmatrix} 0 \\ w_D \end{bmatrix}; \delta, \frac{\rho}{2}\right)$.

◇ **Keypoint of the proof.** **Contradiction argument to show:** for some small $\varepsilon_* > 0$,

$$T^* := \sup\left\{ T > 0 \;\middle|\; \mathscr{F}_0(w(t)) \geq \mathscr{F}_0(w_D) + \varepsilon_*, \ \forall t \in [0, T] \right\} = +\infty,$$

**1°)** smallness in $C(\overline{\Omega})$ of tempearture, based on $H^2$-boundedness, $H^1$-smallness by energy-dissipation and Gagliard-Nirenberg inequality $|z|_{C(\overline{\Omega})} \leq \text{Const.}|z|_{H^1(\Omega)}^{1/2}|z|_{H^2(\Omega)}^{1/2}$;

**2°)** comparison principle for single total variation flow (2nd.eq.);

**3°)** to show the following **stability estimate of interfacial energy**

$\exists C_* > 0$: independent of $W$ and $D$, s.t.:

$$V_W(z) \geq V_W(w_D) - \underline{C_*(1 + \text{diam}(W) + \mathscr{H}^1(\Gamma_D))(\delta + \rho)}, \qquad \boxed{\ll \varepsilon_*}$$
$$\forall z \in D_V \ \text{ satisfying } \ |z - w_D|_{L^\infty(\Omega \setminus \Gamma_D(\rho)_W)} < \delta,$$

by applying **generalized co-area formula** and **variational analysis for the interfaces.**

# 6. Limiting observation

**Definition 4.1 ($\omega$-limit set)**

$$\omega\text{-}\mathcal{S}(\mathscr{P}) := \left\{ \bar{w} \in D_V \;\middle|\; \begin{array}{l} \exists\,\{P_n\} \subset \mathscr{P},\ \exists\,\{\bar{w}_n \,|\, \bar{w}_n \in \mathcal{S}(P_n),\ n \in \mathbb{N}\},\ \text{ s.t.} \\ \left.\begin{array}{l} \bullet\ d_*(\partial P_n, \mathbb{S}^1) \to 0 \\ \bullet\ \bar{w}_n \to \bar{w} \ \text{ in } L^2(\Omega) \end{array}\right\} \text{ as } n \to +\infty \end{array} \right\}$$

**Main Theorem 4. ( Limiting observation from (Case 1) to (Case 0) )**

( I ) (Upper bound) $\omega\text{-}\mathcal{S}_{\mathscr{P}} \subset \mathcal{S}^*$ , where :

$$\mathcal{S}^* := \left\{ w_D := \chi_D - \chi_{D^{\mathrm{ex}}} \;\middle|\; \begin{array}{l} \text{(A1)}_0 \;\; \Gamma_D := \partial D \cap \Omega\text{: Jordan curve.} \\ \text{(A2)}'_0 \;\; \exists r \geq 2\kappa, \text{ s.t.} \\ \quad D = \bigcup_{x \in D, W_0(x;r) \subset D} W_0(x;r), \;\; D^{\mathrm{ex}} = \bigcup_{x \in D^{\mathrm{ex}}, W_0(x;r) \subset D^{\mathrm{ex}}} W_0(x;r). \end{array} \right\}.$$

†. [Attouch](1984) $\omega\text{-}\mathcal{S}_{\mathscr{P}} \subset \mathcal{X}_0$

(II) (Lower bound) $\omega\text{-}\mathcal{S}_{\mathscr{P}} \supset \mathcal{S}_*$ and $\mathcal{S}_* \supsetneqq \mathcal{S}(\mathbb{D}^2)$ , where :

$$\mathcal{S}_* := \left\{ w_D := \chi_D - \chi_{D^{\mathrm{ex}}} \;\middle|\; \begin{array}{l} \text{(A1)}_0 \;\; \Gamma_D := \partial D \cap \Omega\text{: Jordan curve.} \\ \text{(A2)}_0 \;\; \exists r > 2\kappa, \text{ s.t.} \\ \quad D = \bigcup_{x \in D, W_0(x;r) \subset D} W_0(x;r), \;\; D^{\mathrm{ex}} = \bigcup_{x \in D^{\mathrm{ex}}, W_0(x;r) \subset D^{\mathrm{ex}}} W_0(x;r). \end{array} \right\}.$$

(III) (Stability) $\forall w_D \in \mathcal{S}_*$, the steady-state solution $[0, w_D]$ shows the stability, just as in $(*)$ of Main Theorem 3.

**Remark 6.1.** As is expected, the two conditions (A3)$_0$-(A4)$_0$, listed below, are eventually unnecessary in the argument of stability analysis.

(A3)$_0$ $\Gamma_D(r)_{\mathbb{D}^2}$ is $C^2$-diffeomorphic with the rectangle $[0,1]\times[-1,1]$.

(A4)$_0$ $\Gamma_D$ has at most finite number of inflection points.

## ◇ Keypoints of the proof of ( I )-(II) Differential geometry

**Key-Lemma 3.** [S.](2009)

$\forall w_D = \chi_D - \chi_{D^{\mathrm{ex}}} \in \mathcal{S}_*$ with the domain $D$, fulfilling (A1)$_0$-(A2)$_0$, $\exists\{P_n\}\subset\mathscr{P}$, $\exists\{D_n\}$ : sequence of domains in $\Omega$, s.t. :

- $w_{D_n} := \chi_{D_n} - \chi_{D_n^{\mathrm{ex}}} \in \mathcal{S}(P_n)$, $n = 1, 2, 3, \cdots$,

$$\left.\begin{array}{l} \bullet\ d_*(\partial P_n, \mathbb{S}^1) \to 0,\ d_*(\partial D_n, \partial D) \to 0, \\ \bullet\ V_{P_n}(w_{D_n}) = \mathscr{H}^1(\partial D_n) \to \mathscr{H}^1(\partial D) = V_{\mathbb{D}^2}(w_D), \\ \bullet\ w_{D_n} \to w_D \text{ in } L^2(\Omega), \end{array}\right\} \text{as } n \to +\infty.$$

## ◇ Sketch of the proof of (III) of Main Theorem 4

**Assume that:**

- $0 < \bar{\delta} < 1$, s.t. $(r - \bar{\delta})(1 - 3\bar{\delta}) > 2\kappa$,
- $0 < \delta, \rho < \bar{\delta}$, and $\begin{bmatrix} \theta_0 \\ w_0 \end{bmatrix} \in U_W \left( \begin{bmatrix} 0 \\ w_D \end{bmatrix}; \delta, \frac{\rho}{2} \right)$.

**Let us prepare:**

- $\{P_n\} \subset \mathscr{P}$, s.t. $d_*(\partial P_n, \mathbb{S}^1) \longrightarrow 0$ as $n \to +\infty$,
- $\{w_{D_n} \in \mathcal{S}(P_n) \,|\, n \in \mathbb{N}\}$: sequence as in Key Lemma 3,
- $[\theta_n, w_n]$: solutions of $(S)_{P_n}$, $n \in \mathbb{N}$ (resp. $[\theta, w]$: solution of $(S)_{\mathbb{D}^2}$), having the common initial value $[\theta_0, w_0]$.

Then, by Key Lemma 3, $\exists \bar{n} = \bar{n}(\rho) \in \mathbb{N}$, s.t. :

$$\Omega_n(\rho) := \left\{ \; x \in \Omega \; \middle| \; \inf_{y \in \Gamma_{D_n}} f_{P_n}(x - y) \geq \rho \; \right\} \subset \Omega \setminus \Gamma_D\left(\frac{\rho}{2}\right), \; \forall n \geq \bar{n}.$$

$\forall n \geq \bar{n}$, $\forall z \in D_V$ satisfying $|z - w_D|_{L^\infty(\Omega \setminus \Gamma(\rho/2))} \leq \delta$, it follows from topological equivalence of $V_{P_n}$ and $V_{\mathbb{D}^2}$ that:

$$V_{\mathbb{D}^2}(z) \geq \frac{1}{1 + d_*(\partial P_n, \mathbb{S}^1)} V_{P_n}(z)$$

$$\geq \frac{1}{1 + d_*(\partial P_n, \mathbb{S}^1)} \Big( V_{P_n}(w_{D_n}) - C_*(1 + \operatorname{diam}(P_n) + \mathscr{H}^1(\Gamma_{D_n}))(\delta + \rho) \Big);$$

where $C_*$ is the positive constant as in the proof of Main Theorem 3.

Letting $n \to +\infty$ yields that:

$$V_{\mathbb{D}^2}(z) \geq V_{\mathbb{D}^2}(w_D) - C_*(1 + \operatorname{diam}(\mathbb{D}^2) + \mathscr{H}^1(\Gamma_D))(\delta + \rho),$$
$$\forall z \in D_V \text{ satisfying } |z - w_D|_{L^\infty(\Omega \setminus \Gamma(\rho/2))} \leq \delta$$

**without helps from (A3)$_0$-(A4)$_0$**

Now, the remaining proof will be an analogy of that as in the previous stability theory, adopting **contradiction argument**. ■

# 7. Future problems

( I ) Analogous observations for the case of $\mu = 0$.

**System $(S)_W$ when $\mu = 0$:**

$$\begin{cases} (\theta + w)_t - \Delta\theta = 0 \ \text{ in } Q, \quad \text{(there is no relaxation } -\Delta(\mu\theta_t)\text{ )} \\ \theta = 0 \ \text{ on } \Sigma := (0, +\infty) \times \Gamma, \quad \theta(0, x) = \theta_0(x), \quad x \in \Omega; \end{cases} \qquad \text{(1st.eq.)}$$

$$\begin{cases} w_t - \kappa \, \mathrm{div}\, (\partial f_W^\circ(Dw)) + \partial I_{[-1,1]}(w) \ni w + \theta \text{ in } Q, \\ \boxed{\text{anisotropic total variation flow}} \\ \text{(homogeneous Neumann type B.C.)}, \quad w(0, x) = w_0(x), \quad x \in \Omega. \end{cases} \qquad \text{(2nd.eq.)}$$

†. Keypoint will be the $L^\infty$-estimate of temperature.

(II) Joint to control problems, based on $(S)_W$, and their numerical simulations.

**Jointwork with:**

- Prof. Yamazaki (Kanagawa Univ.) Theoretical approach for optimal controls;
- Prof. Okazaki (Gifu Nat. Col. Tech.) Numerical approach for optimal controls.