The notion of an almost classical solutions to the total variation flow and its usefulness

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The equation which is the topic of my talk

$$u_t - \frac{d}{dx}\left(\operatorname{sgn}\left(u_x\right)\right) = 0, \qquad u(a) = A, \quad u(b) = B. \tag{1}$$

with Dirichlet boundary conditions

$$u(a) = A, \qquad\qquad u(b) = B$$

is a one -dimensional example of the total-variation flow.

The motivations to study this problem is twofold:

a) image analysis;

b) crystal growth problems.

Phenomenon:

sudden directional diffusion – so strong nonlinearity that it causes nonlocal effects

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Basic equations are:

the regularity

optimal regularity

“natural” regularity

and regularization

A key point of the novel look is the Yosida approximation,

or the semi-discratization in time

$$nu - \frac{d}{dx}\left(\operatorname{sgn}\left(u_x\right)\right) = nv \tag{2}$$

The nonlinearity is connected to the following functional

$$\mathcal{J}(u) = \begin{cases} \int_a^b |\partial_x u| & \text{if } u \in D(\mathcal{J}) \equiv \{u \in BV(a,b),\ u(a) = A,\ u(b) = B\}, \\ +\infty & L_2(a,b) \setminus D(\mathcal{J}), \end{cases}$$

where $\int_a^b |\partial_x u|$ is the total variation of measure $\partial_x u$.

Then, (2) may be seen as

$$v \in u + \frac{1}{n} \partial \mathcal{J}(u), \tag{3}$$

where $\partial \mathcal{J}$ is the subdifferential of $\mathcal{J}$.

Observation

Proposition.

$$\frac{1}{2}x^2 \text{ \underline{do not} belong to } \mathcal{D}(\partial\mathcal{J})$$

In regular case areas of monotonicity must be separated by a flat region

Let us recall the definition on the domain of subdifferential:

We say that $w \in \partial\mathcal{J}(u)$ iff $w \in L_2(a,b)$ and for all $h \in L_2(a,b)$ the inequality holds,

$$\mathcal{J}(u+h) - \mathcal{J}(u) \geq (w,h)_2. \tag{4}$$

Here $(f,g)_2$ stands for a regular inner product in $L_2(a,b)$.

If $\frac{1}{2}x^2 \in D(\partial\mathcal{J})$, then there exists $w \in L_2(-1,1)$ such that for all $\phi \in C_0^\infty(-1,1)$ and $t \in \mathbb{R}$

$$\int_{(-1,1)} (|x + t\phi_x| - |x|)dx \geq t \int_{(-1,1)} w\phi dx. \tag{5}$$

We restrict ourself to ϕ such that

$$\phi \in C_0^\infty(-\delta, \delta) \quad \text{and} \quad \operatorname{supp} \phi_x \subset [-\delta, -\delta/2] \cup [\delta/2, \delta]. \tag{6}$$

Additionally

$$\phi_x(t) > 0 \quad \text{for} \quad t \in (-\delta, -\delta/2) \quad \text{and} \quad \phi_x(t) < 0 \quad \text{for} \quad t \in (\delta/2, -\delta) \tag{7}$$

and

$$\phi(-\delta) = \phi(\delta) = 0 \quad \text{and} \quad \phi(t) = 1 \quad \text{for} \quad t \in (-\delta/2, \delta/2). \tag{8}$$

Next, let us observe that

$$|x + t\phi_x(x)| - |x| = t\phi_x(x)\operatorname{sgn} x \quad \text{for } |t\phi_x(x)| \le \delta/2; \tag{9}$$

we keep in mind that $\phi_x(t) = 0$ for $t \in (-\delta/2, \delta/2)$

Thus for such ϕ and t the r.h.s. of (5) equals

$$\int_{(-\delta/2,\delta/2)} (|x + t\phi_x(x)| - |x|)dx =$$

$$\int_{(-\delta,-\delta/2)} t\phi_x \cdot (-1)dx + \int_{(\delta/2,\delta)} t\phi_x \cdot (1)dx = -2t.$$

Hence we get

$$-2t \geq t \int_{(-\delta,\delta)} w\phi dx, \tag{10}$$

what implies for $t > 0$

$$2 \leq -\int_{(-\delta,\delta)} w\phi dx \leq \int_{(-\delta,\delta)} |w| dx \to 0, \tag{11}$$

since $w \in L_2(-1,1)$. And here we meet the contradiction, hence $\frac{1}{2}x^2$ can not belong to $D(\partial\mathcal{J})$.

And we are done.

Now we are in a good position to introduce the space where the solution will be constructed

First:

We say that a real valued function u defined over a closed interval $[a, b]$ *belongs to* $BV[a, b]$ provided that

$$\|u\|_{BV[a,b]} = \int_a^b |\partial_x u| < \infty,$$

where $|\partial_x u|$ is the total variation of the measure $\partial_x u$.

Without any loss of generality we assume that $[a, b] = [0, 1]$.

Additionally, we can treat BV functions as multi-valued function such that for each $x_0 \in (0, 1)$

$$u(x_0) = [\lim_{x \to x_0^-} u(x), \lim_{x \to x_0^+} u(x)]_{or},$$

where $[a, b]_{or} = [a, b]$ for $a \le b$ and $[a, b]_{or} = [b, a]$ for $b > a$.

However, the simple description of regularity as a BV function is not sufficient, we are required to restrict ourselves to a subclass of the BV function. There is a need to control the facets.

A facet of u, F is a piece of graph of u with zero slope, i.e.

$$F = F(\xi^-, \xi^+) = \{(x, y) : \; y = u(x), \; x \in [\xi^-, \xi^+]\},$$

maximal with respect to inclusion of sets.

We shall also distinguish a subclass of facets. We shall say that a facet

$$F(\xi^-, \xi^+),$$

has zero curvature, if and only if there is such $\epsilon > 0$, function u restricted to $[\xi^- - \epsilon, \xi^+ + \epsilon]$ is monotone.

In case the function is increasing, it means that

$$u(\xi^- - \epsilon) < u(\xi^-) = u(\xi^+) < u(\xi^+ + \epsilon)$$

We shall see that zero curvature facets do not move as long as they are zero curvature. There may be even an infinite number of them. They have no influence on the evolution of the system.

For that reason we introduce the following objects, capturing the essential things.

We shall say that

$$F_{ess} = F_{ess}(\zeta^-, \zeta^+) = \{(x, u(x)) : x \in [\zeta^-, \zeta^+]\}$$

is an *essential facet* of w, if

there exists $\epsilon > 0$ such that

either

u is decreasing on $(\zeta^- - \epsilon, \zeta^-)$

and $u(t) > u(\zeta^-)$ for $t \in (\zeta^- - \epsilon, \zeta^-)$;

and u is increasing on $(\zeta^+, \zeta^+ + \epsilon)$

and $u(t) > u(\zeta^+)$ for $t \in (\zeta^+, \zeta^+ + \epsilon)$;

Then we call such facet “convex”.

or

u is increasing on $(\zeta^- - \epsilon, \zeta^-)$

and $u(t) < u(\zeta^-)$ for $t \in (\zeta^- - \epsilon, \zeta^-)$;

and u is decreasing on $(\zeta^+, \zeta^+ + \epsilon)$

and $u(t) < u(\zeta^+)$ for $t \in (\zeta^+, \zeta^+ + \epsilon)$.

Then we call such facet "concave".

We start with a definition

Let us suppose that $w \in BV[0,1]$ treated as a maximal BV function.

We define the set

$$\Xi(w) = \{x \in [0,1] : 0 \in w(x)\}. \tag{12}$$

We say that $w \in BV[0,1]$ is *J-regular*, or shorter $w \in$ J-R
iff there exists a set $\Xi_{ess}(w) \subset \Xi(w)$ such that

$\Xi_{ess}(w)$ consits of a finite number of closed nondegenerated intervals, ie.

$$\Xi_{ess}(w) = \bigcup_{i=1,\ldots,K_{ess}(w)} [a_i, b_i] \quad \text{for} \quad a_i < b_i. \tag{13}$$

and each of $[a^i, b^i]$ is an essential facet.

In particular $\Xi(w) \setminus \Xi_{ess}(w)$ consits only of zero curvature facets.

We also define the following quantity

$$\|w\|_{\text{J-R}[0,1]} = \|w\|_{BV[0,1]} + K_{ess}(w), \tag{14}$$

where $K_{ess}(w)$ is the number of connected parts of $\Xi_{ess}(w)$,

however, it is not any norm in this space.

The definition of admissible functions

We shall say that a function a is *admissible*, for short $a \in AF[0,1]$, iff $a : [0,1] \rightarrow \mathbb{R}$,

$$\alpha = \partial_x a \quad \text{with} \quad \alpha \in \text{J-R} \quad \text{and} \quad a(0) = a_0, a(1) = a_1. \qquad (15)$$

Here $\partial_x a$ denotes is the set-valued Clarke differential of a.

Moreover on neighborhoods of ends of $[0,1]$ a must be monotone, ie.

Moreover, at each end of interval $[0, 1]$, an additional condition must hold.

END "0"

a is monotone on an interval $[0, x_0)$ for some $x_0 \in (0, 1)$

and either

$$a(x_0) > a(0) \qquad \text{or} \qquad a(x_0) < a(0);$$

END “1”

The same relations hold for the end 1; and one of the conditions below must hold:

a is monotone on an interval $(x_0, 1]$ for some $x_0 \in (0, 1)$

and either

$$a(x_0) > a(1) \qquad \text{or} \qquad a(x_0) < a(1)$$

Note that the Dirichlet boundary condition makes immobile any facet touching the boundary.

Thus, such facets are zero curvature.

A composition of multivalued operators require proper preparations.

Due to needs of our paper we restrict ourselves to a definition of

$$\operatorname{sgn} \bar{\circ}\, \alpha$$

for a suitable class of multivalued operators α.

It is most important to define this composition in the interior of the domain we work with.

Definition

Let $\alpha \in$ J-R$[0,1]$, then for $x \in [0,1] \setminus \Xi_{ess}(\alpha)$ then

either $x \in [a,b]$ such that α is increasing on $[a,b]$, then

$$\operatorname{sgn} \bar{\circ}\, \alpha(x) = 1; \tag{16}$$

or $x \in [a,b]$ such that α is decreasing on $[a,b]$, then

$$\operatorname{sgn} \bar{\circ}\, \alpha(x) = -1. \tag{17}$$

Note that the set $[0,1] \setminus \Xi_{ess}(\alpha)$ consits of finite number connected intervals on each the function is monotone. Hence the ends of $[0,1]$ can not belong to $\Xi_{ess}(\alpha)$.

Now let $x \in \Xi_{ess}(\alpha)$, then if $x \in [p,q]$ and $[p,q]$ is a convex facet of α, we set,

$$\operatorname{sgn} \bar{\circ}\, \alpha(x) = \frac{2}{q-p}x - \frac{2p}{q-p} - 1 \quad \text{for } \; x \in [p,q]; \tag{18}$$

if $[p,q]$ is a concave facet of α, we set,

$$\operatorname{sgn} \bar{\circ}\, \alpha(x) = -\frac{2}{q-p}x + \frac{2p}{q-p} + 1 \quad \text{for } \; x \in [p,q]. \tag{19}$$

We have already mentioned that the Dirichlet boundary condition does not permit any motion of the facet touching the boundary. Thus, effectively, they behave like zero-curvature facets.

Now we are in a position to state the main result being also a justification of the notion of the almost classical solutions to our system.

Theorem

Let $u_0 \in AF[0,1]$ with $u(0) = a_0$ and $a(1) = a_1$, then the system (1) admits unique solution such that

$$u_x \in L_\infty(0,T;\text{J-R}[0,1]) \tag{20}$$

and u is an almost classical solution to (1), i.e. it fulfills the system (1) in the following meaning

$$\begin{array}{llr} u_t - \partial_x \mathrm{sgn}\,\bar{\circ}\, u_x = 0 & in & [0,1]\times(0,T), \\ u(0,t) = a_1, \qquad u(1) = a_1 & for & t \in [0,T), \\ u|_{t=0} = u_0 & on & [0,1] \end{array} \tag{21}$$

Yosida approximation

We call an operator defining the solution to

$$nu - \frac{d}{dx}\mathrm{sgn}\,(u_{,x}) = v \tag{22}$$

with $nu = v$ at the boundary ∂D, *the resolvent of* $A = -\partial_x \mathrm{sgn}\,\partial_x$ and we denote it by

$$u = R(n, A)v.$$

An operator A_n : J-R $\rightarrow$ J-R such that

$$A_n = n(Id - R(n, A))$$

defined for $n > 0$ and $-A = \partial_x \mathrm{sgn}\, \partial_x$ is called
the *Yosida approximation of* A.

For given $w \in$ J-R we define

$$L(w_x) = \text{min}\{b - a \text{ : such that } [a, b] \text{ is an element of } \Xi_{ess}(w_x)\} \quad (23)$$

By definition $L(w_x)$ is greater than zero for $w_x \in$ J-R.

Main lemma

Let $w_x \in JR$, $L(w_{,x}) = d > 0$, then the solution to

$$nu - A(u) = nw \tag{24}$$

exists uniquely, fulfills

$$u_x \in \text{J-R}, \qquad \|u_x\|_{\text{J-R}} \leq \|w_x\|_{\text{J-R}}$$

$$K_{ess}(u) = K_{ess}(w) \quad \text{for} \quad n > n_0$$

the equation (24) can be restated as follows

$$nu - \partial_x \mathrm{sgn}\,\bar{\circ} u_{,x} = nw + V(n,x),$$

where $V(n,x) \to 0$ in L_q for all $q < \infty$. In addition

$$A_n(u^n) \to \partial_x \mathrm{sgn}\,\bar{\circ} w_{,x} \quad \text{in} \quad L_q(0,1) \quad \text{with} \quad q < \infty.$$

Proof.

We want to show the existence to system (24), however we would like to skip the methods of calculus of variation and we want to base on elementary tools only. For this purpose we restrict our self to $w \in \mathcal{D}(A)$ and for sufficiently large n.

Let

$$\Xi_{ess}(w_x) = \bigcup_{i=1,\ldots,K_{ess}(w_x)} [a_*^i, b_*^i] \tag{25}$$

the regularity of w means here that

$$a_*^i < b_*^i. \tag{26}$$

For each i we solve the problem

$$(b^i - a^i)w(a^i) = \int_{a^i}^{b^i} w + 2\frac{1}{n}\operatorname{sgn}\kappa_{[a_*^i, b_*^i]} \quad \text{and}$$

$$\{a^i, b^i\} = w^{-1}(w(a^i)) \quad \text{locally in the neighborhood of } [a_*^i, b_*^i]. \tag{27}$$

The above problem comes from integration over a suitbale neighborhood of the facet $[a_*^i, b_*^i]$.

For sufficiently large n it is always possible to determin the solution.

This way we construct the function u

$$u = \begin{cases} v & \text{for} \quad x \in [0,1] \setminus \Xi_{ess}(u_x) \\ \\ w(a^i) & \text{for} \qquad\qquad\qquad x \in [a^i, b^i] \end{cases} \tag{28}$$

with $\Xi_{ess}(u_x) = \bigcup\limits_{i=1,\ldots,K_{ess}(u_x)} [a^i, b^i]$, where $\{a^i, b^i\}$ are the solutions to (27). The formula (28) holds as $n > n_0$ then $K_{ess}(u_x) = K(w_x)$

In the case $n = n_0$ we have

$$K_{ess}(u_x) < K_{ess}(w_x). \tag{29}$$

And it means that either $a^1 = 0$, either $b^{K(w_x)}$

or for some i it heppends $b^i = a^{i+1}$.

In these cases we have to slightly modify (28), since the structure of $\Xi_{ess}(u_x)$ is different from $\Xi_{ess}(v_x)$.

We just consider the limit given by (28), so we have to control facets related to decomposition for w.

Regularity

It is clear that

$$K_{ess}(u_x) \le K_{ess}(w_x) \tag{30}$$

and by the constraction (28) it is obvious too, that

$$\|u_x\|_{BV[0,1]} \le \|w_x\|_{BV[0,1]}. \tag{31}$$

Now we want to answer of the question concerning
the meaning of solutions.

Namely, if our constructed function u is indeed the solution.

We would like to hire here differential inclusions.

However we can not look at the system in the following way

$$u - w - \frac{1}{n}\frac{d}{dx}\operatorname{sgn} u_x \ni 0, \tag{32}$$

since the is not reasonable definition of the last term in the l.h.s.

But we can use the fact that $u = w$ at $x = 0$.

We propose the following meaning

$$\int_0^x (u - w)dx' - \frac{1}{n}\text{sgn}\, u_x \Big|_0^x \ni 0. \tag{33}$$

Such modification for a similar issue has been proposed in PB Mucha & P Rybka, SIMA 2007

To check whether the equation is fulfilled by the constructed function

it is enough to consider several cases, which are rather elementary

So we claim that we have the solution.

Now we start with analysis of dependence from n. This point is irreplaceable in study of evolutionary system.

Let $L(w_x) = d > 0$. Then we consider

$$nu - A(u) = nw \quad \text{for } n > n_0.$$

The bound n_0 on n is related to the fact that we would like to to have $K_{ess}(w_x) = K_{ess}(u_x)$.

Take $[a^*, b^*]$ an element of $\Xi_{ess}(u_x)$, assume that it is a convex facet.

So

$$\int_{a^*}^{b^*} nu - \int_{a^*}^{b^*} nw = 2. \tag{34}$$

What we can say about the behavior of the following quantity

$$\int_{a^*}^{a} + \int_{b}^{b^*} (nu - nw) \tag{35}$$

where $[a, b]$ is an element of $\Xi(w_x)$ contained in $[a^*, b^*]$.

Since d is fixed and positive we find from (34)

$$2 = \int_{a^*}^{b^*} n(u-v) \geq \int_a^b n(u-v) \geq dn(u-v)|_{[a,b]}$$

so

$$n(u-w)|_{[a^*,b^*]} \leq \frac{2}{d}. \quad (36)$$

Then we conclude that

$$\int_{b}^{b^*} n(u-w) \leq (b^*-b)n[w(b^*)-w(b)], \tag{37}$$

but $w(b^*) - w(b) \leq \frac{2}{dn}$, on the other hand w is strictly monotone on set (b, b^*), hence (36) implies that

$$b^* - b \leq W^{-1}(\frac{2}{dn}), \tag{38}$$

where $W^{-1}(\cdot)$ is a strictly monotone function, equal to w^{-1} plus a constant such that $W^{-1}(0) = 0$.

Eventually, we get

$$\int_{b}^{b^*} n(u-w) \leq W^{-1}(\frac{2}{dn})\frac{2}{d} \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{39}$$

So we know

$$\int_{a^*}^{b^*} n(u-w) = 2 + V(n) \quad \text{with} \quad V(n) \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{40}$$

Observe that

$$u - w|_{[a,b]} = \overline{\partial_x \mathrm{sgn}\, u_x} = const. \tag{41}$$

Clearly by definition

$$\partial_x \mathrm{sgn}\, \bar{\circ} u_x = \frac{2}{(b^* - a^*)}.$$

Hence we have proved that

$$\overline{\partial_x \mathrm{sgn}\, u_x} = \partial_x \mathrm{sgn}\, \bar{\circ} u_x + V(n, x), \tag{42}$$

where $V(n, x) \to 0$ in at least $L_1(I)$.

From that we are able to conclude the thesis.

Construction of the flow

We introduce for given n and t_0

$$u^n(t+t_0) = u^n(t_0) + \int_{t_0}^{t_0+t} A_n(u^n)ds \tag{43}$$

Note that here $L(w) > 0$ is not required:

Let $u^n(t_0) \in \text{J-R}(I)$ then the solution to (43) exists uniquely on the time interval $(t_0, t_0 + \frac{1}{3n})$ in the class

$$u^n \in C(t_0, t_0 + \frac{1}{3n}; L_1(I)) \cap L_\infty(t_0, t_0 + \frac{1}{3n}; \text{J-R}(I))$$

and

$$\sup_{t \in (0, \frac{1}{3n})} \|u^n(t_0 + t)\|_{\text{J-R}} \leq \|u^n(t_0)\|_{\text{J-R}}. \tag{44}$$

Let us construct an approximation of solution on the time interval $[0, 1]$.

Let

$$U^n : [0, 1] \times I \to \mathbb{R}$$

is given as follows

$$U^n = u_k^n \quad \text{for} \quad t \in [\frac{k}{3n}, \frac{k+1}{3n}) \quad \text{and} \quad 0 \leq k < 3n,$$

where function $\{u_k^n\}$ are given by the following relations

$$u_1^n(t) = u_0 + \int_0^t A_n(u_1^n)ds \quad \text{for } t \in (0, \frac{1}{3n}],$$

$$u_2^n(t_1 + t) = u_1(t_1) + \int_{t_1}^{t_1+t} A_n(u_2^n)ds \quad \text{for } t \in (0, \frac{1}{3n}],$$

...

$$u_{k+1}^n(t_k + t) = u_k(t_k) + \int_{t_k}^{t_k+t} A_n(u_{k+1}^n)ds \quad \text{for } t \in (0, \frac{1}{3n}],$$

...

$$u_{3n}^n(t_{3n-1}+t) = u_{3n-1}^n(t_{3n-1})+\int_{t_{3n-1}}^{t_{3n-1}+t} A_n(u_{3n}^n)ds \quad \text{for } t \in (0, \frac{1}{3n}]$$

and $t_k = \frac{k}{3n}$ for $0 \le k < 3n$.

Passing with $n \to \infty$ we get

$$U^n \to U^* \qquad \text{in suitable topology}$$

and

$$U^*(t_0 + t) = U^*(t_0) + \lim_{n\to\infty} \int_{t_0}^{t_0+t} A_n(U^n(t_0 + t))ds$$

So finally we have shown that U^* fulfills

$$\frac{d}{dt^+}U^* = \frac{d}{dx}\operatorname{sgn} \bar{\circ}\, U^* \tag{45}$$

as the almost classical solution.

K. Kielak, P.B. Mucha, P. Rybka, *Almost classical solutions to the total variation flow*, almost done preprint, coming soon `www.mimuw.edu.pl/∼pbmucha`.

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See also: www.mimuw.edu.pl/∼pbmucha