LECTURE 1: A survey of p-harmonious functions in trees and in Euclidean spaces

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A mini semester on evolution of interfaces.

Sapporo, August 5, 2010

Playing tug-of-war

Two players **Henry** and **Theresa** play a board game.
A token is placed in an initial position x on a board (graph)

To play a step in a random tug-of-war game, flip the coin again:

- **H** If the result is heads, Henry gets to move the token from x to anywhere in $S(x)$
- **T** If the result is tails, Theresa moves instead to any point in $S(x)$ of her choice.

Simulation

HEXAMANIA, by David Wilson at `http://dbwilson.com/`

Playing noisy tug-of-war

Two players **Henry** and **Theresa** play a board game.
A token is placed in an initial position x on a board (graph)

STEP 1

A fair coin is flipped.

- **H** If the result is heads, Henry and Theresa play a step in a tug-of-war game. Go to STEP 2.
- **T** If the result is tails, the token moves one step at random.

STEP 2

To play a step in a random tug-of-war game, flip the coin again:

- **H** If the result is heads, Henry gets to move the token from x to anywhere in $S(x)$
- **T** If the result is tails, Theresa moves instead to any point in $S(x)$ of her choice.

Playing noisy tug-of-war, II

We are also given a **pay-off function** F defined on the boundary of the board.

The game ends when one of the players places a token at a boundary point y. Then Henry pays Theresa $\$F(y)$.

Imagine that we play this game many times over.
What is the expected amount of money Theresa will get? or what is the expected amount of money Henry will have to pay?

Of course, the answer will depend on how well they play the game. When is Henry's turn to play, he will try to approach points on the boundary where F is as small (negative) as possible, while Theresa will try to approach points where F is as large (positive) as possible.

Playing noisy tug-of-war, III

It turns out that if Henry and Theresa play according to optimal strategies, the amount of money $u(x)$ they can expect to get (or pay) starting at x is the solution to a difference equation that is the discrete analogue of the p-Laplace equation.

The value function u verifies the equation

$$u(v) = \frac{1}{2} \cdot \frac{1}{2} \left(\max_i \{u(v_i)\} + \min_i \{u(v_i)\} \right) + \frac{1}{2} \left(\frac{\sum u(v_i)}{\#\{v_i\}} \right) .$$

Inspiration: Games Mathematicians Play,

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- A. P. Maitra, and W. D. Sudderth, *Borel stochastic games with* lim sup *payoff.* Ann. Probab., 21(2):861–885, 1996.
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References

MPR1 *An asymptotic mean value property characterization of p-harmonic functions*, Proc. Amer. Math. Soc., 138:881–889, 2010.

MPR2 *On the definition and properties of p-harmonious functions*, Preprint.

MPR3 *Dynamic programming principle for tug-of-war games with noise.* Preprint.

MPR4 *An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games*, to appear in the SIAM Journal of Mathematical Analysis

S Alex Sviridov, *Elliptic Equations in Graphs via Stochastic Games*, 2010 Ph. D. Thesis.

The p-Laplacian Gambling House

$(\mathfrak{X}, \mathcal{B})$ measure space.

$$\mathfrak{X} = X \cup Y$$

disjoint union of two non-empty sets X and Y. X the **interior** and Y the **boundary**.

For $x \in \mathfrak{X}$ we have a nonempty set $S(x) \subset \mathfrak{X}$ of **successors** of x. For points $y \in Y$ we require that $S(y) = \{y\}$. Moreover, the set $S(x)$ comes equipped with a probability measure supported in $S(x)$ denoted by $\mu(x)$. For points $y \in Y$ on the boundary we have that $\mu(y) = \delta_y$

We are given non-negative numbers α and β so that $\alpha + \beta = 1$ and a pay-off function $F\colon\ Y \mapsto \mathbb{R}$.

The p-Laplacian Gambling House

Definition of the Gambling House

At every point $x \in \mathfrak{X}$ we have a family of probability measures $\Gamma(x)$ in $(\mathfrak{X}, \mathcal{B})$ given by

$$\Gamma(x) = \left\{ \frac{\alpha}{2} (\delta_{x_I} + \delta_{x_{II}}) + \beta\,\mu(x) \colon x_I, x_{II} \in S(x) \right\}$$

Playing *Tug-of-War game with noise*

Start at $x_0 \in \mathfrak{X}$ and choose a probability $\gamma_0[x_0] \in \Gamma(x_0)$. The next position $x_1 \in S(x_0)$ is is selected according to $\gamma_0[x_0]$. Once x_0 and x_1 are chosen, we pick a probability $\gamma_1[x_0, x_1] \in \Gamma(x_1)$ to determine the next game position $x_2 \in S(x_1)$. In this manner we determine a particular history

$$\mathrm{x} = (x_0, x_1, x_2, \dots) \in \mathfrak{X} \times \mathfrak{X} \times \cdots \times \mathfrak{X} \times \cdots = \mathfrak{X}^\infty.$$

Ending the game

The game ends when we reach the boundary Y since once $x_j \in Y$ we have $x_{j+1} \in S(x_j) = \{x_j\}$. We write

$$\tau(x) = \inf\{k : x_k \in Y\}$$

for the first time we hit the boundary with the understanding that $\tau(x) = \infty$ if the boundary is never reached. If the game ends at a point $y \in Y$ the pay-off value is $F(y)$.

$\mathcal{B}^j$ is the product σ-algebra in $\mathfrak{X}^j$ and $\mathcal{B}^\infty$ the σ-algebra in $\mathfrak{X}^\infty$ generated by the cylinder sets

$$A_0 \times A_1 \times \cdots \times A_j \times \mathfrak{X} \times \mathfrak{X} \times \cdots ,$$

where $A_k \in \mathcal{B}^k$ for $k = 0, 1, \ldots, j$.

The p-Laplacian Gambling House

Strategies (Maitra-Sudderth)

The collection of probability measures

$$\sigma = (\gamma_0[x_0], \gamma_1[x_0, x_1] \ldots, \gamma_k[x_0, x_1 \ldots x_k], \ldots)$$

is a *strategy*.

Kolmogorov-Tulcea construction

There exists a unique probability measure $\mathbb{P}_\sigma^{x_0}$ in $(\mathfrak{X}^\infty, \mathcal{B}^\infty)$ with transition probabilities

$$\mathbb{P}_\sigma^{x_0}(\{x_{j+1} \in A\} \mid \mathcal{B}_{j+1}) = \gamma_j[x_0, x_1, \ldots, x_j]$$

Strategies (Peres-Sheffield-Schram-Wilson)

A strategy S is a collection of mappings $\sigma_j : \mathfrak{X}^{j+1} \mapsto \mathfrak{X}$ indicating the next move $x_{j+1} = \sigma_j(x_0, x_1, \ldots, x_j)$ given the partial history $(x_0, x_1, \ldots, x_j)$.

A pair of strategies S_I and S_{II} chosen by each player, and a starting point determine a family of measures

$$\{\mathbb{P}^{x_0}_{S_I, S_{II}}\}_{x_0 \in \mathfrak{X}}$$

that describe the game played under this pair of strategies. Each pair of strategies (S_I, S_{II}), S_I for player I and S_{II} for player II determine a strategy σ and vice versa. We write

$$\sigma = (S_I, S_{II})$$

Assume that the game a. s.

$$\mathbb{P}_\sigma^{x_0}(\tau(x) < \infty) = 1,$$

Expected Pay-off

Average with respect to $\mathbb{P}_\sigma^{x_0}$ to obtain the expected pay-off for the Tug-of-War game starting at x_0

$$u_\sigma(x_0) = \mathbb{E}_\sigma^{x_0}[F(x_\tau)].$$

The Mean Value Property

The value function $u_\sigma(x)$ satisfies the mean value property

$$u_\sigma(x) = \frac{\alpha}{2}\left(u_{\sigma[x_I]}(x_I) + u_{\sigma[x_{II}]}(x_{II})\right) + \beta \int_{S(x_0)} u_{\sigma[y]}(y)\, d\mu(y)$$

The p-Laplacian Gambling House

For $y_0 \in S(x_0)$ the conditional strategy $\sigma[y_0]$ is
$\sigma[y_0] = (\gamma_1[x_0, y_0], \gamma_2[x_0, y_0, y_1], \ldots, \gamma_k[x_0, y_0, y_1 \ldots y_k], \ldots)$

$\mathbb{P}^{y_0}_{\sigma[y_0]}$ is the conditional distribution of the process $(x_2, x_3, \ldots)$ given that $x_1 = y_0$.

Let us stop and consider the particular case when $\alpha = 0$ and $\beta = 1$. In this case –the linear case– the strategies are irrelevant since $\Gamma(x)$ is always $\mu(x)$ so that there is only one family of measures $\{\mathbb{P}^{x_0}\}_{x_0 \in \mathfrak{X}}$. We recover the classical mean value formula

$$u(x) = \int_{S(x)} u(y)\, d\mu(y).$$

The p-Laplacian Gambling House

Value functions when $\alpha \neq 0$

$$u_I(x) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}_\sigma^x [F(x_\tau)]$$

$$u_{II}(x) = \inf_{S_{II}} \sup_{S_I} \mathbb{E}_\sigma^x [F(x_\tau)].$$

Player I lets Player II choose a strategy, presumably to decrease $\mathbb{E}_\sigma^{x_0}[F(x_\tau)]$, and then do as best a possible.

We always have

$u_I(x) \leq u_{II}(x)$ for all $x \in \mathfrak{X}$.

The value of the game

The game has a value when $u_I(x) = u_{II}(x)$ for all $x \in \mathfrak{X}$

The p-Laplacian Gambling House

It turns out that under quite general hypothesis u_I and u_{II} both satisfy the DPP.

DPP, Dynamic Programming Principle

$$u(x) = \frac{\alpha}{2}\left(\sup_{y\in S(x)} u(y) + \inf_{y\in S(x)} u(y)\right) + \beta \int_{S(x)} u(y)\, d\mu(y).$$

Therefore uniqueness for solutions of the DPP with given boundary values implies that $u_I = u_{II}$ so that the game has a value.
We will present two scenarios in which all the details above have been worked out.

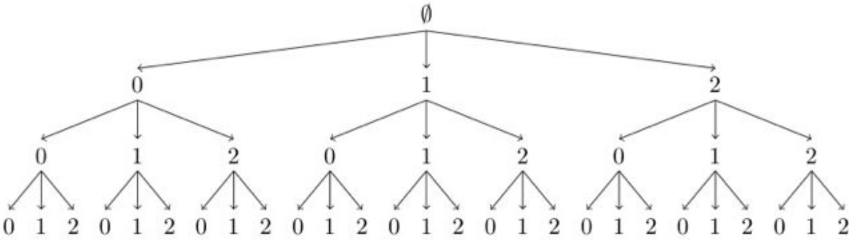
Example: Trees

A directed tree with regular 3-branching T consists of

- the empty set $\emptyset$,
- 3 sequences of length 1 with terms chosen from the set $\{0, 1, 2\}$,
- 9 sequences of length 2 with terms chosen from the set $\{0, 1, 2\}$,
- $\ldots$
- 3^r sequences of length r with terms chosen from the et $\{0, 1, 2\}$

and so on. The elements of T are called *vertices*.

Example: Trees



Calculus on Trees

Each vertex v al level r has three children (successors)

$$v_0, v_1, v_2.$$

Let $u\colon\ T \mapsto \mathbb{R}$ be a real valued function.

Gradient

The gradient of u at the vertex v is the vector in $\mathbb{R}^3$

$$\nabla u(v) = \{u(v_0) - u(v), u(v_1) - u(v), u(v_2) - u(v)\}.$$

Divergence

The averaging operator or *divergence* of a vector $X = (x, y, z) \in \mathbb{R}^3$ as

$$\mathrm{div}(X) = x + y + z.$$

Harmonic functions

A function u is harmonic if satisfies the Laplace equation

$$\mathrm{div}(\nabla u) = 0.$$

The Mean Value Property

A function u is harmonic if and only if it satisfies the mean value property

$$u(v) = \frac{u(v_0) + u(v_1) + u(v_2)}{3}.$$

Thus the values of harmonic function at level r determine its values at all levels smaller than r.

The boundary of the tree

Branches and boundary

A **branch** of T is an infinite sequence of vertices, each followed by on of its immediate successors (this corresponds to the level $r = \infty$.) The collection of all branches forms the boundary of the tree T is denoted by ∂T.

The mapping $g\colon \partial T \mapsto [0,1]$ given by

$$g(b) = \sum_{r=1}^{\infty} \frac{b_r}{3^r} \quad \text{(also denoted by } b\text{)}$$

is a bijection (think of an expansion in base 3 of the numbers in [0,1]).

The Dirichlet problem

• We have a natural metric and natural measure in ∂T inherited from the interval $[0, 1]$.

• The **classical Cantor set** C is the subset of ∂T formed by branches that don't go through any vertex labeled 1.

The Dirichlet problem

Given a (continuous) function $f : \partial T \mapsto \mathbb{R}$ find a harmonic function $u : T \mapsto \mathbb{R}$ such that

$$\lim_{r \to \infty} u(b_r) = f(b)$$

for every branch $b = (b_r) \in \partial T$.

Dirichlet problem, II

Given a vertex $v \in T$ consider the subset of ∂T consisting of all branches that start at v. This is always an interval that we denote by I_v.

Solution to the Dirichlet problem, $p = 2$

The we have

$$u(v) = \frac{1}{|I_v|} \int_{I_v} f(b)\, db.$$

Note that u is a *martingale*.

We see that we can in fact solve the Dirichlet problem for $f \in L^1([0,1])$.

Game interpretation

Random Walk

Start at the top $\emptyset$. Move downward by choosing successors at random with uniform probability. When you get at ∂T at the point b you get paid $f(b)$ dollars.

Two player random Tug-of-War game

A coin is tossed. The player who wins the coin toss chooses the successor vertex (heads for Henry , tails for Theresa.) The game *ends* when we reach ∂T at a point b in which case player II pays $f(b)$ dollars to player I.

More on Random Walk Game interpretation

Every time we run the game we get a sequence of vertices

$$v_1, v_2, \ldots, v_k, \ldots..$$

that determines a point on b the boundary ∂T.
If we are at vertex v_1 and run the game, player II pays $f(b)$ dollars to player I. Let us average out over all possible plays that start at v_1.

The value function is harmonic, $p = 2$.

$$\text{Expected pay-off} = \mathbb{E}^{v_1}[f(t)] = u(v_1) = \frac{1}{|I_{v_1}|} \int_{I_{v_1}} f(b)\, db.$$

Two player random Tug-of-War game, $p = \infty$

In this case, say that f is monotonically increasing. When Theresa moves she tries to move to the right. When Henry moves he moves to the left.These are examples of *strategies*.

Definition of Value functions

$$u^I(v) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}^v[f(b)] \quad \textit{and} \quad u^{II}(v) = \inf_{S_{II}} \sup_{S_I} \mathbb{E}^v[f(b)]$$

DPP (Dynamic Programming Principle)

We have $u^I = u^{II}$. Moreover, if we denote the common function by u, it is the only function on the tree such that:

$$u = f \text{ on } \partial T, \quad u(v) = \frac{1}{2}\left[\max_i\{u(v_i)\} + \min_i\{u(v_i)\}\right].$$

Random Walk + Tug-of-War

Let us combine random choice of successor plus tug of war. Choose $\alpha \geq 0$, $\beta \geq 0$ such that $\alpha + \beta = 1$. Start at $\emptyset$. With probability α the players play Tug-of-War. With probability β move downward by choosing successors at random. When you get at ∂T at the point b player II pays $f(b)$ dollars to player I.

DPP for Tug-of-War with noise, DPP = MVP

The value function u verifies the equation

$$u(v) = \frac{\alpha}{2}\left(\max_i\{u(v_i)\} + \min_i\{u(v_i)\}\right) + \beta\left(\frac{u(v_0) + u(v_1) + u(v_2)}{3}\right).$$

Where are the PDEs?

Setting

$$\mathrm{div}_\infty(X) = \max\{x, y, z\} + \min\{x, y, z\}$$

the value function u of the tug-of-war game satisfies

$$\mathrm{div}_\infty(\nabla u) = 0$$

Setting

$$\mathrm{div}_p(X) = \frac{\alpha}{2}\left(\max\{x, y, z\} + \min\{x, y, z\}\right) + \beta\left(\frac{x + y + z}{3}\right)$$

the value function u of the tug-of-war game with noise satisfies

$$\mathrm{div}_p(\nabla u) = 0.$$

This operator is the homogeneous p-Laplacian.

The (homogeneous) p-Laplacian on trees

The equations

$$\mathrm{div}_2(\nabla u) = 0, \quad \mathrm{div}_p(\nabla u) = 0, \quad \mathrm{div}_\infty(\nabla u) = 0$$

DPP for Tug-of-War with noise

$$u(v) = \frac{\alpha}{2}\left(\max_i\{u(v_i)\} + \min_i\{u(v_i)\}\right) + \beta\left(\frac{u(v_0) + u(v_1) + u(v_2)}{3}\right).$$

1. The case $p = 2$ corresponds to $\alpha = 0$, $\beta = 1$.
2. The case $p = \infty$ corresponds to $\alpha = 1$, $\beta = 0$.
3. In general, there is no explicit solution formula for $p \neq 2$

Formulas for f monotone, $p = \infty$

Suppose that f is monotonically increasing. In this case the best strategy $S_I^\star$ for Theresa is always to move right and the best strategy $S_{II}^\star$ for Henry always to move left. Starting at the vertex v_k at level k

$$v_k = 0.b_1 b_2 \ldots b_k, \qquad b_j \in \{0, 1, 2\}$$

we always move either left (adding a 0) or right (adding a 1). In this case I_v is a Cantor-like set $I_v = \{0.b_1 b_2 \ldots b_k d_1 d_2 \ldots\}$, $d_j \in \{0, 2\}$

Formula for $p = \infty$

$$u(v) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}^v_{S_I, S_{II}}[f(b)] = E^v_{S_I^\star, S_{II}^\star}[f(b)] = \int_{I_v} f(b) d\mathcal{C}_v(b)$$

Formulas for f monotone, $2 \leq p \leq \infty$

The best strategy $S_I^\star$ for Theresa is always to move right and the best strategy $S_{II}^\star$ for Henry always to move left.

Formula for $2 \leq p \leq \infty$

$$u(v) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}^v_{S_I,S_{II}}[f(b)] = E^v_{S_I^\star,S_{II}^\star}[f(b)]$$
$$= \fint_{I_v} f(b) d\mathbb{P}^{\alpha,\beta}(b),$$

for a certain probability measure $\mathbb{P}^{\alpha,\beta}$.

Example: Unique continuation does not hold in the discrete case for p=∞

-31	21	-11	-5	1	3	1	-5	11	-21	23
21	-5	5	-3	-1	1	-1	3	-5	1	21
-11	5	0	1	-1	0	1	-1	0	5	-11
-5	-3	1	0	0	0	0	0	1	-3	-5
3	-1	-1	0	0	0	0	0	-1	-1	3
1	1	0	0	0	0	0	0	0	1	1
3	-1	1	0	0	0	0	0	1	-1	3
-5	3	-1	0	0	0	0	0	-1	3	-5
11	-5	0	1	-1	0	1	-1	0	-5	11
-21	1	5	-3	-1	1	-1	3	-5	5	-21
23	21	-11	-5	1	3	1	-5	11	-21	31

Example: Unique continuation does not hold in the discrete case even for p=2

164	-349	80	163	1	-164	1	163	96	-617	74
-349	-52	-19	28	1	-20	1	28	-38	-9	596
80	-19	-4	1	1	-2	1	1	-1	35	-217
163	28	1	0	0	0	0	0	1	-26	-26
1	1	1	0	0	0	0	0	-2	1	1
-164	-20	-2	0	0	0	0	0	1	7	52
1	1	1	0	0	0	0	0	1	1	1
163	28	1	0	0	0	0	0	-2	1	-53
80	-19	-4	1	1	-2	1	1	-1	-19	80
-349	-52	-19	28	1	-20	1	28	-19	2	-160
164	-349	80	163	1	-164	1	163	77	-403	461

p-harmonious and p-harmonic functions in $\mathbb{R}^n$

1. Asymptotic Mean Value Properties for p-harmonic functions (the technique to connect discrete and continuos).
2. Definition, existence and uniqueness of p-harmonious functions.
3. Strong comparison principle for p-harmonious functions for $2 \leq p < \infty$.
4. Approximation of p-harmonic functions by p-harmonious functions.

1. Asymptotic mean-value properties for p-harmonic functions.

Let $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^N$. Consider the Taylor expansion:

$$u(x+h) = u(x) + \langle \nabla u(x), h \rangle + \frac{1}{2} \langle D^2 u(x) h, h \rangle + o(|h|^2), \text{ as } h \to 0.$$

Averaging on a ball $B_\epsilon(x) \subset \Omega$ we get:

$$\fint_{B_\epsilon(0)} u(x+h)\, dh = u(x) + \frac{1}{2(N+2)}\, \epsilon^2 \Delta(u)(x) + o(\epsilon^2), \text{ as } \epsilon \to 0.$$

Lemma

$u \in C^2(\Omega)$ is harmonic in Ω if and only if for all $x \in \Omega$

$$\fint_{B_\epsilon(0)} u(x+h)\, dh = u(x) + o(\epsilon^2), \text{ as } \epsilon \to 0.$$

The case $p = 2$:

Since viscosity harmonic functions are harmonic in the classical sense, we indeed have:

Lemma

$u \in C(\Omega)$ is harmonic in Ω if and only if for all $x \in \Omega$

$$\fint_{B_\epsilon(0)} u(x+h)\, dh = u(x) + o(\epsilon^2), \text{ as } \epsilon \to 0$$

The case $p = \infty$, $\nabla u(x) \neq 0$

Let $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^N$. In the Taylor expansion, use

$$h = \epsilon \frac{\nabla u(x)}{|\nabla u(x)|} \quad \text{and} \quad h = -\epsilon \frac{\nabla u(x)}{|\nabla u(x)|},$$

add, and compute to get:

$$\frac{1}{2}\left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u \right) = u(x) + \epsilon^2 \Delta_\infty u(x) + o(\epsilon^2) \text{ as } \epsilon \to 0,$$

where

$$\Delta_\infty u(x) = \frac{1}{|\nabla u(x)|^2} \langle D^2 u(x) \nabla u(x), \nabla u(x) \rangle$$

is the *homogeneous* ∞-Laplacian.

The case $p = \infty$, $\nabla u(x) \neq 0$

Lemma

$u \in C^2(\Omega)$, $\nabla u(x) \neq 0$, is ∞-harmonic in Ω if and only if for all $x \in \Omega$

$$\frac{1}{2}\left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u \right) = u(x) + o(\epsilon^2) \text{ as } \epsilon \to 0.$$

Lemma

Let $u \in C(\Omega)$ be just continuous. Suppose that for all $x \in \Omega$ we have

$$\frac{1}{2}\left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u \right) = u(x) + o(\epsilon^2) \text{ as } \epsilon \to 0,$$

then u is ∞-harmonic in Ω.

The case $p = \infty$, $\nabla u(x) \neq 0$

The converse to the previous lemma does not hold.

Example: Aronsson's function near $(x, y) = (1, 0)$

$$u(x, y) = |x|^{4/3} - |y|^{4/3}$$

Aronsson's function is ∞-harmonic in the viscosity sense but it is not of class C^2. A calculation shows that

$$\lim_{\varepsilon \to 0+} \frac{\frac{1}{2}\left\{ \max\limits_{B_\varepsilon(1,0)} u + \min\limits_{B_\varepsilon(1,0)} u \right\} - u(1,0)}{\varepsilon^2} = \frac{1}{18}.$$

But if an asymptotic expansion held in the classical sense, this limit would have to be zero.

The case $1 < p < \infty$, $\nabla u(x) \neq 0$

Let $u \in C^2(\Omega)$ and α, β non-negative such that $\alpha + \beta = 1$.

$$\begin{aligned}\frac{\alpha}{2}\left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u\right) + \beta \fint_{B_\epsilon(x)} u &= u(x) \\ &+ \alpha\Delta_\infty u(x) + \beta\tfrac{1}{(N+2)}\Delta u(x) \\ &+ o(\epsilon^2), \qquad \text{as } \epsilon \to 0,\end{aligned}$$

Let us rewrite the second order operator

$$\alpha\Delta_\infty u(x) + \beta\frac{1}{(N+2)}\Delta u(x) = \beta\frac{1}{(N+2)}\left(\Delta u(x) + \frac{\alpha}{\beta\frac{1}{(N+2)}}\Delta_\infty u(x)\right).$$

The case $1 < p < \infty$, $\nabla u(x) \neq 0$

Next, choose $2 < p < \infty$ such that

$$p - 2 = \frac{\alpha}{\beta \frac{1}{(N+2)}}.$$

We then have

$$\Delta u(x) + \frac{\alpha}{\beta \frac{1}{(N+2)}} \Delta_\infty u(x) = |\nabla u(x)|^{2-p} \mathrm{div} \left(|\nabla u(x)|^{p-2} \nabla u(x) \right).$$

Lemma

$u \in C^2(\Omega)$, $\nabla u(x) \neq 0$, is p-harmonic in Ω if and only if for all $x \in \Omega$

$$\frac{\alpha}{2} \left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u \right) + \beta \fint_{B_\epsilon(x)} u = u(x) + o(\epsilon^2), \qquad \text{as } \epsilon \to 0,$$

The case $1 < p < \infty$

Lemma

Let be $u \in C(\Omega)$. Suppose that for all $x \in \Omega$ we have

$$\frac{\alpha}{2}\left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u\right) + \beta \fint_{B_\epsilon(x)} u = u(x) + o(\epsilon^2), \qquad \text{as } \epsilon \to 0,$$

where $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + \beta = 1$ and

$$\frac{p-2}{N+2} = \frac{\alpha}{\beta},$$

then u is p-harmonic in Ω

Question: Can we modify these lemmas so that they **characterize** p-harmonic functions?

The case $1 < p \leq \infty$

Theorem

$u \in C(\Omega)$ is p-harmonic in Ω if and only if for all $x \in \Omega$ we have that the asymptotic expansion

$$\frac{\alpha}{2}\left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u\right) + \beta \fint_{B_\epsilon(x)} u = u(x) + o(\epsilon^2), \qquad \text{as } \epsilon \to 0,$$

holds in the **VISCOSITY SENSE**, where $\alpha + \beta = 1$ and

$$\frac{p-2}{N+2} = \frac{\alpha}{\beta}.$$

Note that we don't require $\alpha \geq 0$.

2. Definition, $1 < p < \infty$ ($p = \infty$ Le Gruyer)

Let Ω be a (bounded) domain in $\mathbb{R}^N$ and consider

$$\Gamma_\epsilon = \{x \in \mathbb{R}^N \setminus \Omega \;:\; \mathrm{dist}(x, \partial\Omega) \leq \epsilon\}, \qquad \Omega_\varepsilon = \Omega \cup \Gamma_\varepsilon$$

The function u_ε is p-harmonious in Ω with continuous boundary values $F : \Gamma_\varepsilon \to \mathbb{R}$ if $u_\varepsilon(x) = F(x), \;\; x \in \Gamma_\varepsilon$ and

$$u_\varepsilon(x) = \frac{\alpha}{2}\left\{\sup_{\overline{B}_\varepsilon(x)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x)} u_\varepsilon\right\} + \beta \fint_{B_\varepsilon(x)} u_\varepsilon \, dy \quad \text{for every } x \in \Omega,$$

where

$$\alpha = \frac{p-2}{p+N}, \quad \text{and} \quad \beta = \frac{2+N}{p+N}.$$

WARNING! Solutions to this equation may be discontinuous as 1-d examples show.

Tug-of-War Games with Noise $2 \leq p < \infty$

Fix $1 > \alpha \geq 0$, $\beta > 0$ such that $\alpha + \beta = 1$.
Fix $\varepsilon > 0$ and place a token at starting point $x_0 \in \Omega$. Move the token to the next state x_1 as follows:

- With probability α play tug-of war: a fair coin is tossed and the winner of the toss moves the token to any $x_1 \in \overline{B}_\varepsilon(x_0)$.
- With probability β the token moves according to a uniform probability density to a random point in the ball $\overline{B}_\varepsilon(x_0)$.

This procedure yields an infinite sequence of game states $x_0, x_1, \ldots$ where every x_k, except x_0, is a random variable.

Tug-of-War Games with Noise

- A run of the game is $\mathbf{x} = (x_0, x_1, \ldots, x_k, \ldots)$, where $\mathbf{x}(k) = x_k$.
- The game stops the first time it hits Γ_ε. Write

$$\tau(\mathbf{x}) = \min\{k \colon x_k \in \Gamma_\varepsilon\}.$$

  The random variable τ is a STOPPING TIME. We write

$$\mathbf{x}(\tau(\mathbf{x})) = x_\tau.$$

- $F : \Gamma_\varepsilon \to \mathbb{R}$ is a given (Lipschitz, bounded) *payoff function*. The game payoff is $F(\mathbf{x}) = F(x_\tau)$.
- Player I earns \$ $F(x_\tau)$ while Player II earns \$ $-F(x_\tau)$.

Tug-of-War Games with Noise

- Fix strategies S_I and S_{II} for players I and II respectively.
- Start the game at x_0.
- The probability measure $\mathbb{P}^{x_0}_{S_I,S_{II}}$ is defined on the set of all game histories $H \subset \Omega_\varepsilon^\infty$ by the transition probabilities

$$\pi_{S_I,S_{II}}(x_0,\ldots,x_k;A) = \frac{\alpha}{2}\left(\delta_{S_I(x_0,\ldots,x_k)}(A) + \delta_{S_{II}(x_0,\ldots,x_k)}(A)\right) + \beta\frac{\left|A\cap\overline{B}_\varepsilon(x_k)\right|}{\left|\overline{B}_\varepsilon(x_k)\right|}$$

and Kolmogorov's extension theorem.

Tug-of-War Games with Noise, $2 \leq p < \infty$

Games end almost surely

$\mathbb{P}^x_{S_I,S_{II}}(H) = 1$ because $\beta > 0$.

Value of the game for player I

$$u^\varepsilon_I(x) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}^x_{S_I,S_{II}}[F(x_\tau)]$$

Value of the game for player II

$$u^\varepsilon_{II}(x) = \inf_{S_{II}} \sup_{S_I} \mathbb{E}^x_{S_I,S_{II}}[F(x_\tau)]$$

Comparison Principle

$$u^\varepsilon_I(x) \leq u^\varepsilon_{II}(x)$$

THEOREM

The value functions u_I^ε and u_{II}^ε are p-harmonious. They satisfy the equation

$$u(x) = \frac{\alpha}{2}\left\{\sup_{\overline{B}_\varepsilon(x)} u + \inf_{\overline{B}_\varepsilon(x)} u\right\} + \beta \fint_{B_\varepsilon(x)} u(y)\,dy, \qquad x \in \Omega,$$
$$u(x) = F(x), \qquad x \in \Gamma_\varepsilon.$$

(In the case $p = \infty$ Le Gruyer showed that the mapping

$$T(u) = \frac{1}{2}\left\{\sup_{\overline{B}_\varepsilon(x)} u + \inf_{\overline{B}_\varepsilon(x)} u\right\}$$

has a fixed point.)

Comparison I

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set.

- If v_ε is a p-harmonious function with boundary values F_v in Γ_ε such that $F_v(y) \geq u_I^\varepsilon(y)$ for $y \in \Gamma_\varepsilon$, then $v_\varepsilon(x) \geq u_I^\varepsilon(x)$ for $x \in \Omega_\varepsilon$.
- If v_ε is a p-harmonious function with boundary values F_v in Γ_ε such that $F_v(y) \leq u_I^\varepsilon(y)$ for $y \in \Gamma_\varepsilon$, then $v_\varepsilon(x) \leq u_{II}^\varepsilon(x)$ for $x \in \Omega_\varepsilon$.

That is u_I^ε is the smallest p-harmonious function with given boundary values and u_{II}^ε is the largest p-harmonious function with given boundary values.

Comparison I, Proof

Player I arbitrary strategy S_I, player II strategy S_{II}^0 that almost minimizes v_ε, $\quad v_\varepsilon(x_k) \leq \inf_{y\in\overline{B}_\varepsilon(x_{k-1})} v_\varepsilon(y) + \eta 2^{-k}$

Key Point

$$M_k = v_\varepsilon(x_k) + \eta 2^{-k}$$

is a supermartingale for any $\eta > 0$.

$$\begin{aligned}
\mathbb{E}^{x_0}_{S_I,S_{II}^0}[M_k \mid x_0,\dots,x_{k-1}] &= \mathbb{E}^{x_0}_{S_I,S_{II}^0}[v^\varepsilon(x_k) + \eta 2^{-k} \mid x_0,\dots,x_{k-1}] \\
&\leq \frac{\alpha}{2}\left\{ \inf_{y\in\overline{B}_\varepsilon(x_{k-1})} v^\varepsilon(y) + \sup_{y\in\overline{B}_\varepsilon(x_{k-1})} v^\varepsilon(y) + \eta 2^{-k} \right\} \\
&+ \beta \fint_{B_\varepsilon(x_{k-1})} v^\varepsilon\, dy + \eta 2^{-k} \leq v^\varepsilon(x_{k-1}) + \eta 2^{-(k-1)} = M_{k-1}
\end{aligned}$$

Comparison I, Proof

By optimal stopping

$$
\begin{aligned}
u_I^\varepsilon(x_0) &= \sup_{S_I}\inf_{S_{II}} \mathbb{E}^{x}_{S_I,S_{II}}[F(x_\tau)] \\
&\le \sup_{S_I} \mathbb{E}^{x}_{S_I,S_{II}^0}[F(x_\tau)] \\
&\le \sup_{S_I} \mathbb{E}^{x}_{S_I,S_{II}^0}[v_\varepsilon(x_\tau)] \\
&\le \sup_{S_I} \mathbb{E}^{x_0}_{S_I,S_{II}^0}[v_\varepsilon(x_\tau) + \eta 2^{-\tau}] \\
&\le \sup_{S_I} \mathbb{E}^{x_0}_{S_I,S_{II}^0}[M_\tau] \\
\text{(by optimal stopping)} &\le \sup_{S_I} M_0 = v^\varepsilon(x_0) + \eta
\end{aligned}
$$

The game has a value

Theorem

$M_k = u_I^\varepsilon(x_k) + \eta 2^{-k}$ is a supermartingale.
We have $u_I^\varepsilon = u_{II}^\varepsilon$

The proof is a variant of the proof of comparison.
Player II follows a strategy S_{II}^0 such that at $x_{k-1} \in \Omega_\varepsilon$, he always chooses to step to a point that almost minimizes u_I^ε; that is, to a point x_k such that

$$u_I^\varepsilon(x_k) \leq \inf_{y \in \overline{B}_\varepsilon(x_{k-1})} u_I^\varepsilon(y) + \eta 2^{-k}$$

3. Maximum and Comparison Principles

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. If u_ε is p-harmonious in Ω with a boundary data F, then $\sup_{\Gamma_\varepsilon} F \geq \sup_\Omega u_\varepsilon$. Moreover, if there is a point $x_0 \in \Omega$ such that $u_\varepsilon(x_0) = \sup_{\Gamma_\varepsilon} F$, then u_ε is constant in Ω.

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. and let u_ε and v_ε be p-harmonious with boundary data $F_u \geq F_v$ in Γ_ε. Then if there exists a point $x_0 \in \Omega$ such that $u_\varepsilon(x_0) = v_\varepsilon(x_0)$, it follows that $u_\varepsilon = v_\varepsilon \quad \text{in} \quad \Omega$, and, moreover, the boundary values satisfy $F_u = F_v \quad \text{in} \quad \Gamma_\varepsilon$.

Proof of Strong Comparison

The proof uses the fact that $p < \infty$. The strong comparison principle **does not hold** for $p = \infty$.

$$F_u \geq F_v \implies u_\varepsilon \geq v_\varepsilon .$$

We have

$$u_\varepsilon(x_0) = \frac{\alpha}{2}\left\{\sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon\right\} + \beta \fint_{B_\varepsilon(x_0)} u_\varepsilon \, dy$$

and

$$v_\varepsilon(x_0) = \frac{\alpha}{2}\left\{\sup_{\overline{B}_\varepsilon(x_0)} v_\varepsilon + \inf_{\overline{B}_\varepsilon(x_0)} v_\varepsilon\right\} + \beta \fint_{B_\varepsilon(x_0)} v_\varepsilon \, dy .$$

Next we compare the right hand sides. Because $u_\varepsilon \geq v_\varepsilon$, it follows that

Proof of Strong Comparison, II

$$\sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon - \sup_{\overline{B}_\varepsilon(x_0)} v_\varepsilon \geq 0,$$

$$\inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon - \inf_{\overline{B}_\varepsilon(x_0)} v_\varepsilon \geq 0, \quad \text{and}$$

$$\fint_{B_\varepsilon(x_0)} u_\varepsilon \, dy - \fint_{B_\varepsilon(x_0)} v_\varepsilon \, dy \geq 0$$

But since

$$u_\varepsilon(x_0) = v_\varepsilon(x_0),$$

and $\beta > 0$ must have $u_\varepsilon = v_\varepsilon$ almost everywhere in $B_\varepsilon(x_0)$. In particular,

$$F_u = F_v \quad \textbf{everywhere in} \quad \Gamma_\varepsilon$$

since F_u and F_v are continuous. By uniqueness $u_\varepsilon = v_\varepsilon$ everywhere in Ω.

4. Approximation of p-harmonic functions

Boundary Regularity Assumption

Ω bounded domain in $\mathbb{R}^n$ satisfying an exterior sphere condition: For each $y \in \partial\Omega$, there exists $B_\delta(z) \subset \mathbb{R}^n \setminus \Omega$ such that $y \in \partial B_\delta(z)$. $R > 0$ is chosen so that we always have $\Omega \subset B_{R/2}(z)$.

THEOREM

F is Lipschitz in Γ_ε for small $0 < \varepsilon < \varepsilon_0$. Let u be the unique viscosity solution to

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u)(x) = 0, & x \in \Omega \\ u(x) = F(x), & x \in \partial\Omega, \end{cases}$$

and let u_ε be the unique p-harmonious function with boundary data F in Γ_ε, then $u_\varepsilon \to u$ uniformly in Ω as $\varepsilon \to 0$.

Questions

1. What about $1 < p < 2$?

2. Is there a parabolic analogue?

3. What about $p = 1$?