Motion by curvature of planar curves with two free end points

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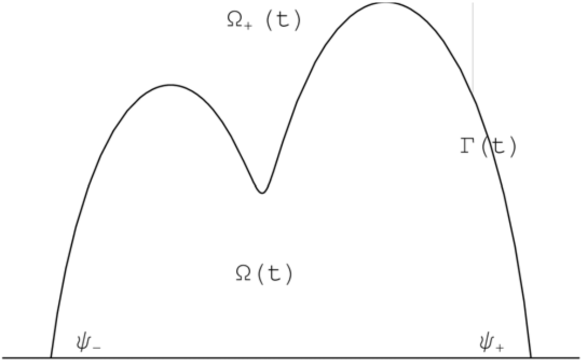
Outline

Problem and motivation

Problem (P): Given an initial curve $\Gamma(0)$, find a family of curves $\{\Gamma(t)\}_{0<t<T}$ that lie on the upper-half plane, have end points on the x-axis with contact angle ψ_- on the left and ψ_+ on the right, and evolve according to the curvature flow.

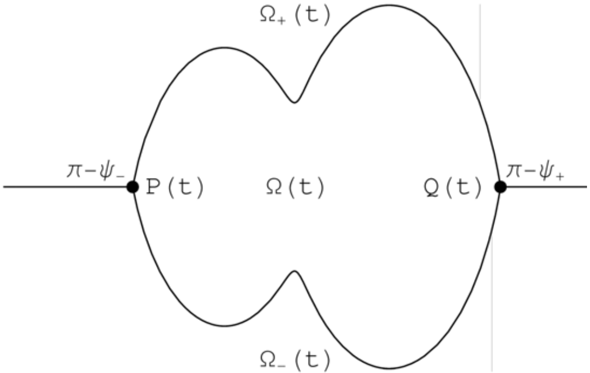
- Curvature flow: $V(\text{normal velocity}) = \kappa(\text{curvature})$.
- This problem arises in the study of the evolution of three grain domains in polycrystals.


Ω+ (t)
Γ(t)
Ω(t)
ψ−
ψ+

- A grain domain is a periodic lattice structure of composite particles of a crystal.
- A grain boundary is the intersection of two grain domains at which orientations of different lattices do not match.
- Grain boundaries are often modelled by the curvature flows.

- A triple junction is the meeting place of three grain domains.
- In general, it is assumed that the three intersection angles (φ_j) at a triple junction and the three interfacial energy densities (σ_j) satisfy the Herring condition, i.e.,
  the sine law of a triangle with φ_j as exterior angles and σ_j as the side lengths.

- The evolution of grain boundaries makes a network of grains topologically simpler and simpler by diminishing of grains. To movie: grains.avi!!
- Our problem is the simplest situation in which two triple junctions are assumed to move only along a straight line; the diminishing grain domain is assumed to be symmetric w.r.t. this straight line and is surrounded by two other grain domains with their grain boundaries on the straight line.


$\Omega_+(t)$
$\pi-\psi_-$
$P(t)$
$\Omega(t)$
$Q(t)$
$\pi-\psi_+$
$\Omega_-(t)$

Three Formulations of (P)

1. Motion of Particles

We can regard $\Gamma(0)$ as the union of the positions of a collection of particles so that $\Gamma(0) = \{(x^0(z), y^0(z)) \mid 0 \leqslant z \leqslant 1\}$ with

$|x_z^0(z)| + |y_z^0(z)| > 0$ for all $z \in [0,1]$,

$y^0(0) = y^0(1) = 0$,

$x_z^0(0) = y_z^0(0) \cot \psi_-$,

$x_z^0(1) = -y_z^0(1) \cot \psi_+$.

Let

$$\Gamma(t) := \{(x(z,t), y(z,t)) \mid z \in [0,1]\}.$$

(P): Find $X = (x, y)$ such that

$$
\begin{cases}
x_t = \dfrac{x_{zz}}{x_z^2 + y_z^2},\ y_t = \dfrac{y_{zz}}{x_z^2 + y_z^2}, z \in (0,1), t \in (0,T), \\
y(0,t) = 0,\ y(1,t) = 0,\ t \in [0,T), \\
x_z(0,t) = y_z(0,t) \cot \psi_-,\ t \in [0,T), \\
x_z(1,t) = -y_z(1,t) \cot \psi_+,\ t \in [0,T), \\
x(z,0) = x^0(z),\ y(z,0) = y^0(z),\ z \in [0,1].
\end{cases} \tag{1}
$$

- This is the most general formulation.

2. Polar Coordinates Formulation

If we can find a reference point $x_0 + 0\mathbf{i} \in \mathbb{C} = \mathbb{R}^2$ such that $\{\Gamma(t)\}$ can be expressed as

$$\Gamma(t) = \{x_0 + R(\varsigma, t)e^{\mathbf{i}\varsigma} \mid 0 \leqslant \varsigma \leqslant \pi\}.$$

Then problem (P) can be expressed as

$$\begin{cases} R_t = \dfrac{RR_{\varsigma\varsigma} - 2R_\varsigma^2 - R^2}{R(R^2 + R_\varsigma^2)}, \ \varsigma \in (0, \pi), t \in (0, T), \\ R_\varsigma(0, t) = -R(0, t) \cot \psi_+, \ \ t \in [0, T) \\ R_\varsigma(\pi, t) = R(\pi, t) \cot \psi_-, \ \ t \in [0, T), \\ R(\varsigma, 0) = R^0(\varsigma), \varsigma \in [0, \pi]. \end{cases} \tag{2}$$

3. Evolution of A Graph

When $\psi_\pm \in (0, \pi/2)$ and $\Gamma(0)$ is a graph $y = u^0(x)$, $x \in [l_-^0, l_+^0]$, one can expect that $\Gamma(t)$ is also a graph given by

$$y = u(x,t),\ x \in [l_-(t), l_+(t)].$$

(P): find unknowns u and $\{l_\pm(t)\}$ such that

$$\begin{cases} u_t = (\arctan u_x)_x,\ x \in (l_-(t), l_+(t)),\ t \in (0,T), \\ u(l_\pm(t), t) = 0,\ t \in [0,T), \\ u_x(l_\pm(t), t) = \mp \tan \psi_\pm,\ t \in [0,T), \\ u(x,0) = u^0(x),\ x \in [l_-(0), l_+(0)] := [l_-^0, l_+^0]. \end{cases} \tag{3}$$

- It is a free boundary problem for a scalar equation.

Studies of Polycrystal

- Physical background: Mullins (1956, 1988), Kobayashi-Warren-Carter (1993, 1998, 2000), group of Adams (1998, 1999, 2001), Kinderlehrer-Liu (2001)
- Books: Woodruff (1973), Brakke (1978), Gurtin (1993), Ilmanen (1994), Sutton-Baluff (1995), Giga (2006)
- Mathematical studies: Angenent-Gurtin (ARMA, 1989), Bronsard-Retich (ARMA, 1993), Mantegazza-Novaga-Tortorelli (Pisa, 2004)

Known Results

- Chen-G.: Self-similar solutions of a grain domain surrounded by any number of grain domains (PhyD, 2007)

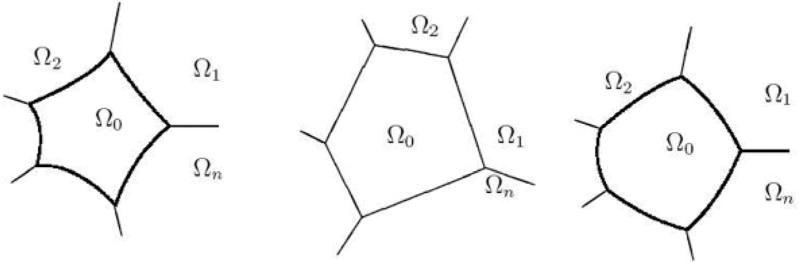
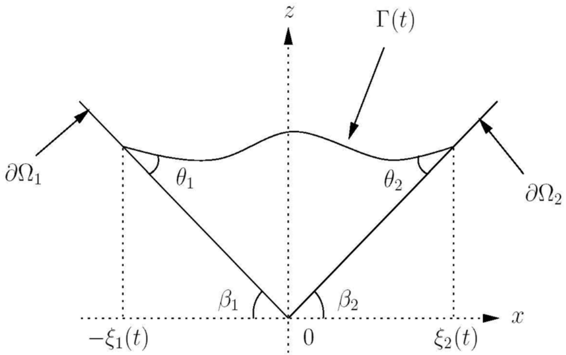


FIGURE 2. Left: self-similar expanding; middle: stationary; right: self-similar shrinking.

- Berlin group led by Oliver Schnürer: (P) with convex curve (to appear in TAMS)
- Bellettini-Novaga: (P) with nonconvex curve and $\psi_{\pm} = \pi/3$ (to appear in JRAM)
- When no triple junctions are involved, it is reduced to the curvature flow of a simple closed curve.
  See Gage-Hamilton (JDG, 1986), Grayson (JDG, 1987), also, Huisken (AJM, 1998).

Known Results for FBP

Study the free boundary formulation in a conical domain with open angle $\pi - (\beta_1 + \beta_2)$, $\beta_i \geq 0$, strictly less than π.



- Set $\varphi_1 := \theta_1 - \beta_1, \varphi_2 := \beta_2 - \theta_2$.
- Chang-G.-Kohsaka (AA, 2003) - expanding case ($\varphi_1 < \varphi_2$).
- G.-Hu (QAM, 2006) - area-preserving case ($\varphi_1 = \varphi_2$); shrinking case ($\varphi_1 > \varphi_2$)
- Global or non-global existence and uniqueness of solutions are established. Moreover, the asymptotic behaviors, as $t \to T^-$, $T \leq \infty$, are also studied.

Existence and Uniqueness of Solution

Theorem 1

Let $\psi_+, \psi_- \in (0,\pi)$ and assume that $\Gamma(0) \in C^{1+\alpha}$ for some $\alpha \in (0,1)$. Then there exists a positive T such that (1) admits a unique solution

$$(x,y) \in C^\infty([0,1]\times(0,T)) \cap C^{1+\alpha,(1+\alpha)/2}([0,1]\times[0,T)),$$

and T is the time of blow-up of curvature:

$$\lim_{t\nearrow T} \|\kappa\|_{L^\infty(\Gamma(t))} = \infty.$$

- Apply a fixed-point argument to the particle formulation.

Geometric Properties

Theorem 2

Assume that $\psi_\pm > 0$, $\psi_+ + \psi_- \leqslant \pi$, *and* $\Gamma(0)$ *is a simple curve whose interior lies in the upper-half plane. Then for each* $t \in (0, T)$, $x(0,t) < x(1,t)$ *and* $\Gamma(t)$ *is also a simple curve with interior lying in the upper-half plane. In addition, the area* $A(t)$ *of the region bounded by* $\Gamma(t)$ *and the* x*-axis is given by*

$$A(t) = A(0) - [\psi_- + \psi_+]t \quad \forall t \in [0, T),$$

so that $T \leqslant T_{\max} := A(0)/(\psi_+ + \psi_-)$.

Self-similar Solution

Theorem 3

Assume that $\psi_\pm \in (0, \pi/2]$. Then there exists a unique self-similar shrinking solution.

Recall the formulation in polar coordinates (2) and set

$$\varphi := \operatorname{arccot} \frac{R_\varsigma}{R}.$$

Then the problem can be written as

$$\begin{cases} RR_t = -1 - \varphi_\zeta, R_\varsigma = R\cot\varphi, \varsigma \in (0,\pi), t \in (0,T), \\ \varphi(0,t) = \pi - \psi_+, \varphi(\pi,t) = \psi_-, t \in [0,T), \\ R(\varsigma,0) = R^0(\varsigma),\ \varsigma \in [0,\pi]. \end{cases}$$

Proof of Theorem 3

A self-similar shrinking solution can be put in the form

$$R(\varsigma,t)=\sqrt{2(T-t)\rho(\varsigma)},$$
$$\varphi(\varsigma,t)=\psi(\varsigma),\ \forall\,\varsigma\in[0,\pi],t\in[0,T).$$

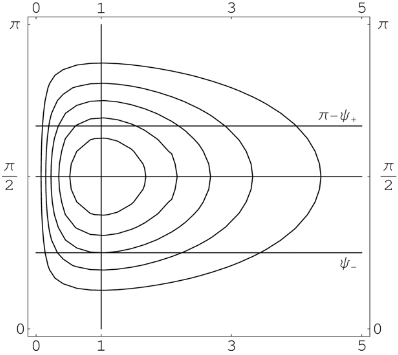
This reduces to to solve the ODE system

$$\rho'=2\rho\cot\psi,\quad \rho>0,\quad \psi'=\rho-1\ \text{ in }\ [0,\pi], \tag{4}$$

subject to the boundary conditions

$$\psi(0)=\pi-\psi_+,\qquad \psi(\pi)=\psi_-. \tag{5}$$

- First integral $\leftrightarrow$ $[\ln \sin^2 \psi + \ln \rho - \rho]' = 0$
- A generic trajectory to (4) is given by $e^{\rho - 1}/\rho = c \sin^2 \psi$ for some constant $c \geq 1 \leftrightarrow \gamma(c)$ (counterclockwise)



- Let $B_1(c,\varphi)$ ($B_2(c,\varphi)$) be the left (right) intersection point of $\psi=\varphi$ with $\gamma(c)$.
- Let $\ell_1(c,\varphi)$ ($\ell_2(c,\varphi)$) be the "time" spent on $\gamma(c)$ from the leftmost point of $\gamma(c)$ to $B_1(c,\varphi)$ ($B_2(c,\varphi)$).
- Key Idea: To evaluate the "time" spent on the trajectory $\gamma(c)$ so that $\ell_i(c,\psi_+)+\ell_j(c,\psi_-)=\pi$, $i,j\in\{1,2\}$.
- For every $\varphi\in(0,\pi/2]$, we have $\ell_2(\infty,\varphi)=\pi/2$, $d\ell_2(c,\varphi)/dc<0$, $\forall\, c\gg 1$, and

$$\frac{d}{dc}\Big(c(c-1)\frac{d}{dc}\ell_2(c,\varphi)\Big)<0,\ \forall\, c>\frac{1}{\sin^2\varphi}.$$

- Let $\omega(c)$ be the whole period of $\gamma(c)$. Then

$$\omega(c) = 2\ell_2(c, \pi/2).$$

- Let $\omega_1(c)$ be the "time" spent on $\gamma(c)$ from the leftmost point to the bottom point. Then we have

$$\lim_{c \searrow 1} \omega(c) = \sqrt{2}\,\pi, \;\; \lim_{c \nearrow \infty} \omega(c) = \pi,$$
$$\lim_{c \searrow 1} \omega_1(c) = \sqrt{2}\,\pi/4, \;\; \lim_{c \nearrow \infty} \omega_1(c) = \pi/2,$$
$$\omega'(c) < 0, \; \omega_1'(c) > 0, \; \forall\, c \in (1, \infty).$$

Asymptotic Behavior

Theorem 4

Assume that $\Gamma(0)$ *is a graph and* $0 < \psi_\pm < \pi/2$ *such that* u^0 *satisfies*

$$u^0 \in C^\infty([l_-^0, l_+^0]),\ u^0(l_\pm^0) = 0,$$
$$u(\cdot) > 0 \text{ in } (l_-^0, l_+^0),\ \mp u_x^0(l_\pm^0) = \gamma_\pm > 0.$$

Then (3) *admits a unique solution with* $T = T_{\max}$, *and as* $t \nearrow T$, $\Gamma(t)$ *shrinks to a point in a self-similar manner.*

- Using the standard blow-up technique in parabolic problem

Proof of Theorem 4

- For every $t \in [0, T)$ and $x \in [l_-(t), l_+(t)]$,

$$|u_x(x,t)| \leqslant M, \ u_t(x,t) \leqslant M, \ u_{xx} \leqslant M$$

  for some positive constant M.
- There exists a constant C that depends only on u^0 such that

$$u_t(x,t) \geqslant -\frac{Ch(0)}{h(t)}, \ u_{xx}(x,t) \geqslant -\frac{Ch(0)}{h(t)},$$

  where $h(t) := \max_{l_-(t)<x<l_+(t)} u(x,t)$.

- There exists $t_* \in [0, T)$ and $\xi \in C^1([t_*, T))$ such that for each $t \in [t_*, T)$,

$$u_x(\cdot, t) > 0 \text{ in } [l_-(t), \xi(t)),$$
$$u_x(\xi(t), t) = 0 > u_{xx}(\xi(t), t),$$
$$u_x(\cdot, t) < 0 \text{ in } (\xi(t), l_+(t)].$$

In addition, $\dot{l}_-(t) > 0$, $\dot{l}_+(t) < 0$ for all $t \in [t_*, T)$.

- There exists a constant M and a time $t_* \in [0, T)$ such that

$$u_{xx}(x, t) \leqslant Mu(x, t), \; \forall x \in (l_-(t), l_+(t)), \; t \in [t_*, T).$$

- Key Estimate: there exists a constant $C > 0$ such that

$$\sqrt{T-t} \leqslant C\ell(t) \leqslant C^2 h(t) \leqslant C^3\sqrt{T-t}$$

for all $t \in [0,T)$, where $\ell(t) := l_+(t) - l_-(t)$.

- By translation, we may assume that

$$l_\pm(T) := \lim_{t\nearrow T} l_\pm(T) = 0.$$

Since $\dot{l}_+(t) < 0 < \dot{l}_-(t)$ for $t \in [t_*, T]$, we have

$$|l_\pm(t)| < \ell(t) \leqslant C\sqrt{T-t} \quad \forall t \in [t_*, T).$$

We make the change of dependent and independent variables:

$$
\begin{aligned}
& z = \frac{x}{\sqrt{2(T-t)}}, \qquad s = -\ln\sqrt{2(T-t)}, \\
& U(z,s) := u(x,t)/\sqrt{2(T-t)}, \\
& L_\pm(s) = l_\pm(t)/\sqrt{2(T-t)}.
\end{aligned}
$$

Set $s_0 = -\ln\sqrt{2T}$. Then the functions $(U, L_\pm)$ satisfies

$$
\begin{cases} U_s = [a(U_z)]_z - zU_z + U, \ z \in (L_-(s), L_+(s)), \ s > s_0, \\ U(L_\pm(s), s) = 0, \ U_z(L_\pm(s), s) = \mp\gamma_\pm, \ s > s_0, \end{cases}
$$

where $a(s) = \arctan(s)$ and $\gamma_\pm = \tan\psi_\pm$.

Conclusion

1. We study the motion by curvature of planar curves having end points moving freely along a line with fixed contact angles to this line.
2. We first prove the existence and uniqueness of self-similar shrinking solution.
3. Then we show that the curve shrinks to a point in a self-similar manner, if initially the curve is a graph.
4. Some Difficulties of Problem (P):
   - Need to eliminate needle-type singularity of solution.
   - Existence and uniqueness of self-similar shrinking solution is highly nontrivial.