

# **Scale-Invariant Extinction Time Estimates for some Singular Diffusion Equations**

Yoshikazu Giga

Graduate School of Mathematical Sciences  
University of Tokyo

Tokyo, Japan

Joint work with R.V. Kohn (Courant Institute)

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# Introduction

Three examples of singular diffusion equations

(1) Second order total variation flow

$$u_t = \operatorname{div}(\nabla u / |\nabla u|)$$

$$L^2 - \text{gradient flow} (u_t = -\operatorname{grad}_{L^2} \Phi)$$

of the total variation

$$\Phi(u) = \int |\nabla u|.$$

(2) Fourth order total variation flow

$$u_t = -\Delta \operatorname{div}(\nabla u / |\nabla u|)$$

$H^{-1}$  gradient flow of total variation

(3) Fourth order surface diffusion law

$H^{-1}$  gradient flow of

$$\Phi^q(u) = \Phi(u) + \frac{\nu}{q} \int |\nabla u|^q dx$$

for  $q > 1$ , and  $\nu > 0$ .

# Sources of problems: Image Processing/ Crystal Growth

- (1) Rudin-Osher-Fathemi(1992): denoising image  
Angenent – Gurtin(1989),  
Taylor(1991) : crystalline flow  
(M.-H. Giga – Y.G.(1998-), Bellettini et al ....  
(1999-) R. Kobayashi- Carter – Warren(2000)  
model for grain motion)
- (2) Vese-Osher(2002): denoising image. ( C.M. Elliott  
et al(2007)(2009))
- (3) Model for evolution of crystal surface under  
roughening temperature (relaxation dynamics)  
H. Spohn(1993), V. B. Shevy et al(2003),(2004)  
inconsistent with step motion model (microscopic)

Note: The problem (1)-(3) should not be taken literally, since right hand sides are undefined when  $\nabla u = 0$  .

(Subdifferential formulation  $u_t \in -\partial\varphi(u)$ )

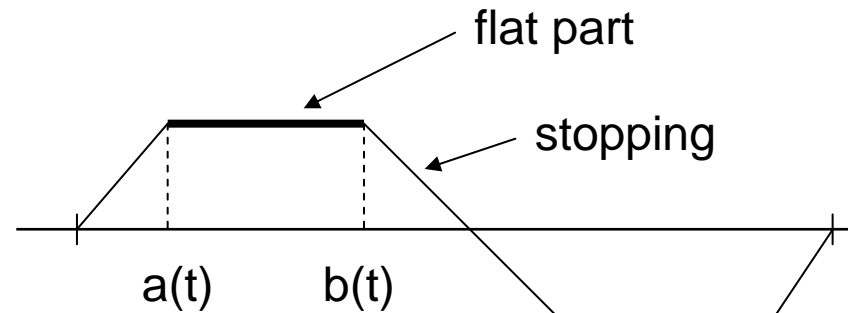
Initiated by Y. Kōmura developed by H. Brezis fits well. See e.g. a nice book by

[ACM] F. Andreu-Vaillio, V. Caselles, J. M. Mazón, Parabolic Quasilinear Equations Minimizing Linear Growth Functionals, 223 Birkhauser(2004) or a review for heuristic explanation

R. Kobayashi and Y. Giga, J. Stat. Phys 95(1999), 1187-1220

Actually, the speed is determined by nonlocal quantity.

Example: 
$$\begin{cases} u_t = (\operatorname{sgn} u_x)_x \\ u|_{t=0} = u_0 \end{cases}$$



with periodic B C

The speed of the flat part

$$u_t = -\frac{2}{b(t) - a(t)};$$

(Formally obtained by integrating over  $a - \delta$ ,  $b + \delta$  for small  $\delta > 0$  and sending  $\delta \rightarrow 0$  assuming

$$u_t = \text{const} \quad \text{on } (a, b).)$$

$b - a$  : the length of facet.

Problem: Solutions are expected to become zero in finite time. Does one estimate upper bound for extinction time? In physical literature for (3) it is shown numerically that solutions become flat in finite time. Does one get the estimate with size-free or scale-invariant?

What does it mean scale-invariant estimate?

Example. Fourth order total variation flow

$$u_t = (-\Delta) \operatorname{div} (\nabla u / |\nabla u|), \quad u|_{t=0} = u_0.$$

This has two scale invariance

$$t \rightarrow \lambda t, u \rightarrow \lambda u, x \rightarrow x/t \rightarrow \lambda^4 t, u \rightarrow \lambda u, x \rightarrow \lambda x.$$

First invariance implies  $T^*(u_0)$  is positively homogeneous of degree one – i.e.

$$T^*(\lambda u_0) = \lambda T^*(u_0) \text{ for } \lambda > 0.$$

Here  $T^*(u_0)$  is the extinction time with initial data  $u_0$ .

This suggests an extinction time estimate

$$T^*(u_0) \leq C \|u_0\|_X.$$

Second invariance restricts the character of the norm  $\|\cdot\|_X$ .

## 2. Second Order Total variation Flow

For Simplicity we impose periodic B.C.

Formal Argument(1) Simplest case 2-D

Multiply  $u$  with  $u_t = \operatorname{div} \left( \nabla u / |\nabla u| \right)$   
and integrate by parts in a period cell.

$$\frac{1}{2} \frac{d}{dt} \int |u|^2 = - \int |\nabla u|$$

where integration is one period cell  $\mathbf{T}^2$

By (isoperimetric) Sobolev inequality

$$\left( \int |u|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq S_n \int |\nabla u|$$

(assuming the average of  $u_0$  equals zero so that  $\int u dx = 0$ )

with space dimension  $n=2$  implies

$$\frac{1}{2} \frac{d}{dt} \int |u|^2 \leq -S_2 \left( \int |u|^2 \right)^{\frac{1}{2}}.$$

This implies

$$\|u : L^2(\mathbf{T}^2)\| (t) \leq \|u_0 : L^2(\mathbf{T}^2)\| - S_2^{-1} t,$$

which deduces

$$T^*(u_0) \leq S_2^{-1} \|u_0 : L^2(\mathbf{T}^2)\|.$$

# Formal Argument :(2) General case

Theorem 1.

For the periodic problem in dimension  $n \geq 2$   
with initial data  $u_0 \in L^2_{av}(\mathbf{T}^n)$  ,  
the extinction time satisfies

$$T^*(u_0) \leq S_n \left\| u_0 : L^n(\mathbf{T}^n) \right\|.$$

Here

$$L^2_{av}(\mathbf{T}^n) = \left\{ u_0 \in L^2(\mathbf{T}^n) : \int u_0 dx = 0 \right\}.$$

$n \geq 2$  even : Multiply  $u^{n-1}$  with  $u_t = \operatorname{div}(\nabla u / |\nabla u|)$   
and integrate by parts yields

$$\begin{aligned} \frac{1}{n} \frac{d}{dx} \int u^n &= \int u^{n-1} u_t = \int u^{n-1} \operatorname{div}(\nabla u / |\nabla u|) \\ &= -(n-1) \int u^{n-2} |\nabla u| = - \int |\nabla u^{n-1}|. \end{aligned}$$

The Sobolev inequality implies

$$\frac{1}{n} \frac{d}{dt} \int u^n \leq -S_n^{-1} \left( \int u^n \right)^{(n-1)/n},$$

which implies

$$\left\| u : L^n(\mathbf{T}^n) \right\| \leq \left\| u_0 : L^n(\mathbf{T}^n) \right\| - S_n^{-1} t.$$

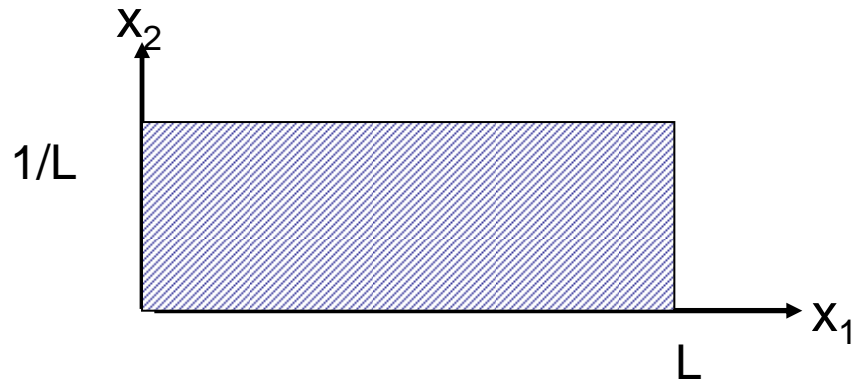
# A. Sobolev inequality

Sobolev constant  $S_n$  is independent of the dilation. In that sense it is scale invariant. However, it still depends on the shape of domain.

Let  $C_L$  be the best Sobolev constant

$$\|u : L^2(\mathbf{T}^2)\| \leq C_L \int |\nabla u|$$

for  $u$  with average zero where the period cell is  $[0, L) \times [0, 1/L)$ .



We restrict our attention to  $a(x) = f(x_1)$   
(depending only on  $x_1$ ). The Sobolev inequality  
implies

$$\left( \int_0^L |f(x_1)|^2 dx_1 \right)^{\frac{1}{2}} L^{-1/2} \leq C_L L^{-1} \int_0^L |f'(x_1)|$$

for any  $f$  with mean value zero.

Changing variables by  $x_1 = Lz$  gives

$$\left( \int_0^1 |g(z)|^2 dz \right)^{1/2} \leq C_L L^{-1} \int_0^1 |g'(z)|$$

for any  $g$  with zero mean value on  $(0,1)$ .

This implies  $\lim_{L \rightarrow \infty} C_L = \infty$ .

## B. Definition of solutions

The solution should be interpreted by subdifferential equations.

$$\Phi_{\pi}(u) = \begin{cases} \int_{\mathbf{T}^n} |\nabla u|, & u \in BV(\mathbf{T}^n) \\ \infty & , \text{otherwise} \end{cases}$$

$H = L^2_{av}(\mathbf{T}^n)$  Hilbert space

$\Phi_{\pi}$  is a convex, lower semicontinuous function  
on  $H$ .

Abstract theory (Kōmura(1967), Brezis(1973)).

Let  $\varphi$  be a convex, lower semicontinuous function in a Hilbert space  $H$  .

Then for each  $u_0 \in H$  there is a **unique** solution  $u \in C([0, \infty), H) \cap AC([\delta, T], H)$  for all  $T > \delta > 0$  such that  $du/dt \in -\partial\varphi(u)(t)$  for *a.e.*  $t > 0$  with  $u|_{t=0} = u_0 \in H$  .

Note: AC denotes absolute continuity so  $u$  is differentiable almost all  $t > 0$ .

For application we take  $\varphi = \Phi_\pi$ ,  $H = L^2_{av}(\mathbf{T}^n)$ .

# C. Other boundary conditions

Dirichlet :  $H = L^2(\Omega)$ ,  $\Omega \subset \mathbf{R}^n$

$$\Phi_D(u) = \begin{cases} \int_{\Omega} |\nabla u| + \int_{\partial\Omega} |u| ds \\ \quad \text{for } u \in BV(\Omega) \\ \infty \quad \text{otherwise} \end{cases}$$

Here  $u$  on  $\partial\Omega$  is the trace of  $u$  on  $\partial\Omega$ .

Neumann:  $H = L^2_{av}(\Omega)$

$$\Phi_N(u) = \begin{cases} \int_{\Omega} |\nabla u|, & u \in BV(\Omega) \\ \infty, & \text{otherwise} \end{cases}$$

Parallel results hold for  $\Phi_D$  and  $\Phi_N$ .

# D. Indication for rigorous arguments

(i) Use characterization of subdifferential

Truncate:

$$u_k = \begin{cases} u & |u| \leq k \\ k & u > k \\ -k & u < -k. \end{cases}$$

Multiply  $u_k^{n-1}$  with  $u_t = \operatorname{div}(\nabla u / |\nabla u|)$  and integrate by parts to get

$$\frac{1}{n} \frac{d}{dt} \int_{\mathbb{T}^n} |u_k|^n = - \int_{\mathbb{T}^n} |\nabla |u_k|^{n-1}|.$$

Not straightforward: use characterization of subdifferential.

Treatment of  $|\nabla |u_k|^{n-1}|$  difficult.

(ii) Approximation by a smoother energy

$$\Phi_{\pi}^{\varepsilon}(u) = \int_{\mathbf{T}^n} \sqrt{|\nabla u|^2 + \varepsilon^2} + \frac{\varepsilon^2}{2} \int_{\mathbf{T}^n} |\nabla u|^2$$

Equation  $u_t \in -\partial\Phi_{\pi}^{\varepsilon}$  formally reads

$$u_t = \operatorname{div} \left( \frac{\nabla u}{\sqrt{|\nabla u|^2 + \varepsilon^2}} \right) + \varepsilon^2 \Delta u.$$

- Derive a priori estimate on  $(0, T)$ ,  $T < T^*(u_0)$  for any  $\delta > \varepsilon / 2$

$$\begin{aligned} \left\| u^\varepsilon : L^n(\mathbf{T}^n) \right\| (t) &\leq \left\| u_0^\varepsilon : L^n(\mathbf{T}^n) \right\| - S_n t \\ &\quad + \delta(n-1)\eta^{-(n-1)} \int_0^t \left\| u^\varepsilon : L^{n-2} \right\|^{n-2}(\tau) d\tau. \end{aligned}$$

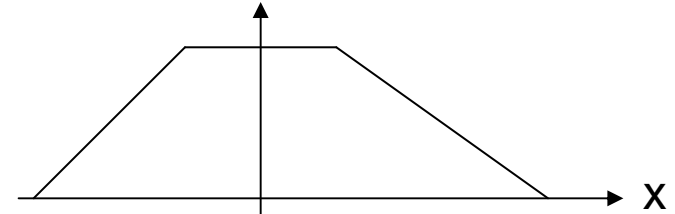
Here  $\left\| u^\varepsilon : L^n \right\| (t)$  is decreasing in time and

$$\eta = \inf \left\{ \left\| u^\varepsilon : L^n \right\| (t), 0 < t < T \right\}.$$

- Use stability theory for subdifferentials.

# E. One-dimensional problem

The proof is more involved. We approximate  $|u|$  by  $\sqrt{|u|^2 + \varepsilon^2}$ .



$$\| u : L^1(\mathbf{T}^1) \| (t) \leq \| u_0 : L^1(\mathbf{T}^1) \| - t.$$

Note for Cauchy problems  $T^*(u_0) = \infty$  if  $u_0 \notin L^1(\mathbf{R})$  for  $u_0$ ; positive, even with  $u_x \leq 0$  for  $x > 0$  and  $u_0(x) = A$  for  $|x| < r_0$  and  $u_0(x) < A$  with  $u_x < 0$  for  $|x| > r_0$ .

## F. Comparison with other result

(i) Andreu – Caselles – Diaz - Mazón(2002)

$$T^*(u_0) \leq \frac{d(\Omega)}{n} \|u_0\|_\infty$$

Dirichlet or Neumann in  $\Omega$  with Lipschitz boundary.

$d(\Omega)$ : diameter of  $\Omega$

Comparison with evolution of characteristic functions  $\lambda(t)\chi_B$  where  $B$  is a ball

(There is an explicit solution of this form.)

(ii) E. DiBenedetto (1993)

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = 0, \quad 1 < p < 2$$
$$T^*(u_0) \leq C \left\| u_0 : L^s \right\|^{2-p}, \quad s = \frac{n(2-p)}{p}, \quad n \geq 2$$
$$1 < p < \frac{2n}{n+2}$$

Cauchy problem, Dirichlet problem

His estimate reduces ours by taking  $p \downarrow 1$ .

Critical exponent:  $p < 2 - 2/(n+1) \Leftrightarrow$  finite time extinction  
(Vazquez, Section 11.5.4, 2006 Smoothing and decay estimates for nonlinear diffusion equations)

(cf the method by Benilan-Crandall(1981))

(iii) Benilan - Crandall (1981)

$$u_t - \Delta(|u|^m \operatorname{sgn} u) = 0 \quad (\text{Cauchy problem})$$

$$u_0 \in L^{1+\beta} \cap L^1 \quad \beta = \frac{2 - 2^* m}{2^* - 2}, \quad \frac{1}{2^*} = \frac{1}{2} - \frac{1}{n}$$

( $k > 0 \Leftrightarrow m < (n-2)/n$ ) subcritically fast diffusion)

$$T^*(u_0) \leq C \left( \int_{\mathbf{R}^n} u_0^{\beta+1} dx \right)^{2/n}$$

Note if  $m \geq (n-2)/n$ , then finite time extinction never occurs for  $u_0 \in L^1(\mathbf{R}^n)$ ,  $u_0 \neq 0$ .

### 3. Fourth Order Total Variation Flow

$$u_t = -\Delta \left[ \operatorname{div} \left( \nabla u / |\nabla u| \right) \right]$$

$$H_{av}^1(\mathbf{T}^n) = \left\{ u \in H^1(\mathbf{T}^n) : \int u \, dx = 0 \right\}$$

inner product  $((f, g))_1 = \int_{\mathbf{T}^n} \nabla f \cdot \nabla g \, dx$

$$H_{av}^{-1}(\mathbf{T}^n) = \text{dual of } H_{av}^1$$

inner product  $((f, g))_{-1} = \int_{\mathbf{T}^n} (-\Delta)^{-1} f \cdot g \, dx$

$$-\Delta : H_{av}^1 \rightarrow H_{av}^{-1} \quad f \mapsto ((f, \bullet))_1$$

(isometry)

Rigorous fourth order total variation flow

$$u_t \in -\partial \varphi(u)$$

$$H = H_{av}^{-1}, \quad \varphi(u) = \Phi_{\pi}(u).$$

## A. Fundamental energy identity

**Lemma 1.** Let  $u$  be the solution. Then

$$\frac{1}{2} \frac{d}{dt} \|u\|_{H^{-1}}^2 = - \int |\nabla u|.$$

Formally multiply  $(-\Delta)^{-1}u$  with

$$u_t = -\Delta \operatorname{div} \left( \nabla u / |\nabla u| \right) \quad \text{and integrate by parts.}$$

General Principle : 'Euler equation'

$$\langle u, \partial \Phi(u) \rangle = d\Phi(u)$$

for  $d$  - homogeneous function  $\Phi$ .

We take the inner product of  $u$  and the equation

$$\left( (u, u_t) \right)_{-1} = - \left( (u, f) \right) \quad \text{for } f \in \partial \varphi(u)$$

to get the desired identity.

# Easy Case : 4-dimensional case

$$\begin{aligned} \|u : H^{-1}\| &= \|(-\Delta)^{-1/2} u : L^2\| \\ &\leq A' \|\nabla(-\Delta)^{-1/2} u : L^p\|, \quad 1/2 = 1/p - 1/n \end{aligned}$$

by Sobolev

$$(n = 4 \Leftrightarrow p = 4/3)$$

$$\leq A_p \|u : L^p\|$$

by Calderón-Zygmund

If  $n = 4$ , then

$$\begin{aligned} \left\| u : H^{-1} \right\| &\leq A_{4/3} \left\| u : L^{4/3} \right\| \\ &\leq A_{4/3} S_4 \int_{\mathbf{T}^n} |\nabla u| \end{aligned}$$

for any  $u \in BV(\mathbf{T}^n)$ .

Then Lemma 1 yields

$$\frac{1}{2} \frac{d}{dt} \left\| u : H^{-1} \right\|^2 (t) \leq - (A_{4/3} S_4)^{-1} \left\| u : H^{-1} \right\|.$$

**Theorem 2.** For the periodic problem with dimension 4 with initial data  $u_0 \in H_{av}^{-1}(\mathbf{T}^4)$

$$\|u : H^{-1}\|(t) \leq \|u_0 : H^{-1}\| - (A_{4/3}S_4)^{-1}t \text{ for}$$

all  $t < T^*(u_0)$ . In particular

$$T^*(u_0) \leq A_{4/3}S_4 \|u_0 : H^{-1}\|$$

How about other dimensions? We have extinction time estimates for  $n \leq 4$

But no estimate is available for  $n \geq 5$ .

# General case

**Theorem 3.** Assume  $1 \leq n \leq 4, 1 \leq p \leq \infty, 1/2 < \theta \leq 1$  satisfying  $1 + \frac{n}{2} = \theta(n-1) + (1-\theta)\left(3 + \frac{n}{p}\right)$

Let  $u$  be the solution of fourth order total variation flow with  $u_0 \in H_{av}^{-1}(\mathbf{T}^n)$ . Then

$$\|u\|_{H^{-1}}(t)^\mu \leq \|u_0\|_{H^{-1}}^\mu - \mu C_*^{-1/\theta} \int_0^t A(s)^{\mu-1} ds$$

for all  $t < T^*(u_0)$  with  $\mu = 2 - (1/\theta)$  and

$$A(t) = |\mathbf{T}^n|^{1/p} t + \left\| (-\Delta)^{-1} u_0 \right\|_{\dot{W}^{-1,p}}.$$

As a consequence

$$T^*(u_0) \leq \mu^{-1} C_*^{1/\theta} \left\| (-\Delta)^{-1} u_0 \right\|_{\dot{W}^{-1,p}}^{1-\mu} \|u_0\|_{H^{-1}}^\mu.$$

# Notation

$$\cdot \left\| w \right\|_{\dot{W}^{-1,p}} = \sup \left\{ \int_{\mathbf{T}^n} \varphi w dx : \varphi \in C_0^1(\mathbf{T}^n), \right. \\ \left. \left\| \nabla \varphi \right\|_{L^{p'}} \leq 1 \right\} \\ 1/p + 1/p' = 1$$

- $C_*$  is a constant appearing in our key interpolation inequality.
- $|\mathbf{T}^n|$  denotes the volume of the period cell.

# B. Ingredients of the proof

(i) Fundamental energy identity (Lemma 1)

(ii) Weak growth bound for  $\|(-\Delta)^{-1} u\|_{\dot{W}^{-1,p}}$

(iii) Interpolation inequalities

(ii)

**Lemma 2.** Let  $\mathcal{U}$  be the solution of the fourth order total variation flow with

$u_0 \in H_{av}^{-1}(\mathbf{T}^n)$ . Then for  $1 \leq p \leq \infty$  we have

$$\left\| (-\Delta)^{-1} u \right\|_{\dot{W}^{-1,p}}(t) \leq |\mathbf{T}^n|^{1/p} t + \left\| (-\Delta)^{-1} u_0 \right\|_{\dot{W}^{-1,p}}$$

for all  $t > 0$ .

Formally 
$$\frac{d}{dt} \left\| (-\Delta)^{-1} u \right\|_{\dot{W}^{-1,p}} \leq \left\| (-\Delta)^{-1} u_t \right\|_{\dot{W}^{-1,p}}$$

$$= \left\| \operatorname{div} \frac{\nabla u}{|\nabla u|} \right\|_{\dot{W}^{-1,p}} \leq \left\| \frac{\nabla u}{|\nabla u|} \right\|_{L^p} \leq |\mathbf{T}^n|^{1/p} .$$

(iii)

**Lemma 3.** (interpolation inequality)  $1 \leq n \leq 4, 1 \leq p \leq \infty$   
 $1/2 \leq \theta \leq 1$  Assume that  $1 + \frac{n}{2} = \theta(n-1) + (1-\theta)\left(3 + \frac{n}{p}\right)$

Then  $\exists C_*$  s.t.

$$\|u\|_{H^{-1}} \leq C_* \left\| (-\Delta)^{-1} u \right\|_{\dot{W}^{-1,p}}^{1-\theta} \left( \int_{\mathbf{T}^n} |\nabla u| \right)^\theta$$

for all  $u \in H_{av}^{-1}(\mathbf{T}^n) \cap BV(\mathbf{T}^n)$ .

The constant  $C_*$  is scale – invariant.

## C. Proof of Theorem 2

We set  $y(t) = \|u\|_{H^{-1}}(t)$ . Energy identity and interpolation inequality implies

$$y \frac{dy}{dt} \leq -\int |\nabla u|^2 \leq -My^{1/\theta} \left\| (-\Delta)^{-1} u \right\|_{\dot{W}^{-1,p}}^{(1-\theta)/\theta},$$

$$M = C_*^{-1/\theta}.$$

The growth estimate implies

$$y^{1-(1/\theta)} \frac{dy}{dt} \leq -M (A(t))^{(\theta-1)/\theta}.$$

Integrating over  $(0, t)$  gives

$$\frac{1}{\mu} \left( y(t)^\mu - y(0)^\mu \right) \leq -M \int_0^t A(s)^{1-1/\theta} ds .$$

This implies extinction time estimate.

$$\text{Observe } \int_0^t A(s)^{\mu-1} ds = \frac{1}{\mu a} \left( (at + A_0)^\mu - A_0^\mu \right)$$

with  $a = |\mathbf{T}^n|^{1/p}$ ,  $A_0 = A(0)$ . We end up with

$$y(0)^\mu \geq M \left[ (aT^* + A_0)^\mu - A_0^\mu \right] / a$$

$$= MA_0^\mu \left( \left( 1 + aT^* A_0^{-1} \right)^\mu - 1 \right) / a$$

$$\geq MA_0^\mu \mu T^* A_0^{-1} . \quad \square$$

# D. Idea of the proof of interpolation inequality

easy case  $p = \infty$

$$\begin{aligned}\|u\|_{H^{-1}}^2 &= ((u, u))_{-1} \\ &= \int_{\mathbf{T}^n} (-\Delta)^{-1} u \cdot u \, dx\end{aligned}$$

By definition of  $\dot{W}^{-1,\infty}$  we have

$$\|u\|_{H^{-1}}^2 \leq \int_{\mathbf{T}^n} |\nabla u| \cdot \left\| (-\Delta)^{-1} u \right\|_{\dot{W}^{-1,\infty}}$$

(This is the case  $\theta=1/2$ ,  $p=\infty$ ,  $C_* = 1$  )

(  $p = \infty$ ,  $n < 3 \Rightarrow \theta = 1/2$  )

(  $p = \theta$ ,  $n = 4$ ,  $1/2 \leq \theta \leq 1$

$$\|u\|_{H^{-1}} \leq \int |\nabla u| )$$

Remaining case  $n \leq 3$ ,  $1 \leq p \leq \infty$ .

## Strategy

$$u = u_r + u_s \quad u_r : \text{regular parts}$$

$$u_s : \text{singular parts}$$

(This approach is convenient for negative norm.)

$$[\text{R. Kohn-F. Otto(2002)}] \left( \int |u| \right)^2 \leq C \|\nabla u\|_{L^1} \|\nabla^{-1} u\|_{L^1}$$

$u_r$  is given as convolution with mollifier]

Here we take  $u_r = e^{t\Delta} u$ . Good for proving

Gagliardo-Nirenberg type inequality

(cf. M.-H. Giga, Y. Giga and J. Saal, Nonlinear PDEs, Asymptotic Behavior of Solutions and Self-Similar Solutions, Birkhauser(2010). Chap.6.)

$$\begin{aligned}
\|u\|_{H^{-1}}^2 &= \int (-\Delta)^{-1} u \cdot u dx \\
&\leq \left| \int (-\Delta)^{-1} u u_r dx \right| + \left| \int (-\Delta)^{-1} u u_s dx \right| \\
&= I_1 + I_2, \quad u_s = -\int_0^t \Delta e^{\tau \Delta} u d\tau
\end{aligned}$$

$$\cdot \quad I_1 \leq \left\| \nabla e^{\tau \Delta} u \right\|_{L^{p'}} \left\| (-\Delta)^{-1} u \right\|_{\dot{W}^{-1,p}}.$$

$L^p - L^1$  estimate for the heat semigroup yields

$$\left\| \nabla e^{\tau \Delta} u \right\|_{L^{p'}} = \left\| e^{t \Delta} \nabla u \right\|_{L^{p'}} \leq M_1 t^{-n/2p} \int |\nabla u|.$$

Thus

$$I_1 \leq M_1 t^{-n/2p} \left( \int |\nabla u| \right) \|(-\Delta)^{-1} u\|_{\dot{W}^{-1,p}}.$$

• For  $I_2$  we have

$$I_2 \leq \|u\|_{H^{-1}} \int_0^t \|\nabla e^{\tau\Delta} u\|_{L^2} d\tau.$$

Again we use

$$\|\nabla e^{s\Delta} u\|_{L^2} \leq M_2 s^{-n/4} \int |\nabla u|$$

Since  $1 \leq n \leq 3$ , this is integrable on  $(0, t)$ .

We thus obtain

$$I_2 \leq M_3 t^{1-n/4} \|u\|_{H^{-1}} \int |\nabla u|.$$

$$\text{Thus } \|u\|_{H^{-1}}^2 \leq \left( M, t^{-n/2p} \|(-\Delta)^{-1} u\|_{\dot{W}^{-1,p}} + M_3 t^{1-n/4} \|u\|_{H^{-1}} \right) \\ \times \int |\nabla u| \quad \text{for all } t > 0.$$

Taking  $t$  suitably we get the desired estimate.  $\square$

Remark (i) For the Neumann problem or Dirichlet problem it is likely we have similar extinction time estimate. In fact, conclusion in Theorem 2 (4-dim result) is still valid for these problems. To extend Theorem 3 we have growth estimates. However, interpolation inequality needs more effort to extend because simple replacement of

$e^{t\Delta}$  by the heat semigroup with Dirichlet and Neumann Laplacian does not work in  $L^1$  setting. Another way to extend is to extend functions to a big periodic cells controlling norms.

(ii) To get a growth estimate we invoke a characterization of subdifferential.

**Lemma 4 [ACM]** Assume that  $v \in D(\Phi_\pi) \subset L^2_{av}(\mathbf{T}^n)$ .

$$f \in \partial \Phi_\pi(v)$$

$$\Leftrightarrow \exists z \in L^\infty(\mathbf{T}^n, \mathbf{R}^n) \text{ such that}$$

$$f = -\operatorname{div} z, \quad \|z\|_{L^\infty} \leq 1$$

$$\int_{\mathbf{T}^n} f v dx = \Phi_\pi(u)$$

Lemma 5 (Characterization of  $\partial_{H^{-1}} \Phi_{\pi}(v)$ )

$$v \in D(\Phi_{\pi}) \subset H_{av}^{-1}(\mathbf{T}^n), f \in H_{av}^{-1}(\mathbf{T}^n)$$

$$f \in \partial_{H^{-1}} \Phi_{\pi}(v)$$

$$\Leftrightarrow \exists z \in L^{\infty}(\mathbf{T}^n, \mathbf{R}^n)$$

$$(-\Delta)^{-1} f = \operatorname{div} z, \|z\|_{L^{\infty}} \leq 1$$

$$\int_{\mathbf{T}^n} (-\Delta)^{-1} f \cdot v dx = \Phi_{\pi}(v)$$

Lemma 5 follows from Lemma 4

Lemma 6  $f \in \partial_{H^{-1}} \Phi_{\pi}(v)$

$$\Leftrightarrow \begin{cases} \tilde{\Phi}_{\pi L^2} \left( (-\Delta)^{-1} f \right) \leq 1 \\ \int_{\mathbf{T}^n} (-\Delta)^{-1} f \cdot v dx = \Phi_{\pi}(v) \end{cases}$$

$$\begin{aligned} & \tilde{\Phi}_{\pi L^2}(v) \\ &= \sup \left\{ \int_{\mathbf{T}^n} v w dx / \Phi_{\pi}(w) : w \in L^2_{av}(\mathbf{T}^n) \right\} \end{aligned}$$

# 4. Fourth Order Surface Diffusion Law

$$u_t = (-\Delta)[\operatorname{div}(\nabla u / |\nabla u| + \nu |\nabla u|^{q-2} \nabla u)]$$

$$q > 1, \nu > 0$$

energy

$$\Phi^q(u) = \begin{cases} \Phi_\pi(u) + \frac{\nu}{q} \int_{\mathbf{T}^n} |\nabla u|^q dx \\ \infty & \text{otherwise} \end{cases}$$

a characterization of subdifferential in  $H^{-1}$

Y. Kashima(2004), Adv. Math. Sci. Appl.

**Theorem 4.** Assume  $1 \leq n \leq 4$ ,  $1 \leq p \leq q/(q-1)$

$1/2 < \theta \leq 1$  satisfying  $1 + \frac{n}{2} = \theta(n-1) + (1-\theta)\left(3 - \frac{n}{p}\right)$

Let  $u$  be the solution of fourth order total variation flow with  $u_0 \in H_{av}^{-1}(\mathbf{T}^n)$ .

Then  $\|u\|_{H^{-1}}(t)^\mu \leq \|u_0\|_{H^{-1}}^\mu - \mu C_*^{-1/\theta} \int_0^t A(s)^{\mu-1} ds$

for all  $t < T^*(u)$  with  $\mu = 2 - 1/\theta$  and

$$A(t) = |\mathbf{T}^n|^{1/p} + \nu^{1/q} |\mathbf{T}^n|^{1/r} \left( q \Phi_{\frac{q}{\pi}}(u_0) \right)^{1-1/q}$$

with  $1/r = 1/p + 1/q - 1$  ( $q/(q-1) \geq p \Leftrightarrow r \geq 1$ )

As a consequence we have

$$T^*(u_0) \leq \frac{1}{\mu} C_*^{1/\theta} \left\| (-\Delta)^{-1} u_0 \right\|_{\dot{W}^{-1,p}}^{1-\mu} \|u_0\|_{H^{-1}}^\mu.$$

# Remarks

(i) Final extinction time estimate is the same as Theorem 3. By approximation we actually need not assume  $\Phi_{\pi}^q(u_0) < \infty$  to get extinction time estimate.

(ii) The growth estimate is slightly different

$$\left\| (-\Delta)^{-1} u \right\|_{\dot{W}^{-1,p}}(t) \leq A(t).$$

(iii) Energy identity as

$$\frac{1}{2} \frac{d}{dt} \|u\|_{H^{-1}}^2(t) = - \int_{\mathbf{T}^n} |\nabla u(\cdot, t)| - \mu \int_{\mathbf{T}^n} |\nabla u(\cdot, t)|^q dx$$

for a.e.  $t > 0$

(iv) The second term does not help or does not hurt to get extinction time estimate.

## (v) Open problems

(a) What happens for  $n \geq 5$  ?

(b) Consider more general equations which often appear in relaxation dynamics below roughening temperature

$$u_t = -\operatorname{div} \left[ M(\nabla u) \nabla \operatorname{div} \left( \frac{\nabla u}{|\nabla u|} + \nu |\nabla u|^{q-2} \nabla u \right) \right]$$

with mobility tensor  $M \geq 0$ . Even existence of solution is not known unless  $M$  is a constant.

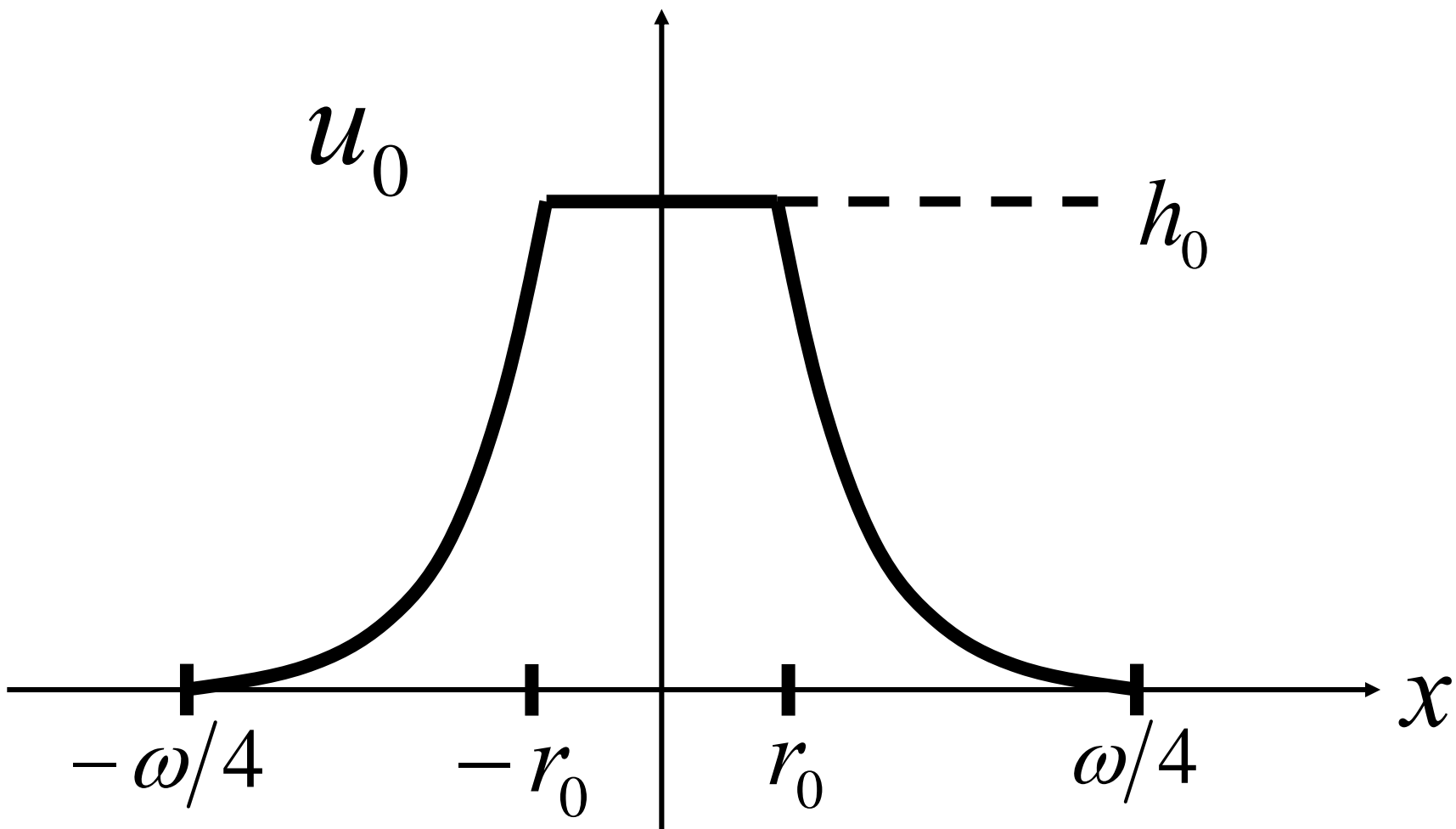
(vi) What is the main difference between second and fourth order total variation flow?

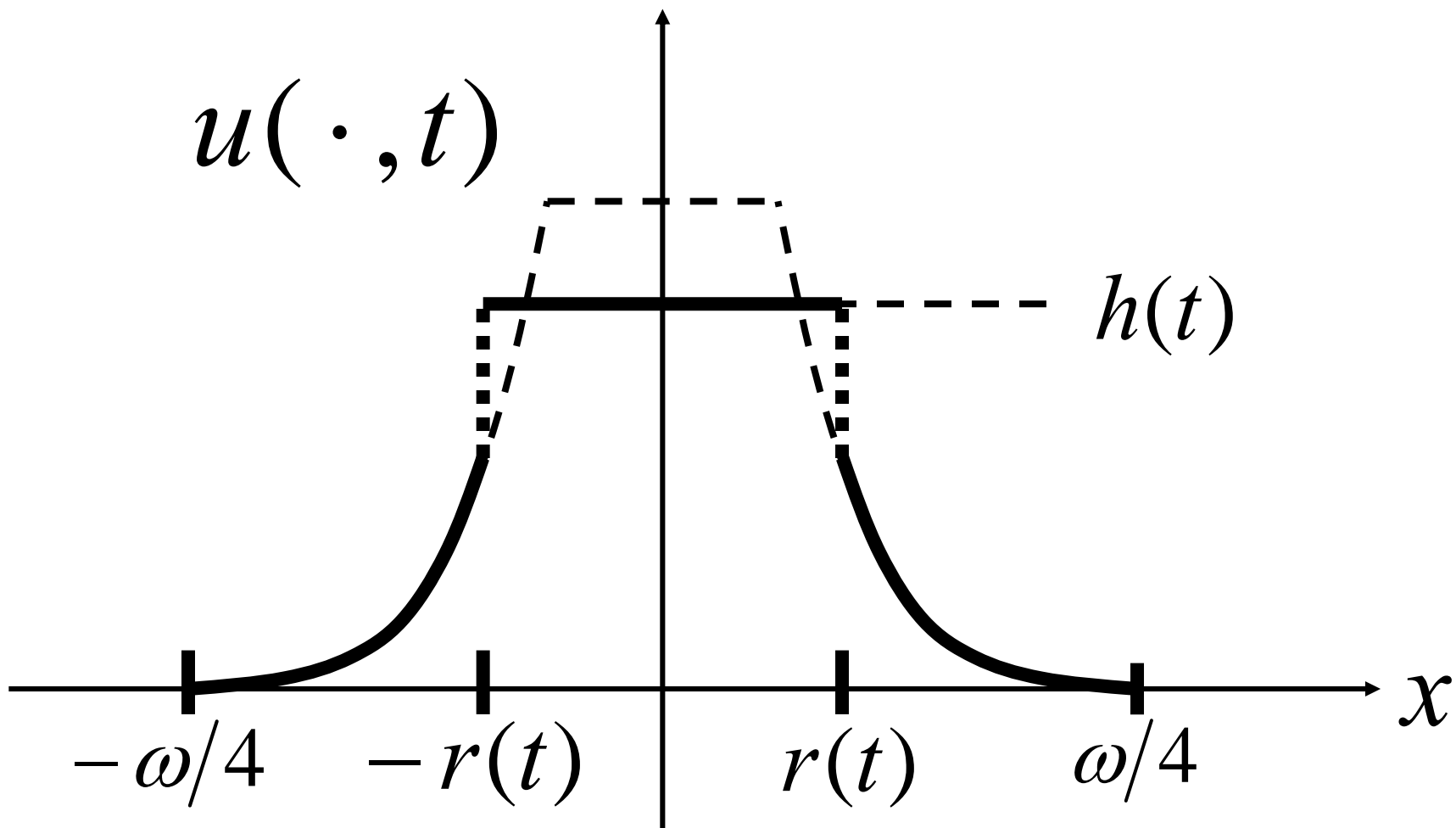
[Mi-Ho Giga - Y.G 2010]

Failure of comparison Principle

Failure of conservation of Continuity

(Formation of jump discontinuities)





# References

Y. Giga and R. V. Kohn, Scale-invariant extinction time estimates for some singular diffusion equations, *Discrete and Continuous Dynamical Systems*, to appear.

Hokkaido University Preprint Series in Math.#963(2010)

<http://eprints3.math.sci.hokudai.ac.jp/2105/>

M.-H. Giga and Y. Giga, Very singular diffusion equations – second and fourth order problems, *Japanese J. Ind. Appl. Math.*, to appear

Hokkaido University Preprint Series in Math.#960(2010)

<http://eprints3.math.sci.hokudai.ac.jp/2053/>