# Total variation for image denoising: Theory and examples in the anisotropic case.

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[1] Joint work with Kevin Vixie

# Preliminaries.

Let $n$ be a positive integer and let $\Omega$ be an open subset of $\mathbb{R}^n$. Given an integer $k$, a $k$-current $T$ on $\Omega$ is a continuous linear functional

$$T : \mathcal{D}^k(\Omega) \to \mathbb{R};$$

here $\mathcal{D}^k(\Omega)$ is the vector space of smooth compactly supported differential forms of degree $k$ on $\Omega$. We let

$$\mathcal{D}_k(\Omega)$$

be the vector space of $k$-currents on $\Omega$. The boundary $\partial T$ is the $(k-1)$-current defined by

$$\partial T(\psi) = T(d\psi) \quad \text{for } \psi \in \mathcal{D}^{k-1}(\Omega);$$

here $d : \mathcal{D}^{k-1}(\Omega) \to \mathcal{D}^k(\Omega)$ is exterior differentiation.

# A simple example.

Suppose $P : [a, b] \to \Omega$ is Lipschitzian. For each smooth compactly supported 1-form $\psi$ on $\Omega$ we set

$$T(\psi) = \int_a^b \psi(P(t))(P'(t))\, d\mathcal{L}^1 t;$$

so $T \in \mathcal{D}_1(\Omega)$. Note that the same $T$ results if we reparameterize $P$ in an orientation preserving way. Also, by the Fundamental Theorem of Calculus,

$$\partial T(g) = g(P(b)) - g(P(a)) \quad \text{for } g \in \mathcal{D}^0(\Omega).$$

In other words, $\partial$ is defined so as to make Stokes' Theorem a definition.

# Representability by integration.

We say the $k$-current $T$ is representable by integration if there exist a Radon measure $\mu$ on $\Omega$ and a locally $\mu$ summable $k$-vector field $\xi$ such that

$$T(\psi) = \int \psi(\xi(x))\, d\mu x \quad \text{for } \psi \in \mathcal{D}^k(\Omega).$$

This will be the case if and only if

$$\sup\{|T(\psi)| : \psi \in \mathcal{D}^k(\Omega) \text{ and } ||\psi|| \leq f\} < \infty$$

whenever $f : \Omega \to [0, \infty)$ is continuous and compactly supported.

# Representability by integration, continued.

Suppose

$$\Phi : \bigwedge_k \mathbb{R}^n \to [0, \infty)$$

is continuous and satisfies

$$\Phi(t\xi) = t\Phi(\xi) \quad \text{whenever } t \in [0, \infty) \text{ and } \xi \in \bigwedge_k \mathbb{R}^n;$$

we call such a function $\Phi$ a integrand (of degree $k$); we say $\Phi$ is positive if $\Phi(\xi) > 0$ for $\xi \in: \bigwedge_k \mathbb{R}^n \sim \{0\}$.
Suppose $\Phi$ is a positive integrand and $\mu$, $\xi$ are as above; we set

$$\int_T \Phi = \int \Phi(\xi(x))\, d\mu x \in [0, \infty].$$

Owing to the homogeneity of $\Phi$ the quantity $\int_T \Phi$ is well defined.

# Mass.

We let

$$\mathbf{M}(T) = \int_T \|\cdot\| \in [0, \infty);$$

here $\|\cdot\|$ is the comass on $\bigwedge_k \mathbb{R}^n$. One calls $\mathbf{M}(T)$ the mass of $T$. Note that

$$\begin{aligned} \mathbf{M}(T) < \infty \; &\Leftrightarrow \int_T \Phi < \infty \quad \text{for any positive integrand } \Phi \\ &\Leftrightarrow \int_T \Phi < \infty \quad \text{for some positive integrand } \Phi. \end{aligned}$$

# The current associated to a function.

Suppose $f \in \mathbf{L}_1^{loc}(\Omega)$. We associate an $n$-dimensional current

$$[f] : \mathcal{D}^n(\Omega) \to \mathbb{R}$$

by requiring that

$$[f](\phi \mathbf{V}^n) = \int f \phi \, d\mathcal{L}^n \quad \text{for } \phi \in \mathcal{D}(\Omega).$$

Here

$$\mathbf{V}^n = \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^n \in \bigwedge^n \mathbb{R}^n$$

is the standard volume $n$-form on $\mathbb{R}^n$.

# Bounded variation.

We say $f$ is of locally bounded variation and write $f \in \mathbf{BV}^{loc}(\Omega)$ if $\partial[f]$ is representable by integration. We say $f$ is of bounded variation and write $f \in \mathbf{BV}(\Omega)$ if $\mathbf{M}([f]) + \mathbf{M}(\partial[T]) < \infty$. Note that this condition is depends on neither the choice of volume form nor the choice of inner product on $\mathbb{R}^n$.

# Some examples.

Suppose $f : \Omega \to \mathbb{R}$ is Lipschitzian. Then $f \in \mathbf{L}_1^{loc}(\Omega)$ and

$$\mathbf{M}(\partial[f]) = \int_\Omega |\nabla f| \, d\mathcal{L}^n.$$

More generally, for any integrand $\Phi$,

$$\int_{\partial[f]} \Phi = \int_\Omega \Phi(\mathbf{V}_n \llcorner df) \, d\mathcal{L}^n;$$

this quantity *does not* depend on a choice of inner product but *does*, obviously, depend on the volume form.

Suppose $E$ is an open subset of $\Omega$ whose boundary is a smooth embedded hypersurface. Then $E \in \mathbf{BV}^{loc}(\Omega)$ and

$$\mathbf{M}(\partial[E]) = \mathcal{H}^{n-1}(E);$$

*here and in what follows we identify a subset of $\Omega$ with its indicator function $1_\Omega$.*

# Slicing formulae.

Indispensable in the study of functions of bounded variation is the slicing (*aka* "coarea") formula:

$$\mathbf{M}(\partial[f] \llcorner B) = \int_{-\infty}^{\infty} \mathbf{M}(\partial[\{f \geq y\}] \llcorner B)\, d\mathcal{L}^1 y$$

where $B$ is any Borel set. We have generalized this formula as follows. Suppose $\Phi$ is a *convex and positive* integrand. Then

$$\int_{\partial[f] \llcorner B} \Phi = \int_{-\infty}^{\infty} \left( \int_{\partial[\{f \geq y\}] \llcorner B} \Phi \right) d\mathcal{L}^1 y$$

where $B$ is any Borel set.

# Some convenient notations.

Suppose $n$ is a positive integer and $\Omega$ is an open subset of $\mathbb{R}^n$. We let

$$\mathcal{F}(\Omega) = \mathbf{L}_1(\Omega) \cap \mathbf{L}_\infty(\Omega)$$

and we let

$$\mathcal{M}(\Omega)$$

be the family of Lebesgue measurable subsets of $\Omega$ with finite Lebesgue measure. Thus a subset $E$ of $\Omega$ belongs to $\mathcal{M}(\Omega)$ if and only if $1_E \in \mathcal{F}(\Omega)$.

# The integrand $\Phi$.

We now fix a positive and convex integrand

$$\Phi : \bigwedge_{n-1} \mathbb{R}^n \to [0,\infty).$$

For each $\xi \in \{\Phi = 1\}$ we let

$$\nu(\xi) = \left\{\psi \in \bigwedge^{n-1} \mathbb{R}^n : \{\Phi = \psi\} \neq \emptyset \text{ and } \psi \leq \Phi\right\}$$

and we let

$$\mathbf{F}(\xi) = \{\eta \in \{\Phi = 1\} : \nu(\xi) = \nu(\eta)\}.$$

Thus $\nu(\xi)$ corresponds to the family of support hyperplanes of the convex set $\{\Phi \leq 1\}$ at $\xi$. We call $\mathbf{F}(\xi)$ the face of $\{\Phi \leq 1\}$ at $\xi$; it is elementary that each face is convex and that $\{\Phi \leq 1\}$ is the disjoint union of the family $\{\mathbf{F}(\xi) : \xi \in \{\Phi = 1\}\}$.

# Total variation regularization.

Suppose

$$F : \mathcal{F}(\Omega) \to \mathbb{R}.$$

Given $\epsilon > 0$ we set

$$F_{\Phi,\epsilon}(f) = \epsilon \int_{\partial[f]} \Phi + F(f), \quad f \in \mathbf{BV}(\Omega) \cap \mathcal{F}(\Omega);$$

we call $F_{\Phi,\epsilon}$ a **total variation regularization** of $F$. We let

$$\mathbf{m}^{loc}_{\Phi,\epsilon}(F)$$

be the set of $f \in \mathbf{BV}(\Omega) \cap \mathcal{F}(\Omega)$ such that

$$F_{\Phi,\epsilon}(f) \leq F_{\Phi,\epsilon}(g)$$

whenever $g \in \mathbf{BV}(\Omega) \cap \mathcal{F}(\Omega)$ and $g$ equals $f$ outside some compact subset of $\Omega$.

# Total variation regularization, continued.

Suppose
$$M : \mathcal{M}(\Omega) \to \mathbb{R}.$$

Given $\epsilon > 0$ we set
$$M_{\Phi,\epsilon}(E) = \epsilon \int_{\partial[E]} \Phi + M(E) \quad \text{for } E \in \mathcal{M}(\Omega);$$

we call $M_{\Phi,\epsilon}$ a **total variation regularization of** $M$. We let
$$\mathbf{n}^{loc}_{\Phi,\epsilon}(M)$$

be the set of $D \in \mathcal{M}(\Omega)$ such that $M_{\Phi,\epsilon}(D) \leq M_{\Phi,\epsilon}(E)$ whenever $E \in \mathcal{F}(\Omega)$ and $D = E$ outside some compact subset of $\Omega$.

# Admissibility.

Suppose

$$F : \mathcal{F}(\Omega) \to \mathbb{R}.$$

For each $Y \in (0, \infty)$ we let

$$\mathbf{I}(F, Y)$$

be the infimum of those $L \in (0, \infty)$ such that

$$|F(f) - F(g)| \leq L \int_\Omega |f - g| \, d\mathcal{L}^n$$

whenever $f, g \in \{h \in \mathcal{F}(\Omega) : ||h||_{\mathbf{L}_\infty(\Omega)} \leq Y\}$.
We say $F$ is admissible if $\mathbf{I}(F, Y) < \infty$ whenever $0 < Y < \infty$.
Thus $F$ is admissible if $F$ is Lipschitzian with respect to $|| \cdot ||_{\mathbf{L}_1(\Omega)}$ on subsets of $\mathcal{F}(\Omega)$ which are bounded with respect to $|| \cdot ||_{\mathbf{L}_\infty(\Omega)}$.

# Admissibility for functionals on sets.

Suppose

$$M : \mathcal{F}(\Omega) \to \mathbb{R}.$$

We let

$$\mathbf{I}(M)$$

be the infimum of those $L \in (0, \infty)$ such that

$$|M(D) - M(E)| \leq L \int_\Omega |1_D - 1_E|, \quad D, E \in \mathcal{M}(\Omega).$$

We say $M$ is admissible if $\mathbf{I}(M) < \infty$.

## The space $\mathcal{C}_\lambda(\Omega)$, $0 \leq \lambda < \infty$, ...

...consists of those sets $D \in \mathcal{M}(\Omega)$ with locally finite perimeter such that

$$\int_{\partial[D] \llcorner K} \Phi \leq \int_{\partial[E] \llcorner K} \Phi + \lambda \int_{\Omega} |1_D - 1_E|$$

whenever $E \in \mathcal{M}(\Omega)$ and $K$ is a compact subset of $\Omega$ such that $\mathcal{L}^n(((D \sim E) \cup (E \sim D)) \sim K) = 0$.

**Theorem.** If $M : \mathcal{M}(\Omega) \to \mathbb{R}$, $M$ is admissible, $0 < \epsilon < \infty$, and $D \in \mathbf{n}^{loc}_{\Phi,\epsilon}(M)$ then $D \in \mathcal{C}_\lambda(\Omega)$ with $\lambda = \frac{\mathbf{I}(M)}{\epsilon}$.

**Theorem.** If $F : \mathcal{F}(\Omega) \to \mathbb{R}$, $F$ is admissible, $0 < \epsilon < \infty$, and $f \in \mathbf{m}^{loc}_{\Phi,\epsilon}(F)$ then $\{f > y\} \in \mathcal{C}_\lambda(\Omega)$ with $\lambda = \frac{\mathbf{I}(F, ||f||_{\mathbf{L}_\infty(\Omega)})}{\epsilon}$.

# Regularity Theorem for $\mathcal{C}_\lambda(\Omega)$.

We have prove the following Theorem when $n = 2$.
**Theorem.** Suppose

$$0 \leq \lambda < \infty \quad \text{and} \quad 0 < \zeta < 1$$

there is

$$\theta \in (0, 1)$$

such that if $D \in \mathcal{C}_\lambda(\Omega)$, $B$ is the reduced boundary of $D$, $a \in B$,

$$0 < R \leq \textbf{dist}\,(a, \mathbb{R}^n \sim \Omega) \quad \text{and} \quad \lambda R < \theta$$

and $M = B \cap \mathbf{B}(a, \theta R)$ then $M$ is a graph and for each $b \in B$ there are $\xi \in \mathbf{F}(*\mathbf{n}_D(a))$, $\eta \in \mathbf{F}(*\mathbf{n}_D(b))$ such that

$$|\xi - \eta| \leq \zeta.$$

# Locality.

We say $F : \mathcal{F}(\Omega) \to \mathbb{R}$ is local if

$$F(f+g) + F(0) = F(f) + F(g)$$

whenever $f, g \in \mathcal{F}$ and $fg = 0$.

We say $M : \mathcal{M}(\Omega) \to \mathbb{R}$ is local if

$$M(D \cup E) + M(\phi) = M(D) + M(E)$$

whenever $E, F \in \mathcal{M}$ and $D \cap E = \emptyset$.

# Locality, continued.

By soft analysis, $F$ is admissible and local if and only if

$$F(f) = F(0) + \int_\Omega k(x, f(x))\, d\mathcal{L}^n x, \quad f \in \mathcal{F}(\Omega)$$

for some Borel function $k : \Omega \times [0, \infty) \to \mathbb{R}$ such that $\Omega \ni x \mapsto k(x, 0)$ is bounded and such that for each $Y \in (0, \infty)$ there is a constant $L$ such that

$$|k(x, y) - k(x, z)| \le L|y - z| \quad \text{for } x \in \Omega \text{ and } y, z \in [0, Y].$$

By soft analysis, $M$ is admissible and local if and only if

$$M(E) = M(\emptyset) + \int_E m(x)\, d\mathcal{L}^n x, \quad E \in \mathcal{M}(\Omega)$$

for some $m \in \mathbf{L}_\infty(\Omega)$.

# Representing $F(f)$ in terms of superlevel sets of $f$.

Suppose $F : \mathcal{F}(\Omega) \to \mathbb{R}$ is admissible and local. We let

$$l(x,y) = \limsup_{z \downarrow y} \frac{k(x,z) - k(x,y)}{z - y}, \quad (x,y) \in \Omega \times [0,\infty).$$

For each $y \in (0,\infty)$ we let

$$U_y(E) = \int_E l(x,y)\, d\mathcal{L}^n x \quad \text{whenever } E \in \mathcal{M}(\Omega)$$

and note that $U_y$ is admissible and local.

It is elementary that

$$F(f) = F(0) + \int_0^\infty U_y(\{f > y\})\, d\mathcal{L}^1 y \quad \text{for } f \in \mathcal{F}(\Omega).$$

# ...thus, by the slicing (*aka* "coarea") formula,

$$F_{\Phi,\epsilon}(f) = F(0) + \int_0^\infty \epsilon \left( \int_{\partial[\{f>y\}]} \Phi \right) + U_y(\{f > y\})\, d\mathcal{L}^1 y$$

for $f \in \mathbf{BV}(\Omega) \cap \mathcal{F}(\Omega)$.

# Zooming in on the value.

**Theorem.** Suppose $F : \mathcal{F}(\Omega) \to \mathbb{R}$ is admissible, local and convex; $\Phi$ is positive and convex; $0 < \epsilon < \infty$ and

$$f \in \mathbf{m}^{loc}_{\Phi,\epsilon}(F).$$

Then

$$\{f > y\} \in \mathbf{n}^{loc}_{\Phi,\epsilon}(U_y) \quad \text{whenever } 0 < y < \infty.$$

Special cases of this Theorem appear in works of Chambolle and Chan-Esedoglu. As it turns out, minimizers for $(U_y)_\epsilon$, $0 < y < \infty$, are easier to understand than minimizers of $F_{\Phi,\epsilon}$.

# Conversely...

**Theorem.** Suppose $F : \mathcal{M}(\Omega) \to \mathbb{R}$ is admissible, local and convex; $\Phi$ is positive and convex; $0 < \epsilon < \infty$; and $G$ is an $\mathcal{L}^n \times \mathcal{L}^1$ measurable subset of $\Omega \times (0, \infty)$ such that

$$\{x : (x, y) \in G\} \in \mathbf{n}^{loc}_{\Phi,\epsilon}(U_y) \quad \text{for } \mathcal{L}^1 \text{ almost all } y$$

and

$$f(x) = \mathcal{L}^1(\{y : (x, y) \in G\}) \quad \text{for } x \in \Omega.$$

Then

$$f \in \mathbf{m}^{loc}_{\Phi,\epsilon}(F).$$

# Denoising $s$.

For the remainder of this talk, $n = 2$, $\Omega = \mathbb{R}^2$ and

$$\gamma : \mathbb{R} \to [0, \infty)$$

is such that $\gamma(0) = 0$, $\gamma(y) > 0$ if $y \in \mathbb{R} \sim \{0\}$ and such that $\gamma$ is convex. Of particular interest are

$$\gamma(y) = \frac{y^2}{2} \quad \text{and} \quad \gamma(y) = |y| \quad \text{for } y \in \mathbb{R}.$$

Suppose

$$s \in \mathcal{M}(\mathbb{R}^2);$$

$s$ could be a grayscale image which we wish to denoise.
For $f \in \mathcal{F}(\mathbb{R}^2)$ let

$$F(f) = \int \gamma(f(x) - s(x))\, d\mathcal{L}^2 x;$$

Obviously, $F(f) = 0$ iff $f$ and $s$ agree almost everywhere.
Moreover, $F$ is admissible, local and convex.

## $\beta$, r and s.

Let

$$\beta(y) = \limsup_{z \downarrow y} \frac{\gamma(z) - \gamma(y)}{z - y} \quad \text{for } y \in \mathbb{R}.$$

*Note that $\beta$ is nonincreasing and negative on $(-\infty, 0)$ and nondecreasing and positive on $(0, \infty)$.* For example, if $\gamma(y) = |y|$, $y \in \mathbb{R}$, then

$$\beta(y) = \begin{cases} -1 & \text{if } -\infty < y < 0, \\ 1 & \text{if } 0 \leq y < \infty \end{cases}$$

and if $\gamma(y) = y^2/2$, $y \in \mathbb{R}$, then

$$\beta(y) = y \quad \text{for } y \in \mathbb{R}.$$

For each $y \in (0, \infty)$ we have

$$U_y(E) = \int \beta(y - s(x))\, d\mathcal{L}^2 x \quad \text{for } E \in \mathcal{M}(\mathbb{R}^2).$$

# Binary images.

We fix a compact subset

$$S$$

of $\mathbb{R}^2$ with $\mathcal{L}^2(S) > 0$ and let $s = 1_S$. We have not yet attempted to extend the work which follows for more general $s$. We let

$$\mathbf{r}(y) = -\frac{\epsilon}{\beta(y-1)} \quad \text{and we let} \quad \mathbf{s}(y) = \frac{\epsilon}{\beta(y)} \quad \text{for } 0 < y < 1;$$

note that **r** is nondecreasing and **s** is nonincreasing.

# The functionals $M_{r,s}$.

For reasons which will shortly become obvious, we let $P = (0, \infty) \times (0, \infty)$ and for each $(r, s) \in P$ we let

$$M_{r,s}(E) = -\frac{1}{r}\mathcal{L}^2(E \cap S) + \frac{1}{s}\mathcal{L}^2(E \sim S) \quad \text{for } E \in \mathcal{M}(\mathbb{R}^2).$$

Note that

$$M_{r,s}(\emptyset) = 0.$$

Whenever $0 < y < 1$ we have

$$\frac{1}{\epsilon} U_y = M_{\mathbf{r}(y),\mathbf{s}(y)} \quad \text{for } 0 < y < 1.$$

so

$$\mathbf{n}^{loc}_{\Phi,\epsilon}(U_y) = \mathbf{n}^{loc}_{\Phi,1}(M_{\mathbf{r}(y),\mathbf{s}(y)}).$$

# From an earlier result it follows that...

...if

$$\upsilon$$

is a function with domain $(0,1)$ and values in $\mathcal{M}(\mathbb{R}^2)$ such that $\{(x,y) : x \in \upsilon(y)\}$ is $\mathcal{L}^2$ measurable;

$$\upsilon(y) \in \mathbf{n}_1^{loc}(M_{\mathbf{r}(y),\mathbf{s}(y)}) \quad \text{for } 0 < y < 1;$$

and

$$f(x) = \mathcal{L}^1(\{y \in (0,1) : x \in \upsilon(y)\}) \quad \text{for } x \in \mathbb{R}^2$$

then

$$f \in \mathbf{m}_{\Phi,\epsilon}^{loc}(F).$$

Moreover, all minimizers arise in this way.
Thus *if we can determine* $\mathbf{n}_{\Phi,1}^{loc}(M_{r,s})$ *for* $(r,s) \in P$ *we have determined* $\mathbf{m}_{\Phi,\epsilon}^{loc}(F)$ *for any* $\gamma$.

# The function $\Psi$ and the sets $Q$ and $N$.

For each $(r,s) \in P$ we let

$$\Psi(r,s) = \inf \left\{ \int_{\partial[E]} \Phi + M_{r,s}(E) : E \in \mathcal{M}(\mathbb{R}^2) \right\}.$$

We let

$$Q$$

be the set of those $(r,s) \in P$ such that $\mathbf{n}_1^{loc}(M_{r,s})$ contains a set of positive measure and we let

$$N$$

be the set of those $(r,s) \in P$ such that $\mathbf{n}_1^{loc}(M_{r,s})$ has two or more essentially distinct members of positive measure.

# More about Q.

Theorem. $Q$ is closed and there is a nondecreasing function $q : (0, \infty) \to (0, \infty)$ such that

$$Q = \{(r, s) \in P : r \leq q(s)\}.$$

Moreover, for any $(r, s) \in P$ we have

$$r < q(s) \;\Leftrightarrow\; \Psi(r, s) < 0.$$

# More about $N$.

For each $\eta > 0$ let

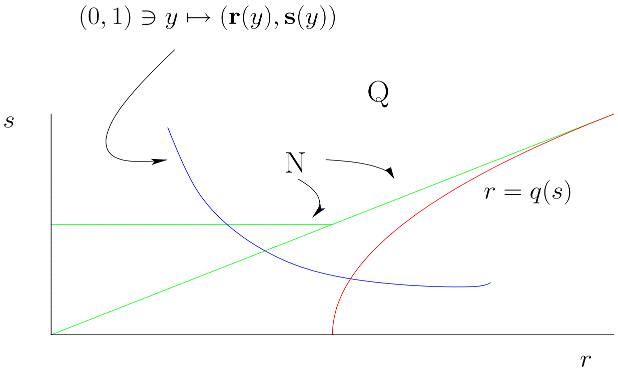
$$N_\eta = \{(r,s) \in N : \Psi(r,s) \leq -\eta\}.$$

**Theorem** Suppose $(r,s)$ is an accumulation point of $N_\eta$. Then

$$\mathbf{Tan}(N_\eta,(r,s)) \subset ([0,\infty) \times [0,\infty)) \cup ((-\infty,0] \times (-\infty,0]).$$

This implies that $N$ is $(\mathcal{H}^1, 1)$ rectifiable.

# The basic picture.



# When $S$ is convex.

For the next ten slides we suppose $\Phi = \mathrm{std}$.

For each $r \in (0, \infty)$ let

$$T_r = \cup\{\mathbf{B}(c,r) : \mathbf{B}(c,r) \subset S\}$$

and let

$$\Gamma(r) = \mathbf{M}(\partial[T_r]) - \frac{1}{r}\mathcal{L}^2(T_r).$$

**Theorem.** $\Gamma$ is increasing and locally Lipschitzian. Moreover,

$$Q = \{(r,s) \in P : \Gamma(r) \leq 0\}.$$

# Two squares.

Now let

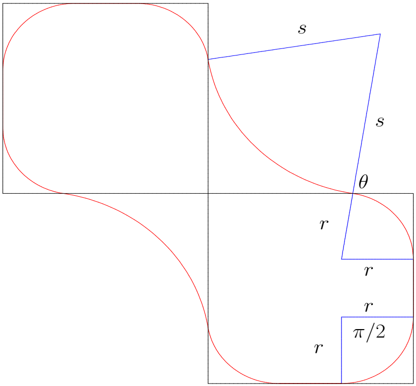
$$S = ([0,1] \times [0,-1]) \cup ([-1,0] \times [0,1]).$$

We will describe $\mathbf{n}_1^{loc}(M_{r,s})$, $(r,s) \in P$ in detail.

# The sets $F_{r,s}$, $r + s > 1$.

Here $r = .35$ and $s = .85$.

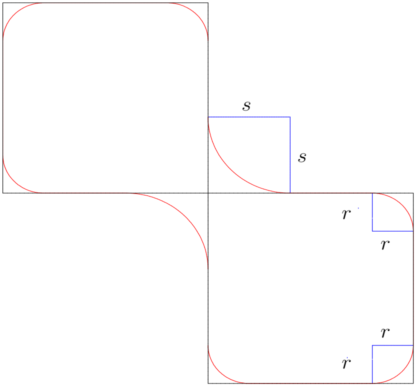
$$\theta = \Theta(r, s), \quad r + s > 1;$$



# The sets $G_{r,s}$, $r + s \leq 1$.

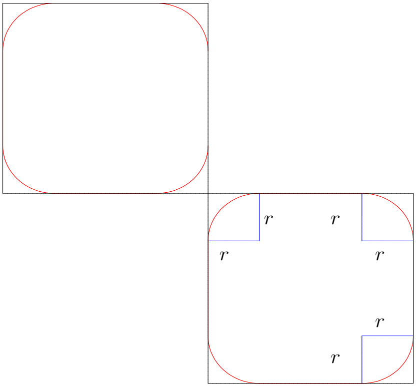
Here $r = .2$ and $s = .4$. $r + s \leq 1$

The blue corner angles are all right angles.



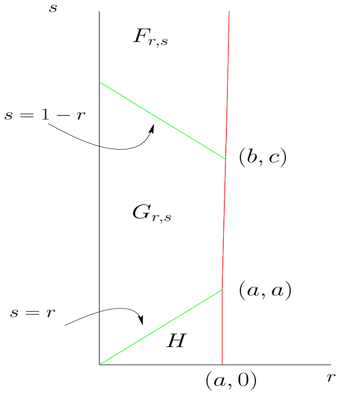
# The sets $H_r$, $r \leq 1/2$.

Here $r = .2$ . The blue angles are right angles.

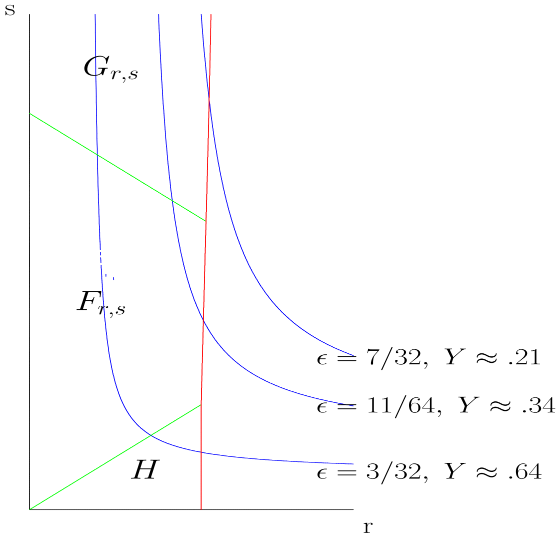


# The whole story for two squares.

$$a = \frac{\sqrt{2-\pi}}{4-\pi} \approx .2650794522$$
$$b = \frac{\sqrt{\pi^2+56\pi+16}-(\pi+12)}{16-4\pi} \approx 0.2725985674$$
$$c = \frac{\sqrt{\pi^2+56\pi+16}-5\pi+4}{16-4\pi} \approx 0.7274014326$$



# The various possibilities.



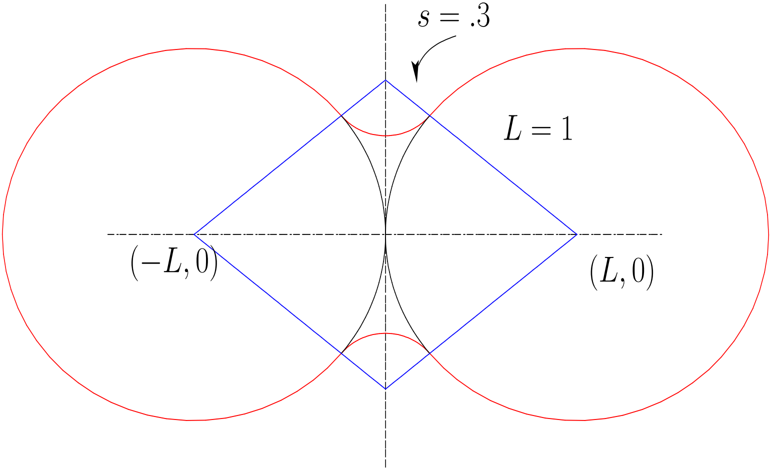
# Two circles.

Suppose $1 \leq L < \infty$. Let $c_{\pm} = (\pm L, 0)$ and let

$$S = \mathbf{B}(c_-, 1) \cup \mathbf{B}(c_+, 1).$$

We will describe $\mathbf{n}_1^{loc}(M_{r,s})$, $(r, s) \in P$, in detail.

# The sets $F_s$, $s \geq 2L/(L^2+1)$.



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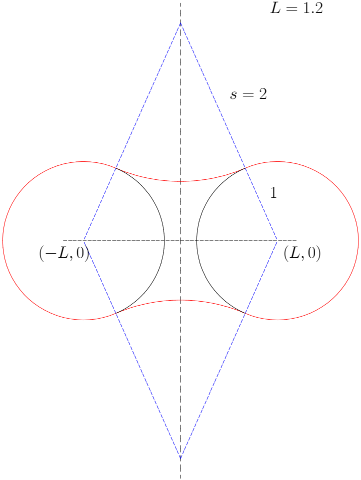


# The functions $\Gamma_F$ and $\Gamma_G$.

Let

$$\Gamma_F(r,s) = \int_{\partial[F_s]} \Phi + M_{r,s}(F_s), \quad 0 < r < \infty, \ \frac{2L}{L^2+1} < s < \infty$$

and let

$$\Gamma_G(r,s) = \int_{\partial[S]} \Phi + M_{r,s}(S), \quad (r,s) \in P.$$

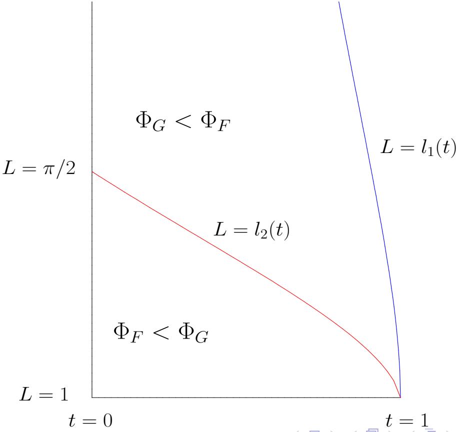
Theorem. Suppose $E \in \mathbf{n}_1^{loc}(M_{r,s})$. Then *either* $E = F_s$ *or* $E = S$. Well, not exactly...

# The winners; here $t = L/(1+s)$.

$$l_1(t) = \frac{1+\sqrt{1-t^2}}{t}$$

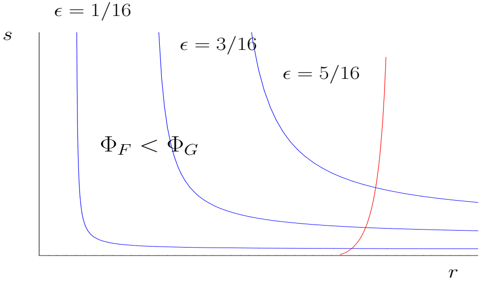
$$l_2(t) = \frac{t(1+\sqrt{1-\beta(t)})}{\beta(t)}$$

$$\beta(t) = \frac{2}{\pi}(\arcsin(t) + t\sqrt{1-t^2})$$



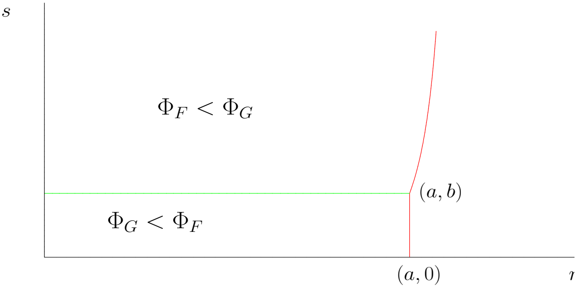
# Two circles touching.

$L = 1$



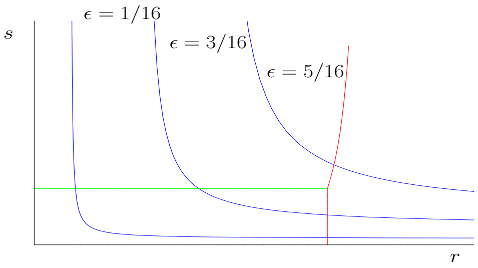
# Two circles separated.

$L = 1.2$ $\qquad$ $a = 1/2, \quad b = 0.5646473918$



# Two circles separated.

$L = 1$ $\gamma(y) = \frac{y^2}{2}$



# Wulff shapes.

Let

$$\Phi^W : \mathbb{R}^n \to [0, \infty)$$

be defined by setting

$$\Phi^W(v) = \sup\{< v \wedge \xi, \mathbf{V}^n >: \xi \in \{\Phi = 1\}\}.$$

It is evident that $\Phi^W$ is positively homogeneous, positive away from zero and convex. Let

$$W_\Phi = \{\Phi^W \leq 1\}.$$

The closed convex set $W_\Phi$ is called the Wulff shape associated to $\Phi$ (and $\mathbf{V}^n$).

# What a minimizer looks like.

We now suppose

$$E \in \mathbf{n}_1^{loc}(M_{r,s}).$$

From our Regularity Theorem it follows that $[E] = [\mathbf{spt}\,[E]]$ and that there is one and only one finite disjointed family

$$\mathcal{C}$$

of simple oriented closed Lipschitz curves such that

$$\partial[E](\omega) = \sum_{C \in \mathcal{C}} \int_C \omega \quad \text{for } \omega \in \mathcal{D}^1(\mathbb{R}^2).$$

Moreover, the oscillation of the tangent to these curves is controlled in an *a priori* manner by $\Phi$, $r$ and $s$.

# The view from the inside.

Suppose

$$b \in C \cap \textbf{int}\, S \quad \text{for some } C \in \mathcal{C}.$$

Then the connected component of $b$ in $C \cap \textbf{int}\, S$ is for some $c \in \mathbb{R}^2$ a subset of

$$B = c + rW_\Phi.$$

Moreover, $\textbf{spt}\,[E]$ near $C \cap \textbf{int}\, S$ lies on the *inside* of $B$, and the aforementioned connected component lies in an open half plane whose boundary contains c.

# The view from the outside.

Suppose

$$b \in C \cap \mathbf{int}\,(\mathbb{R}^2 \sim S) \quad \text{for some } C \in \mathcal{C}.$$

Then the connected component of $b$ in $C \cap \mathbf{int}\,(\mathbb{R}^2 \sim S)$ is for some $c \in \mathbb{R}^2$ a subset of

$$B = c + sW_{\Phi}.$$

Moreover, $\mathbf{spt}\,[E]$ near $C \cap \mathbf{int}\,(\mathbb{R}^2 \sim S)$ lies on the *outside* of $B$, and the aforementioned connected component lies in an open half plane whose boundary contains c.

# Two squares, again.

Suppose

$$\Phi(x) = \max\{|x_1|, |x_2|\} \quad \text{for } x \in \mathbb{R}^2.$$

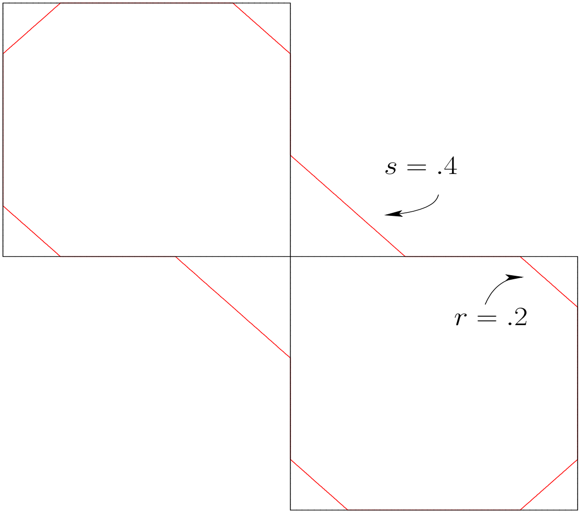
Then

$$W_\Phi = \{x \in \mathbb{R}^2 : |x_1| + |x_2| \leq 1\}.$$

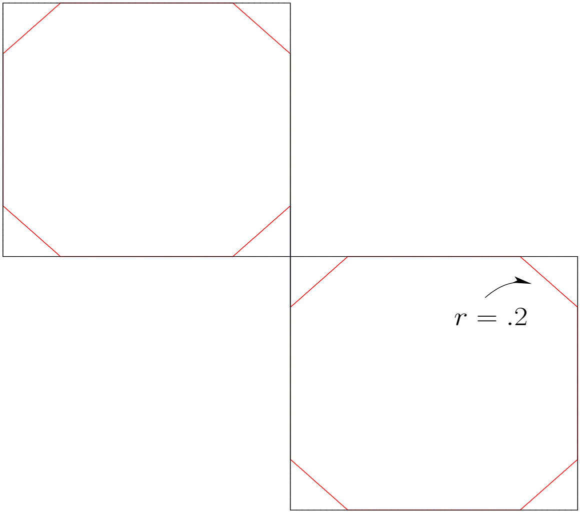
Let

$$S = ([0,1] \times [0,-1]) \cup ([-1,0] \times [0,1]).$$

# The sets $G_{r,s}$.



# The sets $H_r$.



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