# On the Higher Order Numerical Method for Nonlinear Two-Point Boundary Value Problems

Xiao-Yu Zhang and Qing Fang

Department of Mathematical Sciences, Faculty of Science, Yamagata University
Yamagata 990-8560, Japan

## Introduction

Aims of our work

- We are concerned with the numerical solution of a class of nonlinear two-point boundary value problems with general boundary conditions.
- We propose a new finite difference scheme of sixth order accuracy.

Features of our scheme

- By using an appropriately small number of grid points, we get the approximation with higher order accuracy, which is well enough up to the limit of precision under the consideration of rounding-off errors.
- Note that our numerical method is also efficient for layer equations.

## Scheme of our method

Consider the following nonlinear two-point boundary value problem in the form of

$$-(p(x)u')' + f(x,u) = 0, \quad a < x < b, \tag{1}$$

$$B_1[u] = \alpha_1 u(a) - \alpha_2 u'(a) = \alpha, \tag{2}$$

$$B_2[u] = \beta_1 u(b) + \beta_2 u'(b) = \beta, \tag{3}$$

where

$$\alpha_1 \geq 0,\ \alpha_2 \geq 0,\ (\alpha_1, \alpha_2) \neq (0,0),$$
$$\beta_1 \geq 0,\ \beta_2 \geq 0,\ (\beta_1, \beta_2) \neq (0,0),$$

and we assume that

- $p$, $f$ are smooth enough,
- $p(x) > 0$ in $[a,b]$,
- $\frac{\partial f}{\partial u}$ is nonnegative in $[a,b] \times \mathbb{R}$

## Step 1: CFDM of third order accuracy

- $a = x_0 < x_1 < \ldots < x_n < x_{n+1} = b$,
- $x_{i+\frac{1}{2}} = \frac{1}{2}(x_i + x_{i+1})$, $h_{i+1} = x_{i+1} - x_i$, $h = \max\{h_i\}$,
- At the grid point $x_i$, $i = 0, 1, 2, \ldots, n$, the compact finite difference scheme (by L.K.Bieniasz, 2008) is given as an extension of the method of Chawla-Shivakumar by

$$a_i U_{i-1} + b_i U_i + c_i U_{i+1} - (\alpha_i \tilde{F}_{i-\frac{1}{2}} + \beta_i \tilde{F}_i + \gamma_i \tilde{F}_{i+\frac{1}{2}}) = 0,$$

where $U_i$ represents the approximate value of $u(x_i)$,

$$a_i = \frac{2}{h_i(h_i + h_{i+1})},\ b_i = \frac{-2}{h_i h_{i+1}},\ c_i = \frac{2}{h_{i+1}(h_i + h_{i+1})},$$
$$\alpha_i = \frac{2h_i}{3(h_i + h_{i+1})},\quad \beta_i = \frac{1}{3},\quad \gamma_i = \frac{2h_{i+1}}{3(h_i + h_{i+1})},$$
$$\tilde{F}_{i-\frac{1}{2}} = F\left(x_{i-\frac{1}{2}}, \frac{U_{i-1} + U_i}{2}, \frac{U_i - U_{i-1}}{h_i}\right),$$

$$\tilde{F}_i = F(x_i, U_i + \Delta_i, p_i U_{i-1} + q_i U_i + r_i U_{i+1} - \rho_i(\tilde{F}_{i+\frac{1}{2}} - \tilde{F}_{i-\frac{1}{2}})),$$

$$\tilde{F}_{i+\frac{1}{2}} = F\left(x_{i+\frac{1}{2}}, \frac{U_i + U_{i+1}}{2}, \frac{U_{i+1} - U_i}{h_{i+1}}\right),$$

$$p_i = -\frac{h_{i+1}}{h_i(h_i + h_{i+1})},\ q_i = \frac{h_{i+1} - h_i}{h_i h_{i+1}},\ r_i = \frac{h_i}{h_{i+1}(h_i + h_{i+1})},$$
$$\Delta_i = -\eta_i(a_i U_{i-1} + b_i U_i + c_i U_{i+1}),$$
$$\rho_i = \frac{h_i^2 + h_i h_{i+1} + h_{i+1}^2}{6(h_i + h_{i+1})},\quad \eta_i = \frac{h_i^2 - h_i h_{i+1} + h_{i+1}^2}{4},$$

The boundary derivative approximation for $u'(x_0)$ is given by

$$U_0^{(1)} = \frac{U_1 - U_0}{h_1} - \frac{h_1}{6}(\hat{F}_0 + 2\tilde{F}_{\frac{1}{2}}),$$

with

$$\hat{F}_0 = F\left(x_0, U_0 - \frac{h_1^2}{4}\tilde{F}_{\frac{1}{2}}, \frac{U_1 - U_0}{h_1} - \frac{h_1}{2}\tilde{F}_{\frac{1}{2}}\right),$$

If $\alpha_2 \neq 0$, the discretization of (2): $\alpha_1 U_0 - \alpha_2 U_0^{(1)} = \alpha$.

The boundary derivative approximation for $u'(x_{n+1})$ is given by

$$U_{n+1}^{(1)} = \frac{U_{n+1} - U_n}{h_{n+1}} + \frac{h_{n+1}}{6}(\hat{F}_{n+1} + 2\tilde{F}_{n+\frac{1}{2}}),$$

with

$$\hat{F}_{n+1} = F\left(x_{n+1}, U_{n+1} - \frac{h_{n+1}^2}{4}\tilde{F}_{n+\frac{1}{2}}, \frac{U_{n+1} - U_n}{h_{n+1}} + \frac{h_{n+1}}{2}\tilde{F}_{n+\frac{1}{2}}\right),$$

If $\beta_2 \neq 0$, the discretization of (3): $\beta_1 U_{n+1} + \beta_2 U_{n+1}^{(1)} = \beta$.

It is known that $U_i$ is three order accurate and $U_0^{(1)}$ and $U_{n+1}^{(1)}$ are fourth order accurate (Bieniasz, 2008). Let $u^{(0)}(x)$ be the cubic spline function determined by

$$u^{(0)}(x_j) = U_j, \quad j = 0, 1, 2, \ldots, n+1,$$
$$u^{(0)'}(x_0) = U_0^{(1)}, \quad u^{(0)'}(x_{n+1}) = U_{n+1}^{(1)},$$

then there exists a positive constant C independent of $h$ s.t.

$$\|u^{(0)} - u\|_\infty \leq Ch^3$$

## Step 2: Linearized problem of (1)–(3)

$$-(p(x)v')' + f_u(x, u^{(0)}(x))v = f_u(x, u^{(0)}(x))u^{(0)}(x) - f(x, u^{(0)}(x)), \tag{4}$$

$$B_1[v] = \alpha, \tag{5}$$

$$B_2[v] = \beta. \tag{6}$$

Let $u(x)$ and $v(x)$ be the solutions of (1)–(3) and (4)–(6), respectively. Then $v - u$ satisfies the problem

$$-(p(x)(v-u)')' + f_u(x, u^{(0)}(x))(v-u) = f_u(x, u^{(0)}(x))(u^{(0)} - u) + f(x,u) - f(x, u^{(0)}), \tag{7}$$
$$B_1[v-u] = 0,$$
$$B_2[v-u] = 0.$$

## Exact solution of (4)–(6)

- $\alpha_2 \neq 0$, $\beta_2 \neq 0$, we have

$$v(x) = \int_a^b G(x,\xi)[f_u(\xi, u^{(0)}(\xi))u^{(0)}(\xi) - f(\xi, u^{(0)}(\xi))]\, d\xi + \frac{\alpha}{\alpha_2}p(a)G(x,a) + \frac{\beta}{\beta_2}p(b)G(x,b),$$

- $\alpha_2 = 0$, the term $\frac{\alpha}{\alpha_2}p(a)G(x,a) \Longrightarrow \frac{\alpha}{\alpha_1}p(a)G_\xi(x,a)$,
- $\beta_2 = 0$, the term $\frac{\beta}{\beta_2}p(b)G(x,b) \Longrightarrow -\frac{\beta}{\beta_1}p(b)G_\xi(x,b)$.

## Numerical solution of (4)–(6)

We use a sixth order Runge-Kutta (SORK) method to solve ODE system (Refer to J.C.Butcher).

Applying the SORK method to the initial value problem

$$y_1' = y_2, \tag{8}$$

$$y_2' = \frac{1}{p(x)}(f_u(x, u^{(0)}(x))y_1 - p'(x)y_2), \tag{9}$$

$$y_1(a) = \alpha_2, \quad y_2(a) = \alpha_1, \tag{10}$$

at $x_0$, $x_{\frac{1}{4}}$, $x_{\frac{1}{2}}$, $x_{\frac{3}{4}}$, $x_1$, ..., $x_n$, $x_{n+\frac{1}{4}}$, $x_{n+\frac{1}{2}}$, $x_{n+\frac{3}{4}}$, $x_{n+1}$ with step sizes $\frac{h_1}{4}$, $\frac{h_1}{4}$, $\frac{h_1}{4}$, $\frac{h_1}{4}$, $\frac{h_2}{4}$, ..., $\frac{h_{n+1}}{4}$, $\frac{h_{n+1}}{4}$, $\frac{h_{n+1}}{4}$, $\frac{h_{n+1}}{4}$, we denote the numerical solution by $Y_i = \left(Y_i^{(1)}, Y_i^{(2)}\right)$, $i = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \cdots, n, n+\frac{1}{4}, n+\frac{1}{2}, n+\frac{3}{4}, n+1$.

Next, applying the SORK method to the initial value problem (8)–(10) with initial conditions

$$y_1(b) = \beta_2, \quad y_2(b) = -\beta_1,$$

at $x_{n+1}$, $x_{n+\frac{3}{4}}$, $x_{n+\frac{1}{2}}$, $x_{n+\frac{1}{4}}$, $x_n$, ..., $x_1$, $x_{\frac{3}{4}}$, $x_{\frac{1}{2}}$, $x_{\frac{1}{4}}$, $x_0$ with step sizes $-\frac{h_{n+1}}{4}$, $-\frac{h_{n+1}}{4}$, $-\frac{h_{n+1}}{4}$, $-\frac{h_{n+1}}{4}$, ..., $-\frac{h_1}{4}$, $-\frac{h_1}{4}$, $-\frac{h_1}{4}$, $-\frac{h_1}{4}$, we denote the numerical solution by $Z_i = (Z_i^{(1)}, Z_i^{(2)})$, $i = n+1, n+\frac{3}{4}, n+\frac{1}{2}, n+\frac{1}{4}, n, \cdots, 1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}, 0$.

Let

$$g_{ij} = \begin{cases} \frac{Y_i^{(1)} Z_j^{(1)}}{\Delta}, & (i \leq j), \\ \frac{Z_i^{(1)} Y_j^{(1)}}{\Delta}, & (i \geq j), \end{cases}, \quad \text{where } \Delta = -p(b)\begin{vmatrix} Y_{n+1}^{(1)} & \beta_2 \\ Y_{n+1}^{(2)} & -\beta_1 \end{vmatrix},$$

$i, j = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \ldots, n, n+\frac{1}{4}, n+\frac{1}{2}, n+\frac{3}{4}, n+1$.

It can be shown that $g_{ij}$ is an approximation of sixth order accuracy of $G(x_i, x_j)$.

Also, let

$$\varphi_{ij} = g_{ij}\{f_u(x_j, U_j)U_j - f(x_j, U_j)\},$$

$j = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \ldots, n, n+\frac{1}{4}, n+\frac{1}{2}, n+\frac{3}{4}, n+1$. Then, by using a sixth order accurate numerical integration formula, that is,

$$V_i = \sum_{j=0}^{n} \frac{h_{j+1}}{90}(7\varphi_{ij} + 32\varphi_{ij+\frac{1}{4}} + 12\varphi_{ij+\frac{1}{2}} + 32\varphi_{ij+\frac{3}{4}} + 7\varphi_{ij+1}) \quad + \alpha p(a)\frac{Z_i^{(1)}}{\Delta} + \beta p(b)\frac{Y_i^{(1)}}{\Delta},$$

we know that $V_i$ is the **sixth order accurate** approximation of the exact solution of the problem (1)–(3) at each $x_i$, $i = 0, 1, 2, \cdots, n+1$.

## Numerical results

**Example 1** Dirichlet problem of nonlinear reaction-diffusion equation:

$$-(p(x)u')' + f(x,u) = 0, \quad 0 < x < 1,$$
$$u(0) = 0.25, \quad u(1) = -0.25,$$

where $p(x) = e^{-\frac{x^2}{2}}$,
$f(x,u) = ue^{-\frac{x^2}{2}} - e^{-\frac{x^2}{2}}(-e^u + e^{x(0.5-x)+0.25} - 3x^2 + x + 2.25)$
with exact solution $u(x) = x(0.5 - x) + 0.25$.

## Numerical results

Non-uniform partition of $[0,1]$ in Example 1:

- $h = 1/(3m)$, $n = 4m + 1$,
- $x_{2i+1} = x_{2i} + h, \quad x_{2i+2} = x_{2i+1} + h/2,$ $i = 0, 1, 2, \cdots, 2m-1.$

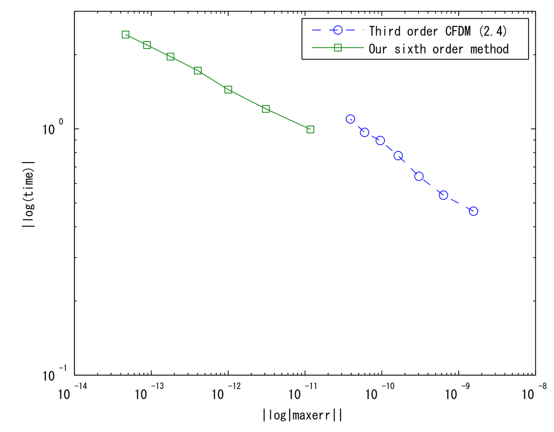
Table 1: Numerical results for Example 1 by using non-uniform partition

| $n$ | $h = \max h_i$ | $\max\lvert u_i - U_i\rvert$ | $\max\lvert u_i - U_i\rvert/h^6$ |
|---|---|---|---|
| 17 | 8.33e-2 | 1.778e-11 | 3.517e-5 |
| 21 | 6.67e-2 | 3.099e-12 | 3.530e-5 |
| 25 | 5.56e-2 | 1.005e-12 | 3.420e-5 |
| 29 | 4.76e-2 | 4.030e-13 | 3.456e-5 |
| 33 | 4.17e-2 | 1.780e-13 | 3.402e-5 |
| 37 | 3.70e-2 | 8.810e-14 | 3.413e-5 |
| 41 | 3.33e-2 | 4.630e-14 | 3.375e-5 |

Table 2: Time and maximum error results for Example 1

| Our sixth order method | | Third order compact fdm | |
|---|---|---|---|
| Maximum Error | Time (seconds) | Maximum Error | Time (seconds) |
| 1.178e-11 | 0.76230 | 1.296e-11 | 1.44299 |
| 1.006e-12 | 0.98012 | 1.755e-12 | 2.05615 |
| 1.782e-13 | 1.33596 | 1.447e-13 | 3.03292 |

Figure 1: Plot of |log(time)| with respect to |log|maxerr|| for Example 1




## Numerical results

**Example 2** We consider the following problem of layer equation:

$$-(p(x)u')' + q(x)u - r(x) = 0, \quad -1 < x < 1, \tag{11}$$

$$u(-1) = \frac{1 - \tanh(-10)}{2}, \quad u(1) = \frac{1 - \tanh(10)}{2}, \tag{12}$$

where
$p(x) = \cosh(10x), q(x) = 100e^{10x}, r(x) = 50\frac{\cosh(10x) - \sinh(10x)}{\cosh^2(10x)}$
with exact solution $u(x) = \frac{1}{2}(1 - \tanh(10x))$.

Here we use the following grid generation method with a stretching function

$$x = \begin{cases} y^p & \text{for } y \in [0,1], \\ -(-y)^p & \text{for } y \in [-1,0], \end{cases}$$

for some parameter $p > 0$ to generate a non-uniform partition, so that grid points are dense around $x = 0$ but the total number of grid points unchanged. Here we take $p = 3$.

Figure 2: Plot of the approximate solution when $n = 40$ for Example 2

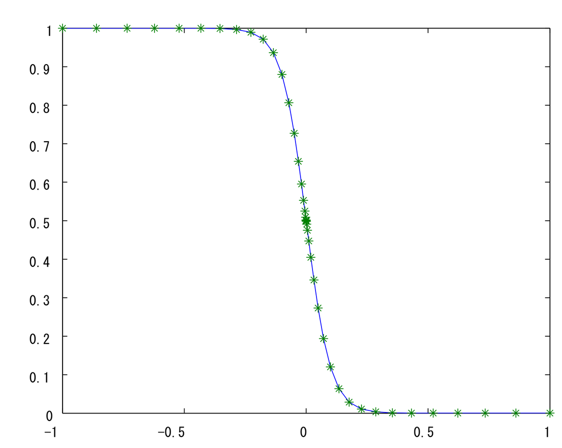


Table 3: Numerical results for Example 2

| $n$ | $\max h_i$ | $\max\lvert u_i - U_i\rvert$ | $\max\frac{\lvert u_i - U_i\rvert}{h^6}$ |
|---|---|---|---|
| 30 | 1.81e-1 | 1.711e-6 | 0.0481 |
| 40 | 1.39e-1 | 1.008e-6 | 0.1378 |
| 45 | 1.25e-1 | 6.554e-7 | 0.1731 |
| 50 | 1.13e-1 | 4.220e-7 | 0.2017 |
| 60 | 9.52e-2 | 1.849e-7 | 0.2489 |
| 70 | 8.21e-2 | 8.757e-8 | 0.2849 |

## References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