Diffusion Equation on Curved Surface with Thin Layer

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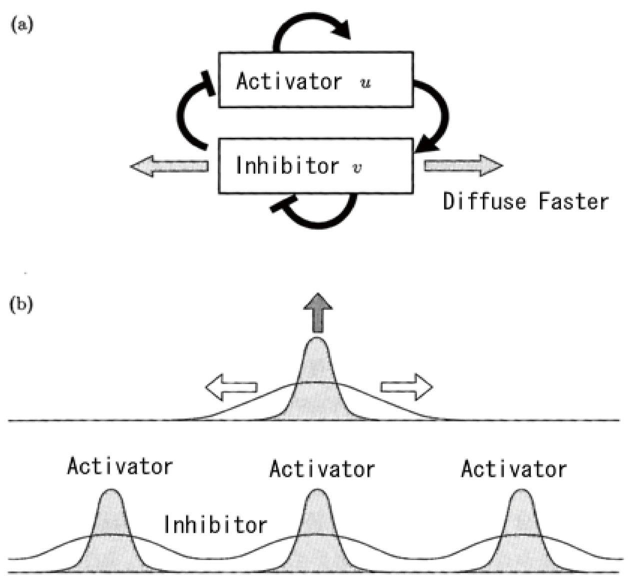
1. Motivation

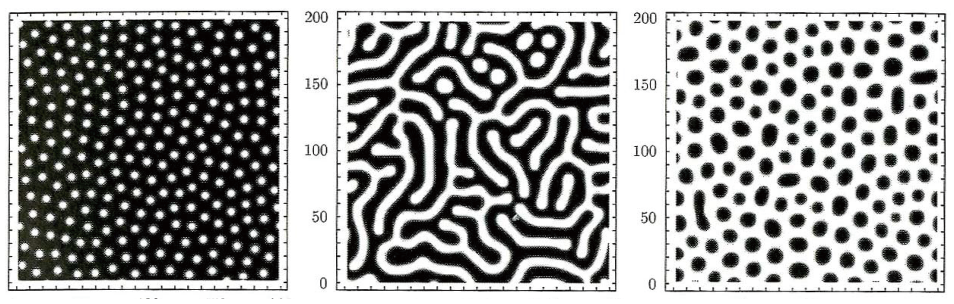


Reaction Diffusion Model (Linearlized)

$$\frac{\partial u}{\partial t} = au - bv + D_u \Delta u,$$

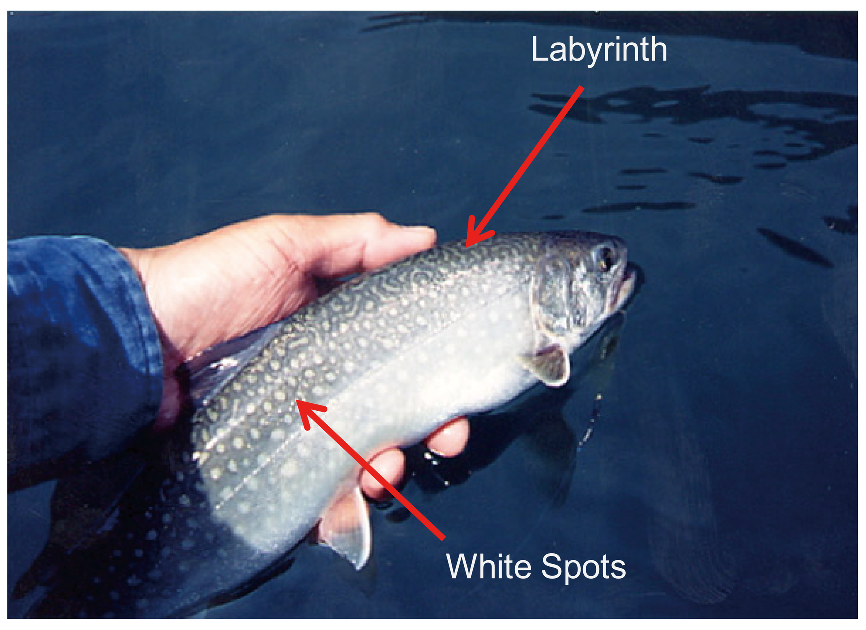
$$\frac{\partial v}{\partial t} = cu - dv + D_v \Delta v.$$

Patterns obtained by reaction diffusion system



White Spots   Labyrinth   Black Spots

Real animates has different patterns even in one individual!



For Char fish, side part has white spots, and dorsal part has Labyrinth pattern.

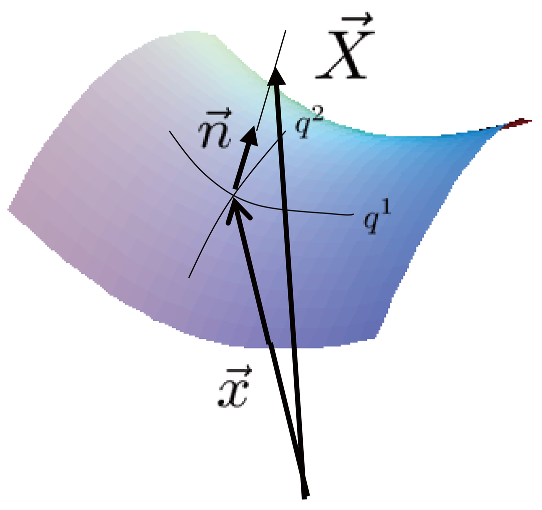
It seems curvature selects the pattern..

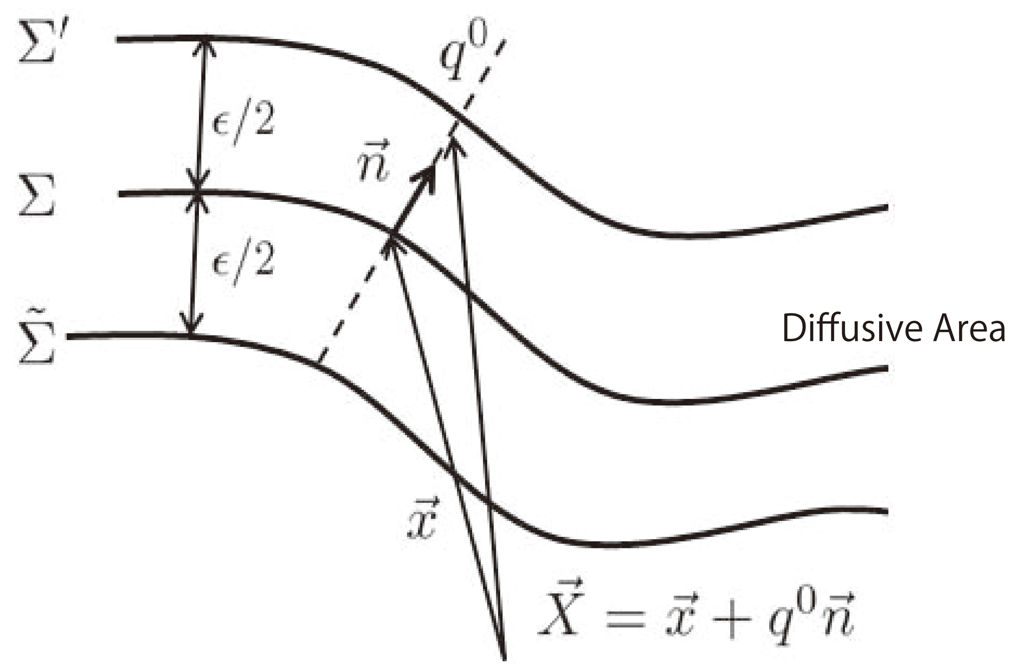
We need diffusion equation which depends not only on Gauss, but also mean curvature.

2. Problem setting

Diffusion occurs not in 2D but in 3D space.

We consider the surface Σ with thick ness ε.





$$\vec{X}(q^0, q^1, q^2) = \vec{x}(q^1, q^2) + q^0 \vec{n} \qquad -\epsilon/2 \le q^0 \le \epsilon/2$$

3. Geometrical quantities around Surface

$$G_{\mu\nu} = \frac{\partial \vec{X}}{\partial q^\mu} \cdot \frac{\partial \vec{X}}{\partial q^\nu} \qquad \mu, \nu = (0,1,2)$$

$$G_{0i} = G_{i0} = 0, \quad G_{00} = 1.$$

$$G_{ij} = g_{ij} + q^0 \left(\frac{\partial \vec{x}}{\partial q^i} \cdot \frac{\partial \vec{n}}{\partial q^j} + \frac{\partial \vec{x}}{\partial q^j} \cdot \frac{\partial \vec{n}}{\partial q^i} \right) + (q^0)^2 \frac{\partial \vec{n}}{\partial q^i} \cdot \frac{\partial \vec{n}}{\partial q^j} \quad (i, j = 1, 2)$$

$$g_{ij} = \vec{B}_i \cdot \vec{B}_j. \qquad \vec{B}_i = \frac{\partial \vec{x}}{\partial q^i}$$

(Metric on surface Σ)   (tangent)

Second fundamental Tensor

$$\kappa_{ij} = \frac{\partial \vec{n}}{\partial q^i} \cdot \vec{B}_j.$$

Mean Curvature

$$\kappa = g^{ij} \kappa_{ij},$$

Ricci Scalar, Gauss Curvature

$$R/2 = G = det(\kappa^i_j).$$

Gauss Equation

$$\frac{\partial \vec{B}_i}{\partial q^j} = -\kappa_{ij} \vec{n} + \Gamma^k_{ij} \vec{B}_k,$$

Weingarten Equation

$$\frac{\partial \vec{n}}{\partial q^j} = \kappa^m_j \vec{B}_m$$

Form of Metric tensor with Curvature inner thin film

$$G_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & G_{ij} \end{pmatrix}. \quad G_{ij} = (g^m_i + q^0 \kappa^m_i)(g_{mj} + q^0 \kappa_{mj})$$

Section 4. Embedding of Diffusion Field

$$\frac{\partial \phi^{(3)}}{\partial t} = D\Delta^{(3)}\phi^{(3)}, \quad 1 = \int \phi^{(3)}(q^0, q^1, q^2)\sqrt{G}\, d^3q$$

↓

$$\frac{\partial \phi^{(2)}}{\partial t} = D\tilde{\Delta}^{(2)}\phi^{(2)}, \quad 1 = \int \phi^{(2)}(q^1, q^2)\sqrt{g}\, d^2q$$

$$1 = \int \phi^{(3)}(q^0, q^1, q^2)\sqrt{G}\, d^3q \qquad \tilde{\phi}^{(3)}$$

$$= \int [\int_{-\epsilon/2}^{\epsilon/2} dq^0 (\phi^{(3)}\sqrt{G/g})] \sqrt{g}\, d^2q$$

$$= \int \phi^{(2)}(q^1, q^2)\sqrt{g}\, d^2q$$

$$\phi^{(2)}(q^1, q^2) = \int_{-\epsilon/2}^{\epsilon/2} \tilde{\phi}^{(3)} dq^0,$$

We multiply $\sqrt{G/g}$ and integrate by q^0 to equation

$$\frac{\partial \phi^{(3)}}{\partial t} = D\Delta^{(3)}\phi^{(3)}$$

$$\frac{\partial \phi^{(2)}}{\partial t} = D\int_{-\epsilon/2}^{\epsilon/2} \tilde{\Delta}^{(3)}\tilde{\phi}^{(3)} dq^0$$

$$\tilde{\Delta}^{(3)} \equiv \sqrt{G/g}\ \ \Delta^{(3)} \sqrt{g/G}.$$

-Decomposition of Laplacian-

$$\tilde{\Delta}^{(3)} = g^{-1/2}\frac{\partial}{\partial q^\mu} G^{1/2} G^{\mu\nu} \frac{\partial}{\partial q^\nu} (g/G)^{1/2} = \tilde{\Delta}^{(2)} + \tilde{\Delta}^{(1)}.$$

$$\tilde{\Delta}^{(2)} \equiv g^{-1/2}\frac{\partial}{\partial q^i} G^{1/2} G^{ij} \frac{\partial}{\partial q^j} (g/G)^{1/2}, \quad \tilde{\Delta}^{(1)} \equiv \frac{\partial}{\partial q^0} G^{1/2} \frac{\partial}{\partial q^0} G^{-1/2}.$$

-Boundary Condition-

Particles do not pass through the boundary

$$\int_{-\epsilon/2}^{\epsilon/2} \tilde{\Delta}^{(1)}\tilde{\phi}^{(3)} dq^0 = g^{-1/2}\int_{-\epsilon/2}^{\epsilon/2} \frac{\partial}{\partial q^0}(G)^{1/2}\frac{\partial}{\partial q^0}\phi^{(3)}\, dq^0$$

$$= g^{-1/2}[\ (G)^{1/2}\frac{\partial \phi^{(3)}}{\partial q^0}]\ |_{-\epsilon/2}^{\epsilon/2} = 0.$$

-Form of resultant Laplacian-

$$\tilde{\Delta}^{(2)} = \Delta^{(2)} + q^0 \hat{A} + (q^0)^2 \hat{B} + \mathcal{O}(\epsilon^3),$$

$$\hat{A} = -g^{-1/2}\frac{\partial}{\partial q^i} g^{1/2}(2\kappa^{ij}\frac{\partial}{\partial q^j} + g^{ij}\frac{\partial \kappa}{\partial q^j}),$$

$$\hat{B} = g^{-1/2}\frac{\partial}{\partial q^i} g^{1/2}(3\kappa^{im}\kappa^j_m\frac{\partial}{\partial q^j} + \frac{1}{2}g^{ij}\frac{\partial(\kappa^2 - R)}{\partial q^j} + 2\kappa^{ij}\frac{\partial \kappa}{\partial q^j})$$

-Equation-

$$\frac{\partial \phi^{(2)}}{\partial t} = D\Delta^{(2)}\phi^{(2)} + D\hat{A}\int_{-\epsilon/2}^{\epsilon/2} q^0 \tilde{\phi}^{(3)} dq^0 + D\hat{B}\int_{-\epsilon/2}^{\epsilon/2} (q^0)^2 \tilde{\phi}^{(3)} dq^0 + \mathcal{O}(\epsilon^3).$$

-Assumption-

Equilibrium condition into normal direction is satisfied in a small time ε^2/D

$$0 = \frac{\partial \phi^{(3)}}{\partial q^0} = g^{1/2}\frac{\partial G^{-1/2}\tilde{\phi}^{(3)}}{\partial q^0}.$$

$$\tilde{\phi}^{(3)} = \frac{1}{N}(G/g)^{1/2}\phi^{(2)}(q^1, q^2), \quad N \equiv \int_{-\epsilon/2}^{\epsilon/2} (G/g)^{1/2} dq^0.$$

Now integration can be done explicitly and we obtain effective 2-dimensional diffusion equation.

5. Anomalous Diffusion Equation

Two Dimensional Effective Diffusion Equation

$$\frac{\partial \phi^{(2)}}{\partial t} = D\Delta^{(2)}\phi^{(2)} + \frac{\epsilon^2}{12}D(\hat{A}\kappa + \hat{B})\phi^{(2)}$$

$$= D\Delta^{(2)}\phi^{(2)} + \tilde{D}g^{-1/2}\frac{\partial}{\partial q^i}\ g^{1/2}\{(3\kappa^{im}\kappa^j_m - 2\kappa\kappa^{ij})\frac{\partial}{\partial q^j} - \frac{1}{2}g^{ij}\frac{\partial R}{\partial q^j}\}\phi^{(2)}$$

Form of Diffusion flow

$$-\frac{\partial \phi^{(2)}}{\partial t} = \nabla_i(J^i_N + J^i_A) = g^{-1/2}\frac{\partial}{\partial q^i}\ g^{1/2}(J^i_N + J^i_A)$$

With $J^i_N = -Dg^{ij}\frac{\partial \phi^{(2)}}{\partial q^j}$,

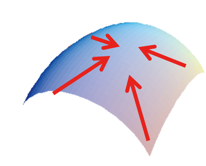
Anomalous Diffusion flow

$$J^i_A = -\frac{\epsilon^2}{12}D\{(3\kappa^{im}\kappa^j_m - 2\kappa\kappa^{ij})\frac{\partial \phi^{(2)}}{\partial q^j} - \frac{1}{2}g^{ij}\frac{\partial R}{\partial q^j}\phi^{(2)}\}.$$

Diffusion vs Concentration   curvature gradient flow

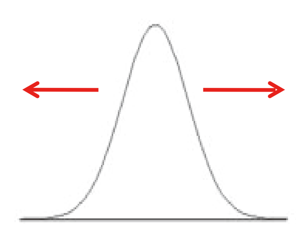
Properties of Anomalous flow

Ricci Scalar gradient flow



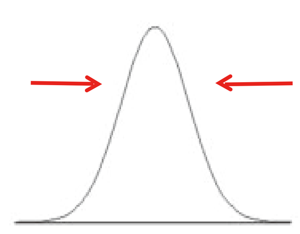
Diffusion

$f^{ij} \equiv 3\kappa^{im}\kappa^j_m - 2\kappa\kappa^{ij} > 0$



Concentration

$f^{ij} \equiv 3\kappa^{im}\kappa^j_m - 2\kappa\kappa^{ij} < 0$



Using the coordinate

$$g_{ij} = \delta_{ij}, \quad \kappa^i_j = diag[1/r_1,\ 1/r_2]$$

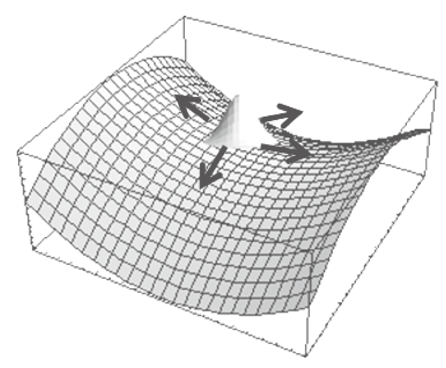
$$f^{ij} = \delta^{ij}\left(\frac{1}{(r_i)^2} - \frac{2}{r_1 r_2}\right)$$

Diffusion vs. Concentration (4 cases)

(1) Hyperbolic Surface R < 0 :

$$f^{11} > 0, \quad f^{22} > 0$$

Diffusion

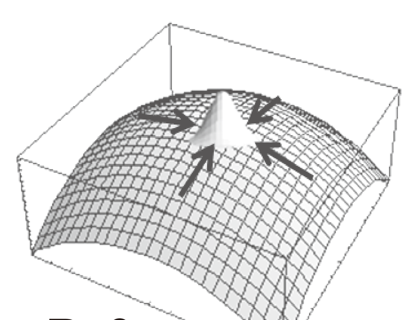


Hyperbolic surface, R<0

(2) Convex or Concave Surface R > 0 :

$$1/2 < \left|\frac{r_2}{r_1}\right| < 2. \quad f^{11} < 0, \quad f^{22} < 0$$

Concentration

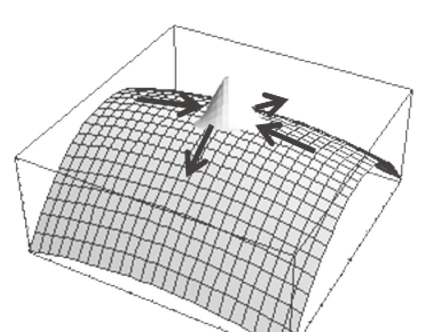


R>0, $1/2 < |\frac{r_2}{r_1}| < 2.$

(3) Convex or Concave Surface R > 0

$$\left|\frac{r_2}{r_1}\right| < 1/2, \text{ or } \left|\frac{r_2}{r_1}\right| > 2. \quad f^{11}f^{22} < 0$$

Diffusion and Concentration

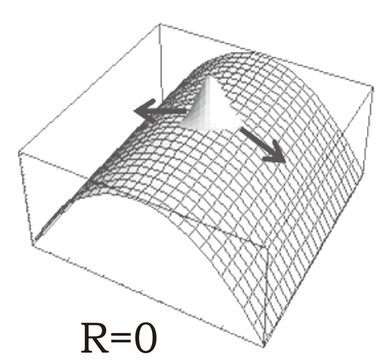


R>0, $|\frac{r_2}{r_1}| < 1/2$, or $|\frac{r_2}{r_1}| > 2.$

(4) Flat R=0:

$$r_2 = \infty, \qquad f^{22} = 0, \quad f^{11} > 0$$

Diffusion in One direction



R=0