

Large-time asymptotics for a class of non-coercive Hamilton-Jacobi equations appearing in crystal growth

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I. Background of Crystal Growth

Our work is inspired by the study of step motions under nonuniformity in supersaturation with microscopic time scale approximation [Yokoyama-Giga-Rybka, '08]. We review their results first.

Hamilton-Jacobi Equations for Step Motions

We consider a one-dimensional model of the motion of steps:

$$\begin{cases} u_t = m(u_x) \sqrt{1 + u_x^2} \sigma(x), & \text{for } (x, t) \in (0, \infty) \times (0, \infty), \\ u(0, t) = ct, & \text{for } t \in (0, \infty), \\ u(x, 0) = 0, & \text{for } x \in (0, \infty). \end{cases} \quad (1)$$

- The interval $(0, \infty)$ denotes a cross-section orthogonal to the edge of surface and the step source is assumed to be the origin.
- The function u of position $x \in [0, \infty)$ and time $t \in (0, \infty)$ represents the growing facet.
- m is the kinetic coefficient in the form

$$m(p) = \frac{p}{p_s} \tanh\left(\frac{p_s}{p}\right) \quad (2)$$

with a criterion of local slope p_s .

- σ denotes the surface supersaturation. Nonuniformity of supersaturation: σ is a decreasing function of x in $[0, \infty)$, due to Berg's effect.
- c is the growth rate at the step source, which actually depends on p_s , the supersaturation and slope at the step source.

Microscopic Time Scale Approximation and Non-coercivity

Let $\varepsilon = p_s$. As ε is in general small, we define $v_\varepsilon(x, t) = \frac{1}{\varepsilon} u(x, \varepsilon t)$ and then we get the microscopic time scale approximation: $v_\varepsilon \rightarrow v$ locally uniformly in $(0, \infty) \times (0, \infty)$, where v is the solution of

$$\begin{cases} v_t = H(x, v_x), & \text{for } (x, t) \in (0, \infty) \times (0, \infty), \\ v(0, t) = ct, & \text{for } t \in (0, \infty), \\ v(x, 0) = 0, & \text{for } x \in (0, \infty), \end{cases} \quad (3)$$

where the Hamiltonian $H(x, p) = p \tanh(1/p) \sigma(x)$ is however **non-coercive**:

$$\liminf_{|p| \rightarrow \infty} |H(x, p)| \neq \infty \text{ for any } x. \quad (4)$$

Numerical Results

Take $\sigma_0 = 1$, $c = \tanh(1)$ and $\sigma(x) = \max\{\sigma_0(1 - x^2), 0\}$. We have the following figures depicting the large-time behavior of solution of (3).

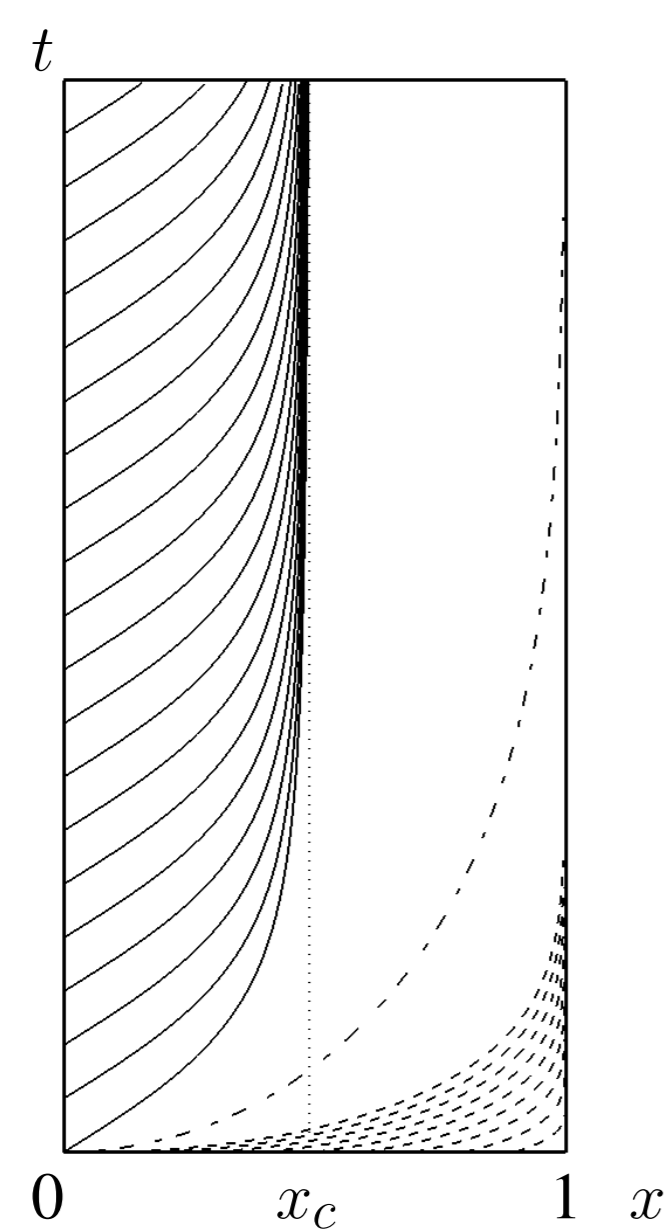


Figure 1: Characteristics of (3) in (x, t) plane

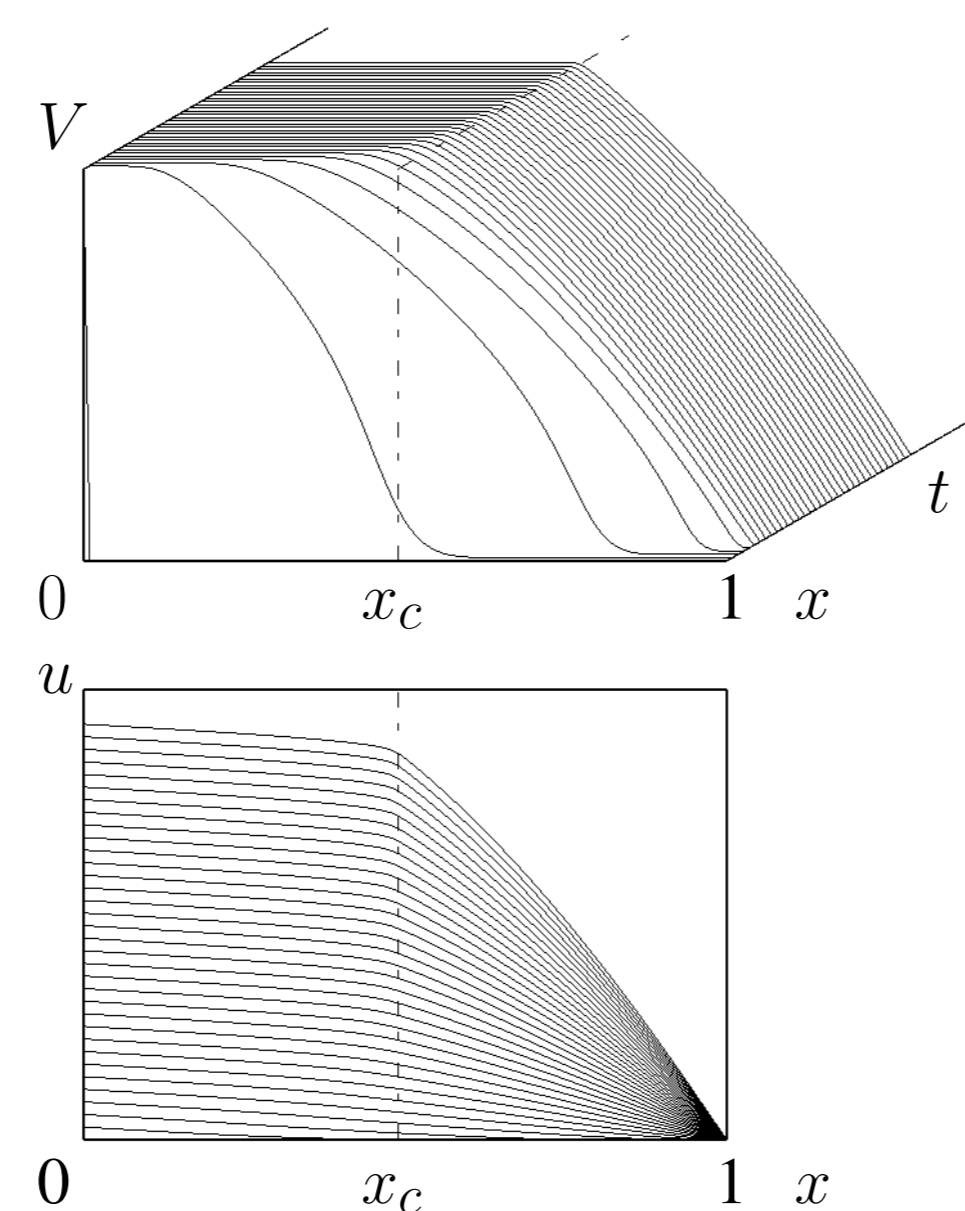


Figure 2: Growth rates of $u(x, t)$ of (3)

Conclusion:

- There is a metastable (critical) point x_c dividing the domain into two parts.
- The interval $(0, x_c)$ is the maximal stable region of a facet, in which the crystal has the same growth rate as the step source.
- The growth outside this interval is slower.

The non-coercivity is essential. Look at the following figures of (1), the coercive case.

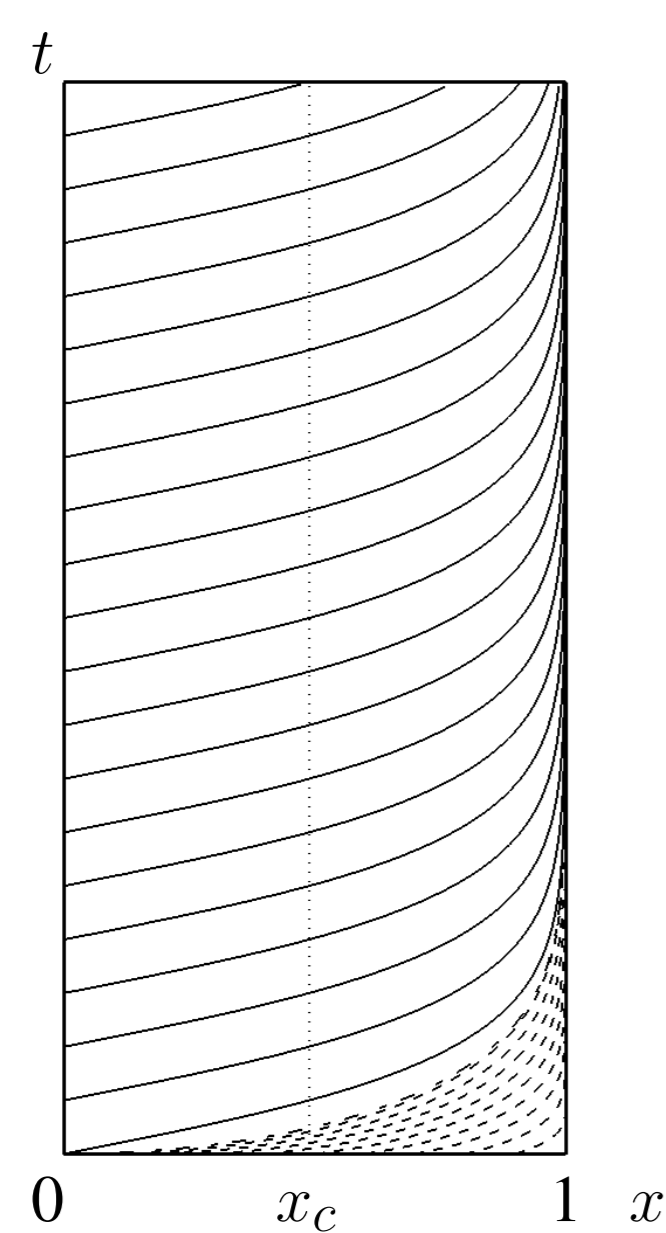


Figure 3: Characteristics of (1) in (x, t) plane

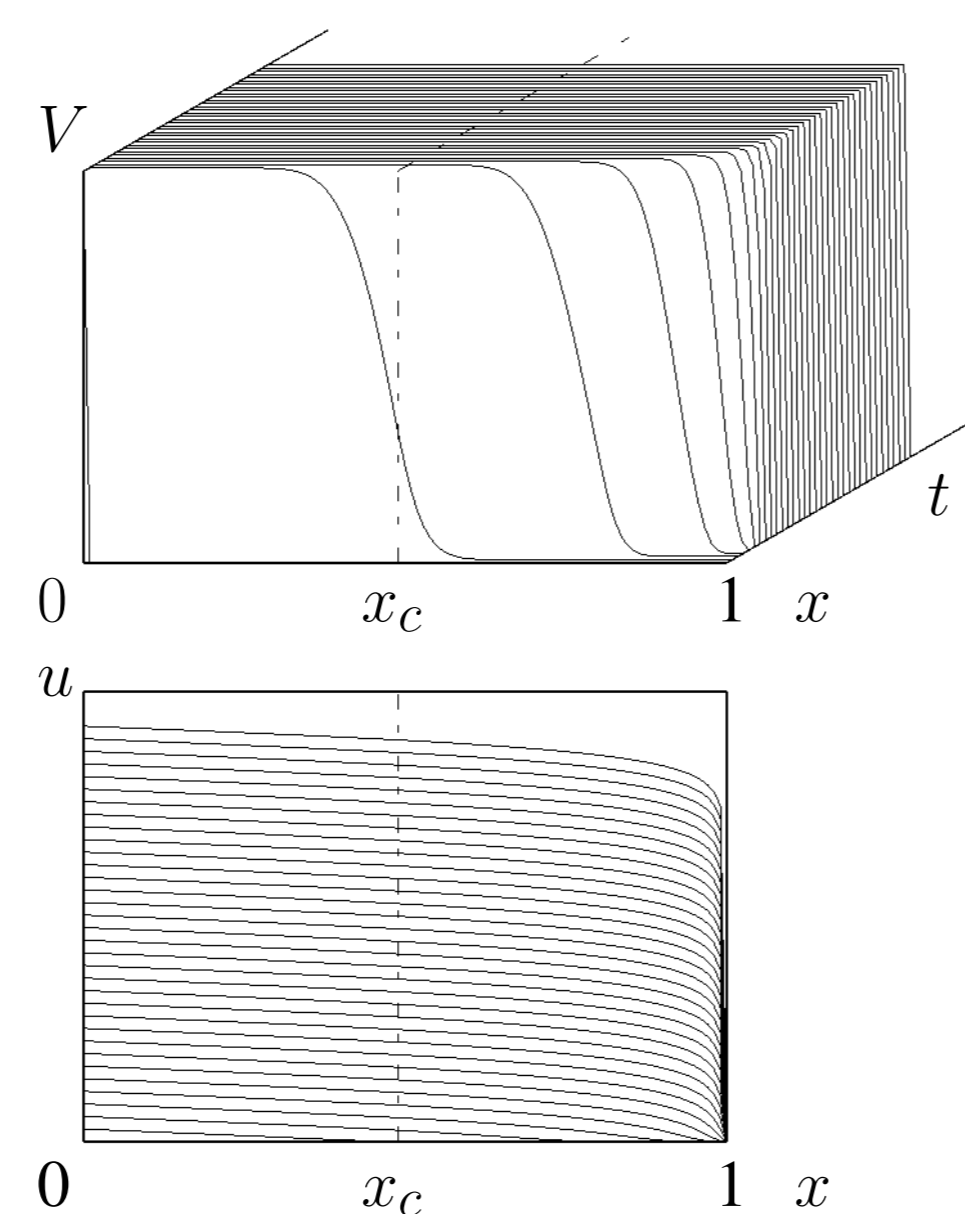


Figure 4: Growth rates of $u(x, t)$ of (1)

(All figures are supplied by E. Yokoyama.)

II. Analysis for More General Problems

We study two general problems in the framework of viscosity solution theory.

General One-dimensional Dirichlet Boundary Problems

$$\begin{cases} u_t + H(x, u_x) = 0, & \text{for } (x, t) \in (0, \infty) \times (0, \infty), \\ u(0, t) = 0, & \text{for } t \in (0, \infty), \\ u(x, 0) = u_0(x), & \text{for } x \in (0, \infty) \end{cases} \quad (5)$$

with $u_0(0) = 0$. It is important to investigate

$$\begin{cases} H(x, v_x) = 0 & \text{in } (0, \infty), \\ v(0) = 0, \end{cases} \quad (S)$$

whose viscosity solutions are not regular. Assume, for simplicity, H satisfies the following:

$$\text{(Continuity)} \quad \begin{cases} \exists C > 0 \text{ such that for } \forall x, y \in [0, \infty) \text{ and } \forall p, q \in \mathbb{R}, \\ |H(x, p) - H(y, q)| \leq C(|x - y| + |p - q|). \end{cases} \quad (A1)$$

$$\text{(Nonuniformity)} \quad \begin{cases} \lim_{|p| \rightarrow \infty} H(x, p) = c(x) \text{ locally uniformly in } x \text{ and } \exists x_c \in \mathbb{R} \text{ satisfying} \\ c > 0 \text{ in } [0, x_c), c < 0 \text{ in } (x_c, \infty) \text{ and } c(x_c) = 0. \end{cases} \quad (A2)$$

$$\text{(Instability)} \quad \forall x \in (x_c, \infty), \sup_{p \in \mathbb{R}} H(x, p) < 0. \quad (A3)$$

$$\text{(Monotonicity)} \quad \begin{cases} \exists l > 0 \text{ such that, for all } x \in (0, x_c] \text{ and } p \in \mathbb{R} \\ \text{satisfying } H(x, p) = 0, H(x_1, p) > H(x_2, p) \text{ holds} \\ \text{whenever } x_1, x_2 \in (x - l, x + l) \cap [0, x_c] \text{ and } x_1 < x_2. \end{cases} \quad (A4)$$

$$\text{(Lower bound)} \quad \exists p_1 \in C([0, x_c]) \cap L^1(0, x_c) \text{ such that } H(x, p_1(x)) \leq 0 \text{ in } x \in [0, x_c), \quad (A5)$$

$$\text{(Upper bound)} \quad \begin{cases} \exists \gamma_0 < 0 \text{ and } C_0 > 0 \text{ such that } H(x, p) \geq 0 \\ \text{for } \forall x \in [0, x_c) \text{ and } p \geq C_0 |x - x_c|^{\gamma_0 - 1}. \end{cases} \quad (A6)$$

Theorem 1. Assume (A1)–(A6). Assume that $u_0 \in C([0, \infty))$ with $u_0(0) = 0$ and satisfies

$$u_0(x) \geq \int_0^x p_1(y) dy \text{ for all } x \in [0, x_c]. \quad (6)$$

Let u be the viscosity solution of (5). Then for all $x \in [0, x_c) \cup (x_c, \infty)$, $u(x, t) \rightarrow v(x)$ as $t \rightarrow \infty$, where v is the unique viscosity solution of (S) and takes value $+\infty$ in (x_c, ∞) . Moreover, the convergence is locally uniform respectively in $[0, x_c)$ and in (x_c, ∞) .

Example 1. Let us see a simple example:

$$\begin{cases} u_t + \arctan |u_x|^\alpha - x = 0 & \text{in } (0, \infty) \times (0, \infty), \\ u(0, t) = 0 & \text{for all } t \in (0, \infty), \\ u(x, 0) = u_0(x) & \text{for } x \in (0, \infty). \end{cases} \quad (7)$$

Now that $x_c = \pi/2$ and the explicit solution of the stationary equation is

$$v(x) = \int_0^x (\tan y)^{\frac{1}{\alpha}} dy$$

for all $x \in [0, \pi/2)$. Theorem 1 gives the long time behavior that as $t \rightarrow \infty$, $u(x, t) \rightarrow v(x)$ for $x \in [0, \pi/2]$ and $u(x, t) \rightarrow \infty$ for $x > \pi/2$. But we only allow the initial data satisfying $u_0(0) = 0$ and the compatibility condition

$$u_0(x) \geq - \int_0^x (\tan y)^{\frac{1}{\alpha}} dy \text{ in } (0, \pi/2).$$

If it is violated, say $\exists x_0$ such that $p_0 = (u_0)_x(x_0) < -(\tan(x_0))^{\frac{1}{\alpha}}$ when $\alpha = 2$, then the characteristics $x(t) = x_0 + \arctan(p_0 + t)^2 - \arctan p_0^2$, solved in the equation

$$\begin{cases} \frac{dx}{dt} = \frac{2p}{1+p^4} \\ \frac{dp}{dt} = 1 \end{cases} \text{ with } \begin{cases} x(0) = x_0 \\ p(0) = p_0. \end{cases}$$

will hit the boundary $x = 0$ at $t = -p_0$ and the above results do not hold any more since the strict Dirichlet condition is broken.

What is New in Our Analysis?

We rigorously prove and improve the physical observation mainly at three aspects.

1. We get the asymptotic profile v in addition to the speed.
2. We allow stronger singularity at $x = x_c$. (In the example above, singularity is strong when $\alpha = 1$.)
3. We allow more general initial data u_0 and find the indispensable compatibility condition (6).

Higher dimensional Cauchy Problems

It is possible to extend the results above to higher dimensions and initial value problems. Similarly, we obtain stable, unstable and critical region. We here just mention key points in this case:

1. The importance of wellposedness of stationary equation like (5) is even more remarkable. We apply a different definition proposed by [Lasry-Lions, '89] and use its solution to construct the bounds for the time-dependent solution.
2. When handling the stationary equation, we use the technique of Aubry-Mather set to compensate the loss of Dirichlet condition.
3. We need to assume $x \rightarrow H(x, p)$ near the boundary of stable region is somewhat monotone along the normal.
4. The asymptotic profile depends only on the initial data in the stable region.