# Lecture 2:
# Mean curvature motion as a limit of dislocation dynamics

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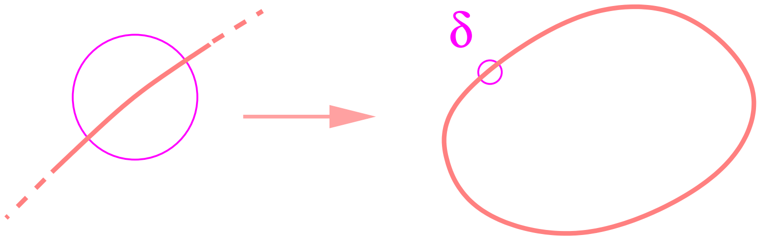



Nomal velocity (by rescaling)

$$c = c_0 \star 1_{\Omega_t} \qquad \Longrightarrow \qquad c = \kappa = \text{curvature}$$

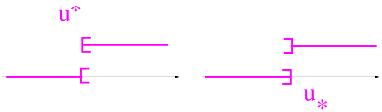
# Notion of viscosity solution

We consider solutions $u(t,x)$ of

$$\begin{cases} u_t = H(t,x,u,Du,D^2u) & \text{on} \quad (0,T)\times\mathbb{R}^N \\ u(0,x) = u_0(x) \end{cases}$$

with $H$ elliptic, i.e. satisfying

$$H(t,x,u,p,M+Q) \geq H(t,x,u,p,M) \quad \text{for all} \quad 0 \leq Q \in \mathbb{R}^{N\times N}_{sym}$$

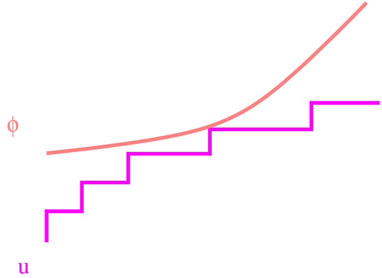



We define the upper and lower semi-continuous envelopes

$$\begin{cases} u^*(x) = \limsup_{y \to x} u(y) \\ u_*(x) = \liminf_{y \to x} u(y) \end{cases}$$

## Definition

- *$u$ is USC iff* $u = u^*$
- *$u$ is LSC iff* $u = u_*$
- *$u$ is continuous iff* $u^* = u_*$



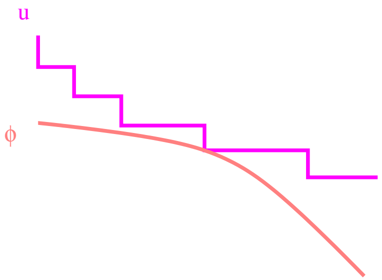

**Definition (Subsolution)**

*Assume that $u$ is USC. Then $u$ is a viscosity subsolution iff $u(0,x) \leq (u_0)^*(x)$ and for every test function $\varphi \in C^2$ satisfying*

$$\begin{cases} u \leq \varphi & \text{in a neighborhood of} \quad (t_0, x_0) \\ u = \varphi & \text{at} \quad (t_0, x_0) \end{cases}$$

*then*

$$\varphi_t \leq H^*(t_0, x_0, \varphi, D\varphi, D^2\varphi) \quad \text{at} \quad (t_0, x_0)$$




## Definition (**Supersolution**)

*Assume that $u$ is LSC. Then $u$ is a viscosity supersolution iff $u(0,x) \geq (u_0)_*(x)$ and for every test function $\varphi \in C^2$ satisfying*

$$\begin{cases} u \geq \varphi & \text{in a neighborhood of} \quad (t_0, x_0) \\ u = \varphi & \text{at} \quad (t_0, x_0) \end{cases}$$

*then*

$$\varphi_t \geq H_*(t_0, x_0, \varphi, D\varphi, D^2\varphi) \quad \text{at} \quad (t_0, x_0)$$

For a subsolution, you test from above!

**Definition (Viscosity solution)**

*A function $u$ is a viscosity solution iff $u^*$ is a subsolution and $u_*$ is a supersolution.*

**Theorem (Comparison principle)**

*Under structural assumptions on $H$,*
*if $u$ is a subsolution and $v$ a supersolution satisfying*

$$u \leq v \quad \text{at} \quad t = 0$$

*then*

$$u \leq v \quad \text{for} \quad t \geq 0$$

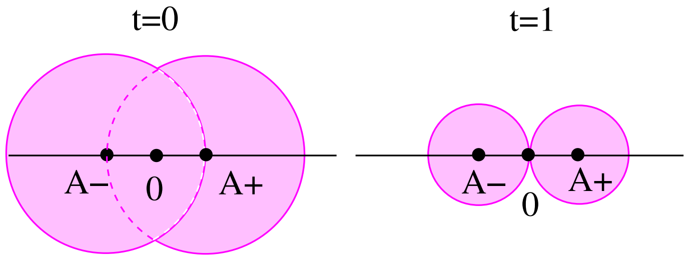
Example for $H = H(p) = -|p|$, and the equation

$$\rho_t = -|\nabla \rho|$$

we have the comparison principle.

We want to show that $\rho^1$ is not a viscosity solution.

# First solution




We set

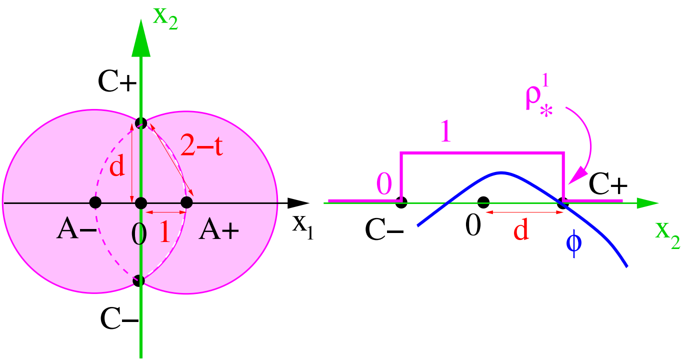
$$\Omega_t^1 = B_{2-t}(A_+) \cup B_{2-t}(A_-)$$

and

$$\rho^1 = 1_{\Omega_t^1}$$

is a distribution solution of

$$\rho_t = c|\nabla \rho| \quad \text{with} \quad c = -1$$




Locally we have

$$\varphi(t,x) = d(t) - x_2 \quad \text{with} \quad d(t) = \sqrt{(2-t)^2 - 1}$$

If $\rho^1_*$ is a supersolution, then

$$\varphi_t \geq -|\nabla \varphi| \quad \Longleftrightarrow \quad 1 \geq -d' = \frac{2-t}{\sqrt{(2-t)^2-1}} > 1$$

Contradiction. Therefore $\rho^1$ is not a viscosity solution.

# Mean curvature motion (MCM)

We consider solutions $u^0(t,x)$ of the general MCM

$$\begin{cases} u^0_t = H(Du^0, D^2u^0) \quad \text{on} \quad (0,T)\times\mathbb{R}^N \\ u^0(0,x) = u_0(x) \end{cases}$$

with

$$H(p,M) = \text{trace} \ \left(M\cdot A\left(\bar{p}\right)\right) \quad \text{with} \quad \bar{p} = \frac{p}{|p|}$$

where

$$\begin{cases} A \in C\left(\mathbb{S}^{N-1}; \mathbb{R}^{N\times N}_{sym}\right) \\ A\cdot Q = A \geq 0 \quad \text{with} \quad Q = Q(\bar{p}) = I - \bar{p}\otimes\bar{p} \end{cases}$$

**Remark**

*If $A\left(\bar{p}\right) = I - \bar{p}\otimes\bar{p}$, then this gives the classical isotropic MCM, i.e. each level set $\Gamma_t^\lambda = \left\{u^0(t,\cdot) = \lambda\right\}$ moves with normal velocity equal to $\sum_{i=1}^{N-1}\kappa_i$ where $\kappa_i$ are the principal curvatures of the hypersurface $\Gamma_t^\lambda$.*

**Theorem (Existence and uniqueness for MCM)**

*Let* $u_0 \in W^{1,\infty}(\mathbb{R}^N)$.
*Then there exists a unique solution* $u^0$ *of the general MCM equation.*

[Chen, Giga, Goto (1991)],
[Evans, Spruck (1991)]

# Dislocation dynamics

We consider solutions $\rho(t,x)$ of the following equation :

$$\rho_t = \left\{ J \star \rho - \frac{1}{2}\left(\int_{\mathbb{R}^N} J\right)\right\} |\nabla \rho| \quad \text{on} \quad (0,T)\times \mathbb{R}^N$$

with (for some normalizations)

$$0 \le J(-z) = J(z) = \frac{g(z/|z|)}{|z|^{N+1}} \cdot 1_{\{|z|\ge 1\}}, \quad g \in C(\mathbb{S}^{N-1};\mathbb{R})$$

We replace the discontinuous function $\rho$ by a continuous function $u$ such that

$$\{\rho = 1\} = \{u \ge 0\} \quad \text{and} \quad \{\rho = 0\} = \{u < 0\}$$

and set

$$u_t = \left\{ J \star 1_{\{u(t,\cdot)\ge 0\}} - \frac{1}{2}\left(\int_{\mathbb{R}^N} J\right)\right\} |\nabla u|$$

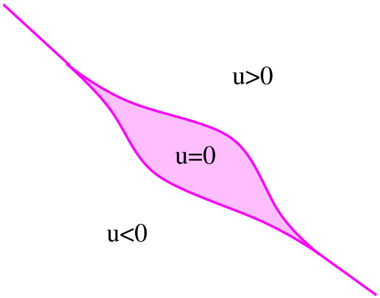
$$u_t = \left\{ \left( J \star 1_{\{u(t,\cdot)\geq 0\}} \right)(x) - \frac{1}{2}\left( \int_{\mathbb{R}^N} J \right) \right\} |\nabla u|$$

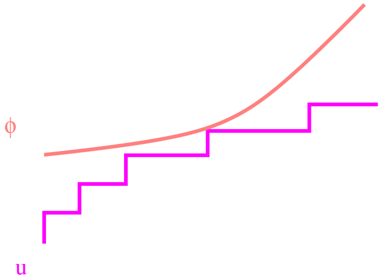
For this equation, all the level sets do not play the same role. We prefer the Slepčev level sets formulation

$$u_t(t,x) = \left\{ \left( J \star 1_{\{u(t,\cdot)\geq u(t,x)\}} \right)(x) - \frac{1}{2}\left( \int_{\mathbb{R}^N} J \right) \right\} |\nabla u(t,x)|$$

We consider solutions $u(t,x)$ solutions of the dislocation dynamics (DD)

$$\begin{cases} u_t = \left\{ \left(J \star 1_{\{u(t,\cdot)\geq u(t,x)\}}\right)(x) - \frac{1}{2}\left(\int_{\mathbb{R}^N} J\right)\right\} |\nabla u| & \text{on} \quad (0,T)\times\mathbb{R}^N \\ u(0,\cdot) = U_0 & \text{at} \quad t=0 \end{cases}$$





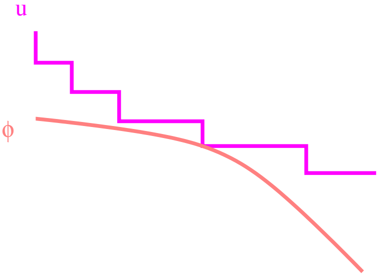

## Definition (**Subsolution**)

*Assume that $u$ is USC. Then $u$ is a viscosity subsolution iff $u(0,x) \leq (U_0)^*(x)$ and for every test function $\varphi \in C^2$ satisfying*

$$\begin{cases} u \leq \varphi & \text{in a neighborhood of} \quad (t_0,x_0) \\ u = \varphi & \text{at} \quad (t_0,x_0) \end{cases}$$

*then*

$$\varphi_t \leq \left\{ \left( J \star 1_{\{u(t_0,\cdot) \geq u(t_0,x_0)\}} \right)(x_0) - \frac{1}{2}\left( \int_{\mathbb{R}^N} J \right) \right\} |\nabla \varphi| \quad \text{at} \quad (t_0,x_0)$$




## Definition (**Supersolution**)

*Assume that* $u$ *is LSC. Then* $u$ *is a viscosity supersolution iff* $u(0,x) \geq (U_0)_*(x)$ *and for every test function* $\varphi \in C^2$ *satisfying*

$$\begin{cases} u \geq \varphi & \text{in a neighborhood of} \quad (t_0, x_0) \\ u = \varphi & \text{at} \quad (t_0, x_0) \end{cases}$$

*then*

$$\varphi_t \geq \left\{ \left( J \star 1_{\{u(t_0,\cdot) > u(t_0,x_0)\}} \right)(x_0) - \frac{1}{2}\left( \int_{\mathbb{R}^N} J \right) \right\} |\nabla \varphi| \quad \text{at} \quad (t_0, x_0)$$

**Definition (Viscosity solution)**

*A function $u$ is a viscosity solution iff $u^*$ is a subsolution and $u_*$ is a supersolution.*

**Theorem (Comparison principle)**

*Under the previous assumptions,*
*if $u$ is a subsolution and $v$ a supersolution satisfying*

$$u \leq v \quad \textit{at} \quad t = 0$$

*then*

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[Da Lio, Forcadel, M. (2008)], assuming that $J \in BV(\mathbb{R}^N)$,

[Ishii, Matsumura (2009)], assuming that $J \in L^1(\mathbb{R}^N)$
and can even deal with kernel $J$, singular at the origin.

**Theorem (Existence and uniqueness for dislocation dynamics)**

*Let $U_0 \in W^{1,\infty}(\mathbb{R}^N)$.*
*Then there exists a unique solution $u$ of (DD).*

# Rescaling dislocation dynamics

We define for $\varepsilon > 0$, the almost parabolic rescaling

$$u^\varepsilon(t,x) = u\left(\frac{t}{\varepsilon^2|\ln\varepsilon|}, \frac{x}{\varepsilon}\right)$$

Then $u^\varepsilon$ solves

$$\begin{cases} u^\varepsilon_t = \left\{\left(J^\varepsilon \star 1_{\{u^\varepsilon(t,\cdot)\geq u^\varepsilon(t,x)\}}\right)(x) - \frac{1}{2}\left(\int_{\mathbb{R}^N} J^\varepsilon\right)\right\}|\nabla u^\varepsilon| & \text{on } (0,T)\times\mathbb{R}^N \\ u^\varepsilon(0,\cdot) = u_0 & \text{at } t = 0 \end{cases}$$

with

$$J^\varepsilon(z) = \frac{1}{\varepsilon^{N+1}|\ln\varepsilon|}J\left(\frac{z}{\varepsilon}\right) = \frac{1}{|\ln\varepsilon|}g(z)\cdot 1_{\mathbb{R}^N\setminus B_\varepsilon}(z)$$

where we set

$$g(z) = \frac{g(z/|z|)}{|z|^{N+1}}$$

**Theorem (MCM as a limit of DD [Da Lio, Forcadel, M. (2008)])**

*Let* $u_0 \in W^{1,\infty}(\mathbb{R}^N)$*.*
*Then the solution* $u^\varepsilon$ *satisfies*

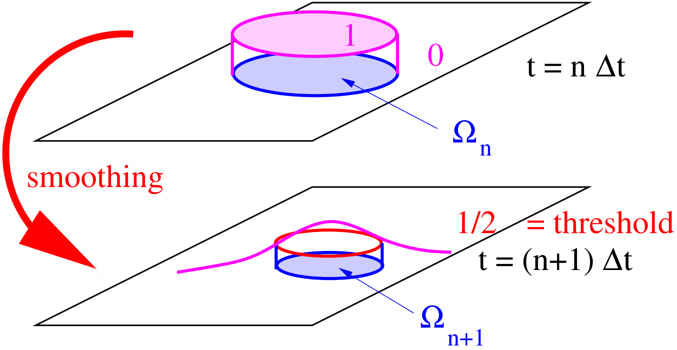
$$u^\varepsilon \to u^0 \quad \textit{in} \quad L^\infty_{loc}([0,T) \times \mathbb{R}^N)$$

*where* $u^0$ *is the solution of MCM with the matrix*

$$A_g(\bar{p}) = \int_{\theta \,\in\, \mathbb{S}^{N-2} \,=\, \mathbb{S}^{N-1} \cap \{\bar{p}^\perp\}} \left( \frac{1}{2} g(\theta)\ \theta \otimes \theta \right) d\theta$$

# Related literature

- Gamma limit for the stationary problem
  [Garroni, Muller (2004)]
- Convergence results for the Merriman, Bence, Osher algorithm
  [Barles, Georgelin (1995)],
  [Chambolle, Novaga (2007)]
  [Evans (1993)],
  [Ishii (1995)],
  [Ishii, Pires, Souganidis (1999)],
  [Merriman, Bence, Osher (1992)],


smoothing
1
0
t = n Δt
Ω n
1/2
= threshold
t = (n+1) Δt
Ω n+1

# Proof of the convergence

The main step is the proof of the following result

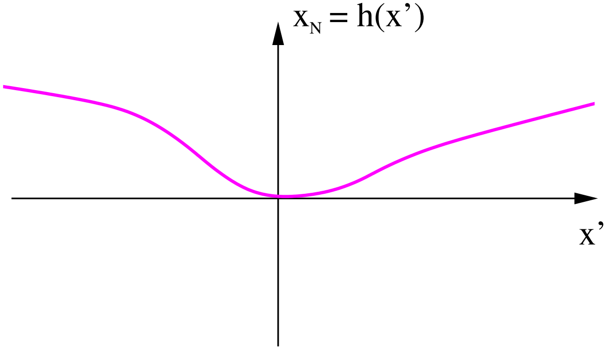
**Proposition (Convergence on a test function)**

*Let* $\varphi \in C^2$ *with* $|D\varphi(0)| = 1$. *Let*

$$c^\varepsilon = \left(J^\varepsilon \star 1_{\{\varphi>\varphi(0)\}}\right)(0) - \frac{1}{2}\int_{\mathbb{R}^N} J^\varepsilon$$

*Then*

$$c^\varepsilon \to H(D\varphi(0), D^2\varphi(0))$$

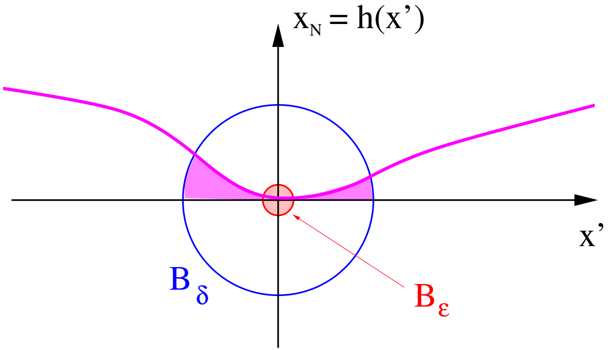



To simplify, we assume that there exists a function $h$ such that for $x = (x', x_N)$

$$\begin{cases} \{\varphi \geq \varphi(0)\} = \{x_N \leq h(x')\} \\ h(x') \geq 0 = h(0) \quad \text{and} \quad Dh(0) = 0 \end{cases}$$

Moreover

$$D\varphi(0) = -e_N \quad \text{and} \quad (I - e_N \otimes e_N) \cdot D^2\varphi \cdot (I - e_N \otimes e_N) = D^2 h(0)$$




We introduce $\delta$ fixed small such that

$$\varepsilon << \delta$$

$$
\begin{aligned}
c^\varepsilon &= \left(J^\varepsilon \star 1_{\{\varphi>\varphi(0)\}}\right)(0) - \tfrac{1}{2}\int_{\mathbb{R}^N} J^\varepsilon \\
&= \int_{\mathbb{R}^N} dx\ J^\varepsilon(-x) 1_{\{x_N \le h(x')\}} - \tfrac{1}{2}\int_{\mathbb{R}^N} J^\varepsilon \\
&= \int_{\mathbb{R}^N} dx\ J^\varepsilon(x) 1_{\{0 \le x_N \le h(x')\}} \\
&= \frac{1}{|\ln\varepsilon|}\int_{\mathbb{R}^N\setminus B_\varepsilon} dx\ g(x) 1_{\{0\le x_N\le h(x')\}} \\
&\simeq \frac{1}{|\ln\varepsilon|}\int_{B_\delta\setminus B_\varepsilon} dx\ g(x) 1_{\{0\le x_N\le \frac{1}{2}D^2h(0)\cdot(x',x')\}} \\
&\simeq \frac{1}{|\ln\varepsilon|}\int_{B_\delta^{N-1}\setminus B_\varepsilon^{N-1}} dx'\ g(x',0)\tfrac{1}{2}D^2h(0)\cdot(x',x') \\
&\simeq \frac{1}{|\ln\varepsilon|}\left(\int_\varepsilon^\delta dr\ r^{N-2}\frac{1}{r^{N+1}}r^2\right)\int_{\theta\in\mathbb{S}^{N-2}} d\theta\ g(\theta)\tfrac{1}{2}D^2h(0)\cdot(\theta,\theta) \\
&\simeq \frac{\ln(\delta/\varepsilon)}{|\ln\varepsilon|}A_g(e_N) : D^2\varphi(0) \qquad \to H(D\varphi(0), D^2\varphi(0))
\end{aligned}
$$

# Variational nature of the limit MCM

Let us define the distribution $L_g$ (with $g(\lambda z) = |\lambda|^{-N-1} g(z)$) as

$$< L_g, \varphi >= \frac{1}{2\pi} \int_{\mathbb{R}^N} g(z) \left\{ \varphi(z) - \varphi(0) - z \cdot \nabla \varphi(0) \cdot 1_{B_1}(z) \right\}$$

**Theorem (Variational MCM [Da Lio, Forcadel, M. (2008)])**

*Let $N \geq 2$ and $g \in C(\mathbb{S}^{N-1}; \mathbb{R})$. Then $\widehat{L_g}(\lambda \xi) = |\lambda| \widehat{L_g}(\xi)$ and*

$$A_g = -D^2_{\xi\xi} \widehat{L_g} \quad \text{on} \quad \mathbb{S}^{N-1}$$

Also applies to the Merriman, Bence, Osher algorithm with general kernel $K_0$ where

$$A(\bar{p}) = \int_{x \in \mathbb{R}^{N-1} = \bar{p}^\perp} \left( \frac{1}{2} K_0(x) \cdot x \otimes x \right) \, dx$$

Let us define the energy (with $L = L_g$)

$$E(u^0) = \int -\widehat{L}(Du^0)$$

and assume that $u^0$ solves

$$\text{normal velocity} = \frac{u^0_t}{|Du^0|} \quad = -E'(u^0)$$

$$= -\text{div}\left\{(D\widehat{L})(Du^0)\right\}$$

$$= -(D^2u^0 : D^2\widehat{L}(p)) \quad \text{with} \quad p = Du^0$$

$$= \text{trace}(D^2u^0 \cdot A_g(\bar{p})) \cdot |Du^0|^{-1}$$

$$= \text{anisotropic mean curvature}$$

Therefore $u^0$ is solution of MCM, which is then variational.

**Step 1**

$$
\begin{aligned}
< -D^2_{\xi\xi}\widehat{L}(\xi)\cdot(\zeta,\zeta),\varphi > \quad &=< \mathcal{F}(-ix\otimes ixL)\cdot(\zeta,\zeta),\varphi > \\
&=< \mathcal{F}((x\cdot\zeta)^2 L),\varphi >) \\
&=< L,(x\cdot\zeta)^2\mathcal{F}(\varphi) > \\
&= \tfrac{1}{2\pi}\int_{\mathbb{R}^N} dx\ g(x)(x\cdot\zeta)^2\mathcal{F}(\varphi)(x) \\
&= \tfrac{1}{2\pi} < \mathcal{F}(g(x)(x\cdot\zeta)^2),\varphi >
\end{aligned}
$$

Then

$$
-D^2_{\xi\xi}\widehat{L}(\xi)\cdot(\zeta,\zeta) = \frac{1}{2\pi}\mathcal{F}(g(x)(x\cdot\zeta)^2)
$$

# Formal proof of the theorem

**Step 2** : for $\theta = x/|x|$, $r = |x|$, $|\xi| = 1$

$$\begin{aligned}
\mathcal{F}(g(x)(x\cdot\zeta)^2) &= \int_{\mathbb{R}^N} dx\ g(x)(x\cdot\zeta)^2 e^{-i\xi\cdot x} \\
&= \int_{\mathbb{S}^{N-1}\times(0,+\infty)} d\theta dr\ g(\theta)(\theta\cdot\zeta)^2 e^{-i\xi\cdot r\theta} \\
&= \int_{\mathbb{S}^{N-1}} d\theta\ g(\theta)(\theta\cdot\zeta)^2 \int_0^{+\infty} dr \left(\frac{e^{-i\xi\cdot r\theta}+e^{i\xi\cdot r\theta}}{2}\right) \\
&= \int_{\mathbb{S}^{N-1}} d\theta\ \frac{1}{2} g(\theta)(\theta\cdot\zeta)^2 \underbrace{\int_{-\infty}^{+\infty} dr e^{-i\xi\cdot r\theta}}_{=2\pi\delta_0(\xi\cdot\theta)}
\end{aligned}$$

Therefore

$$-D^2_{\xi\xi}\widehat{L}(\xi)\cdot(\zeta,\zeta) = \frac{1}{2\pi}\mathcal{F}(g(x)(x\cdot\zeta)^2) = \int_{\mathbb{S}^{N-1}\cap\{\xi^\perp\}} d\theta\ \frac{1}{2} g(\theta)\cdot(\theta\cdot\zeta)^2$$

# Conclusion

- We have

$$g \geq 0 \quad \Longrightarrow \quad A_g \geq 0 \quad \Longrightarrow \quad -\widehat{L_g} \quad \text{is convex}$$

- If $N = 2$, then

$$g \geq 0 \quad \Longleftrightarrow \quad -\widehat{L_g} \quad \text{is convex}$$

- If $N \geq 3$, then

$$\text{There exists} \quad -\widehat{L_g} \quad \text{convex with} \quad g \not\geq 0$$

## Counter-example with $g \not\geq 0$ and $-\widehat{L_g}$ convex

For $\xi = (\xi_1, \xi_2, \xi_3)$, take a smooth version of

$$-\widehat{L_g}(\xi) = |(\xi_1, \xi_2)| + |(\xi_2, \xi_3)| + |(\xi_3, \xi_1)| - \eta|\xi| \quad \text{for} \quad \eta > 0 \quad \text{small enough}$$

This means

$$g(z) = \delta_S(z) - \frac{2}{\pi}\eta$$

with

$$S = \mathbb{S}^2 \cap \{z_1 = 0\} \cap \{z_2 = 0\} \cap \{z_3 = 0\}$$

# Further results

**Theorem (Error estimate [Forcadel (2008)])**

*There exists* $C = C(T, |Du_0|_{L^\infty}, N, g) > 0$*, such that*

$$|u^\varepsilon - u^0|_{L^\infty((0,T)\times\mathbb{R}^N)} \leq \frac{C}{|\ln\varepsilon|^{\frac{1}{6}}}$$

**Theorem (Error estimate for a scheme for MCM [Forcadel (2008)])**

*There exists* $C = C(T, |Du_0|_{L^\infty}, N, g) > 0$, *such that we have*

$$|u^0 - v|_{L^\infty((0,T)\times\mathbb{R}^N)} \leq \frac{C}{|\ln \varepsilon|^{\frac{1}{6}}} \quad \text{for} \quad \varepsilon \geq C(\Delta x + \sqrt{\Delta t})$$

*where* $v$ *is the numerical solution to an implicit scheme discretizing the* $\varepsilon$*-equation.*

Formally we have

$$|\ln\varepsilon| c^\varepsilon = |\ln\varepsilon|\cdot MCM + M_0 + \varepsilon^2\,(\text{4th order derivative of the surface}) + h.o.t.$$

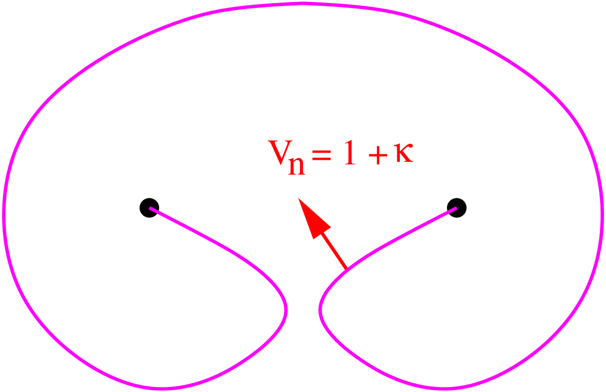
where

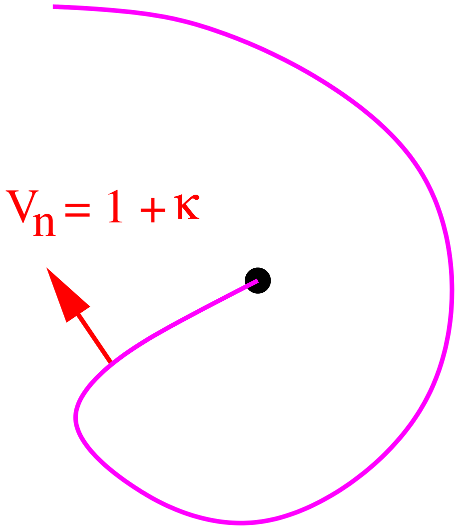
$$M_0 = \lim_{\varepsilon\to 0} M_\varepsilon \quad \text{with} \quad M_\varepsilon = |\ln\varepsilon|\,(c^\varepsilon - MCM)$$

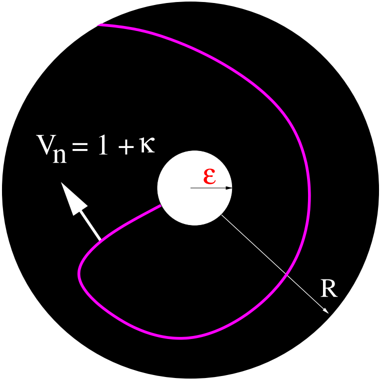
Here $M_0$ is a geometric 2nd order operator (without comparison principle).

# Frank-Read sources and spirals

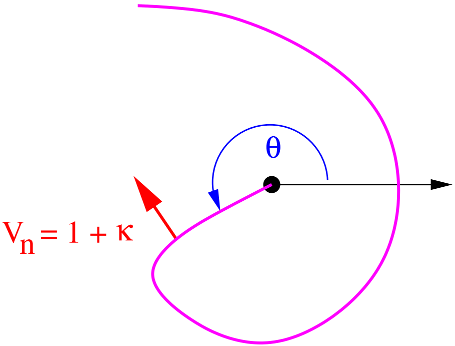
# Frank-Read source




$V_n = 1 + \kappa$




[Giga, Ishimura, Kohsaka (2002)],
[Ohtsuka (2003)]




Then $\theta = -u(t,r)$ with $u$ solution of the spiral equation

$$\begin{cases} r u_t = \sqrt{1 + r^2 u_r^2} + u_r \left( \dfrac{2 + r^2 u_r^2}{1 + r^2 u_r^2} \right) + \dfrac{r u_{rr}}{1 + r^2 u_r^2} & \text{for} \quad (t,r) \in (0,+\infty)^2 \\ u(0,\cdot) = u_0 & \text{at} \quad t = 0 \end{cases}$$

and no condition at $r = 0$ and $r = +\infty$.

**Theorem ([Imbert, Forcadel, M. (preprint 2010)])**

*If the initial data has a curvature $\kappa = -1$ at $r = 0$ and $u_0$ is globally Lipschitz and smooth enough, then there exists a unique viscosity solution of the spiral equation.*