Short Time Uniqueness Result for Solutions of Nonlocal and Non-monotone Geometric Equations

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§1 Introduction

We consider the evolution of compact hypersurfaces $\{\Gamma_t\}_{t\geq 0} \subset \mathbb{R}^N$ moving according to the nonlocal law

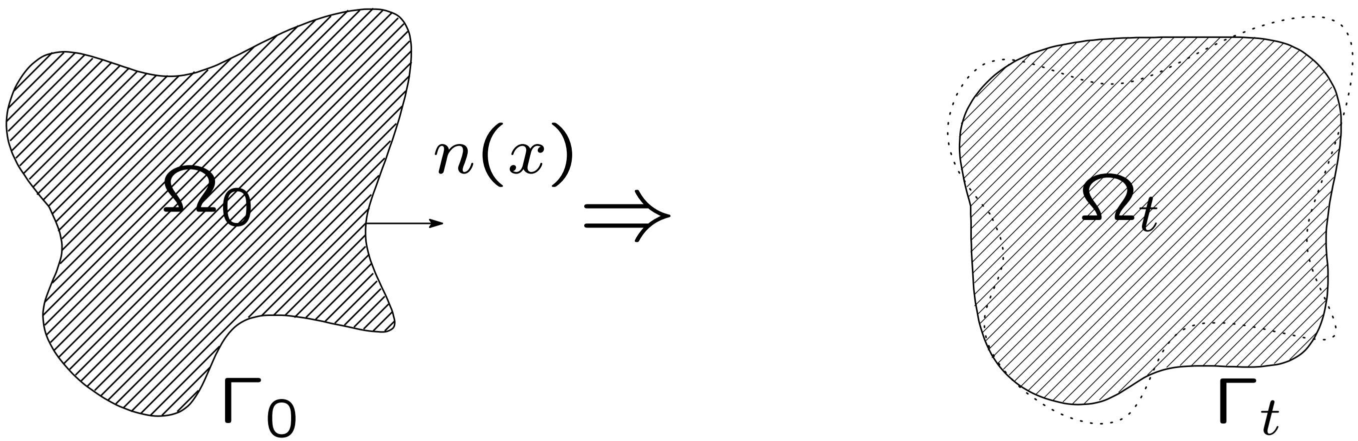
$$V = h(x, t, \Omega_t, n(x), Dn(x)) \quad \text{on } \Gamma_t, \qquad (1)$$

where

V: the normal outward velocity at the point (x, t),

$n(x)$: the outward normal w.r.t. Ω_t to Γ_t at $x \in \Gamma_t$,

$Dn(x)$: "curvature".



One of the main examples we have in mind is the dislocation dynamics (cf. lectures of Prof. Monneau), i.e., Equation (1) with

$$h = M(n(x))\big(c_0(\cdot,t) * \mathbf{1}_{\overline{\Omega_t}} + c_1(\cdot,t)\big) - \mathrm{div}\,(\xi(n(x))) \text{ on } \Gamma_t, \tag{2}$$

mobility PK force exterior force line tension

where

"$*$" denotes a convolution with respect to x variable,

$\mathbf{1}_A$: the indicator function of $A \subset \mathbb{R}^N$.

Assumptions.

$\exists\, L, M \geq 0$ s.t., for $\forall\, x, y \in \mathbb{R}^N$, $t \in [0,T]$

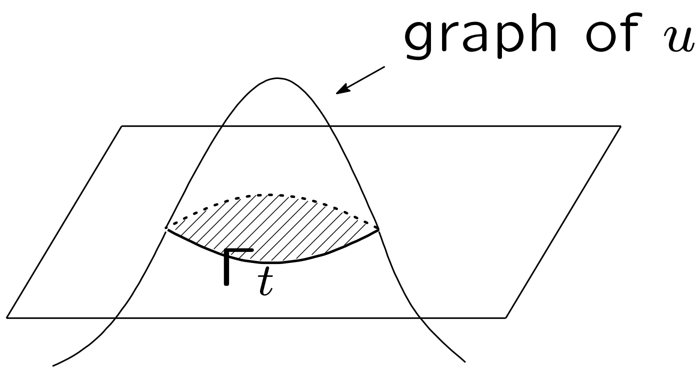
$$\|D_x c_0(\cdot,t)\|_{L^1(\mathbb{R}^N)} \leq L, \ |c_1(x,t) - c_1(y,t)| \leq L|x-y|,$$
$$\|c_0(\cdot,t)\|_{L^1(\mathbb{R}^N)} + |c_0(x,t)| + |c_1(x,t)| \leq M.$$

Rem. The sign of c_0 may change at the request of physics.

How do we construct a (unique) family $\{\Gamma_t\}_{t\geq 0}$ satisfying (1)?

$\Rightarrow$ *Level Set Approach* is one of powerful tools. The analytic foundation of this was established by Chen, Giga and Goto [11] and Evans and Spruck [13] by using *the theory for viscosity solution* (see [12] for instance). See the monograph [16] by Prof. Giga for details.

Cf. minimizing movement method, phase field method, integral varifold.



§2 Level Set Equations and Level Set Approach.

Consider an auxiliary function $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ such that

$$\Gamma_t = \{x \in \mathbb{R}^N \mid u(\cdot,t) = 0\},$$
$$\Omega_t = \{x \in \mathbb{R}^N \mid u(\cdot,t) > 0\}.$$

A classical calculation yields

$$V = \frac{u_t}{|Du|} \text{ and } n = -\frac{Du}{|Du|}.$$

Inserting the above formulae in (1) with (2), we obtain

$$\frac{u_t}{|Du|} = c[\mathbf{1}_{\{u\geq 0\}}](x,t) + \mathrm{div}\left(\frac{Du}{|Du|}\right)$$
$$\text{in } \mathbb{R}^N \times (0,T), \tag{3}$$

where

$$c[\mathbf{1}_{\{u\geq 0\}}](x,t)$$
$$:= \int_{\mathbb{R}^N} c_0(x-y,t)\mathbf{1}_{\{u(\cdot,t)\geq 0\}}(y)\,dy + c_1(x,t).$$

Level Set Approach

1. For a given initial hypersurface Γ_0 in $\mathbb{R}^N$, choose $u_0 : \mathbb{R}^N \to \mathbb{R}$ such that

$$\Gamma_0 = \{x \in \mathbb{R}^N \mid u_0(x) = 0\},$$
$$\Omega_0 = \{x \in \mathbb{R}^N \mid u_0(x) > 0\}.$$

2. Solve

$$\text{(I)} \quad \begin{cases} u_t = \left(c[\mathbf{1}_{\{u\geq 0\}}](x,t) + \operatorname{div}\left(\dfrac{Du}{|Du|}\right)\right)|Du| & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 \in W^{1,\infty}(\mathbb{R}^N) & \text{in } \mathbb{R}^N, \end{cases}$$

and set

$$\Gamma_t = \{x \in \mathbb{R}^N \mid u(\cdot,t) = 0\},$$
$$\Omega_t = \{x \in \mathbb{R}^N \mid u(\cdot,t) > 0\}.$$

We expect that $\{\Gamma_t\}_t$ is a (generalized) solution of (1).

In order to guarantee level set approach, we need to consider the fundamental questions:
(i) existence of viscosity solutions of (I);
(ii) uniqueness of viscosity solutions of (I) and
(iii) whether Γ_t depends only on Γ_0 and not on the shape of u_0 outside of Γ_0, i.e.,

$$\{u_0 = 0\} = \{v_0 = 0\} \text{ and } \{u_0 > 0\} = \{v_0 > 0\}$$
$$\Downarrow$$
$$\{u(\cdot,t) = 0\} = \{v(\cdot,t) = 0\} \text{ and } \{u(\cdot,t) > 0\} = \{v(\cdot,t) > 0\}$$

We can give a positive answer for (i) and (ii) for short time. In this talk, we shall only address question (ii).

§3 Non-Monotone Motion

It is worth mentioning that if such motion (1) is local,

i.e., when h does not depend on Ω_t,

then it is *monotone*. Then it is proved in [9] that the motion can be defined and studied by the level set approach.

$\{\Omega_t\}_t$ is *monotone*

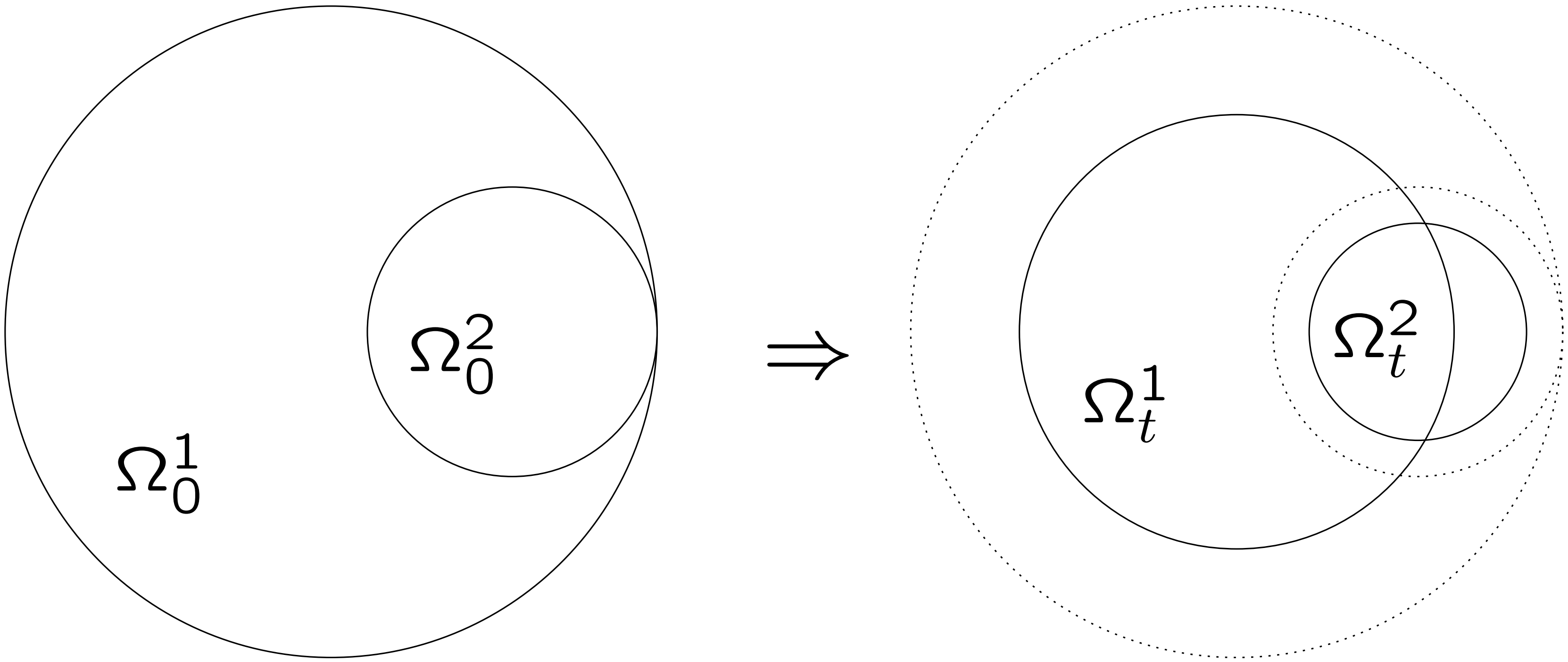
$\Longleftrightarrow$

The *inclusion principle* holds, i.e.,

$$\Omega_0^1 \subset \Omega_0^2 \quad \Rightarrow \quad \Omega_t^1 \subset \Omega_t^2 \text{ for any } t > 0.$$

We consider the equation

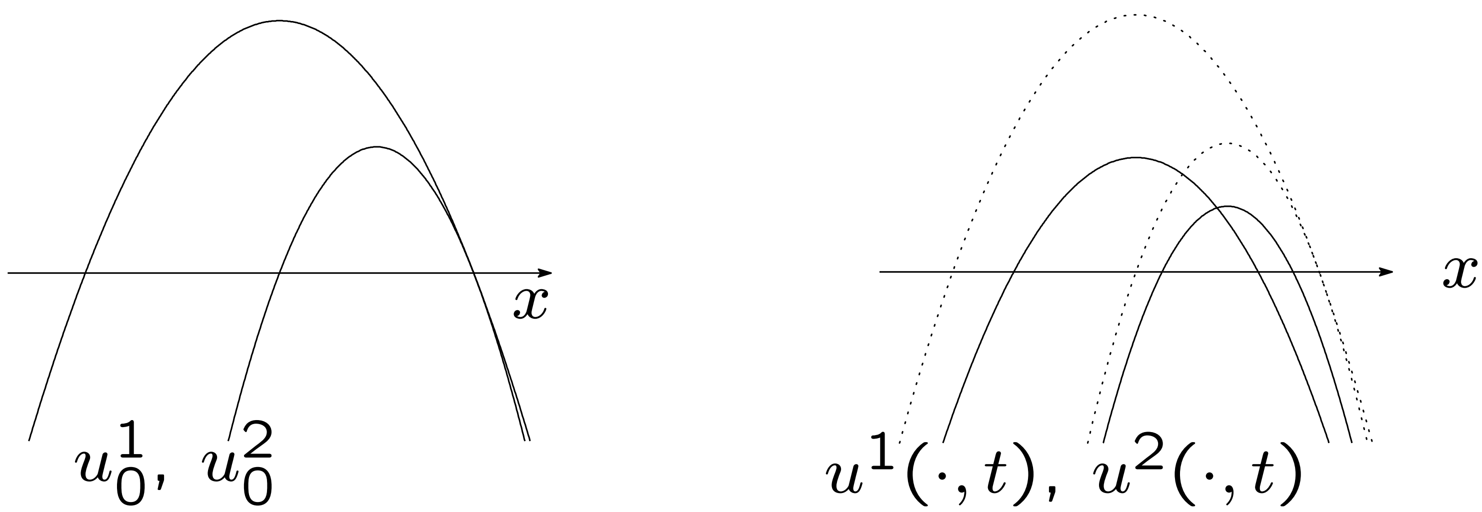
$$\color{blue}{V = -\mathrm{Vol}\,(\Omega_t)} \quad \text{on } \Gamma_t.$$



$\Rightarrow$ *Non-monotone phenomenon* of surface evolution

$$\Omega_0^1 \subset \Omega_0^2 \;\not\Rightarrow\; \Omega_t^1 \subset \Omega_t^2.$$

From viewpoint of auxiliary functions u^1, u^2, we observe the followings.



This observation implies that we cannot expect the comparison principle, i.e.,

$$u_0^2 \le u_0^1 \text{ in } \mathbb{R}^N \;\Rightarrow\; u^2(\cdot, t) \le u^1(\cdot, t) \text{ in } \mathbb{R}^N$$

for any $t \in [0, T]$.

Preceding Studies

1st order non-local equation (Existence and Uniqueness):

Alvarez-Hoch-LeBouar-Monneau '06 [2] (dislo, short time),

Alvarez-Cardaliaguet-Monneau '05 [1] (dislo),

Barles-Ley ('06) [7],

Barles-Cardaliaguet-Ley-Monneau '08 [3] (dislo),

Barles-Cardaliaguet-Ley-Monteillet '09 [4] (dislo+FN).

2nd order non-local equation (Existence):

Giga-Goto-Ishii '92 [17] (coupled interface equations),

Soravia-Souganidis '96 [20] (FN, asymptotic behavior),

Barles-Cardaliaguet-Ley-Monteillet '09 [5] (general).

2nd order non-local equation (Existence and Uniqueness):

Forcadel '08 [14] (general, short time, u_0 is a graph-like function),

Forcadel-Monteillet '09 [15] (dislo, mean curvature, short time).

§4 Fattening Difficulty

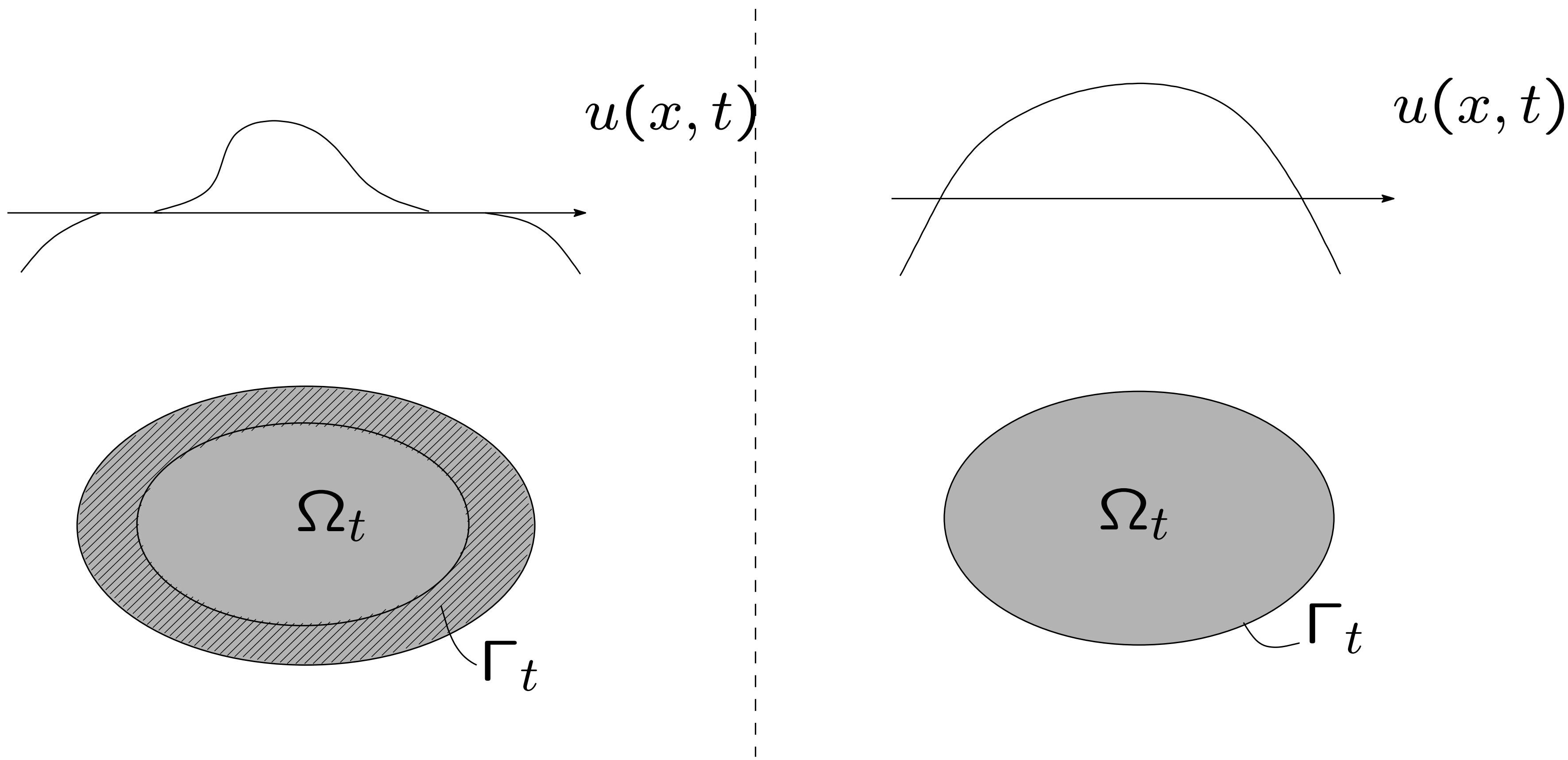
We are led to the study the (local) initial-value problem

$$\begin{cases} u_t = \Big(c(x,t) + \mathrm{div}\,\big(\frac{Du}{|Du|}\big)\Big)|Du| \text{ in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 \in W^{1,\infty}(\mathbb{R}^N) \qquad \text{in } \mathbb{R}^N, \end{cases} \tag{4}$$

where $c \in C(\mathbb{R}^N \times [0,T])$ are bounded and Lipschitz continuous with respect to the x variable.

One of our main results is a short time *lower gradient estimate* (Non-Fattening) for the viscosity solution of (4), i.e.,

$$|Du(x,t)| \geq \eta(t) > 0$$
$$\text{in a neighborhood of } \{u(\cdot,t) = 0\}. \tag{5}$$



Fattening Non-Fattening

It is easy to see that (5) implies *Non-Fattening*.

How do we get a lower gradient estimate (Non-Fattening)?

We assume that $u_0 \in \mathrm{Lip}\,(\mathbb{R}^N)$, $u_0 \equiv -1$ on $B(0, R_0)^c$.

$\exists\, \lambda_0 \in (0,1)$, $\eta_0 > 0$ and $\nu \in C(\mathbb{R}^N, \mathbb{R}^N)$ s.t.

$$u_0(x) - u_0(x + \lambda\nu(x)) \leq -\lambda\eta_0 \text{ in nbd of } \{u_0(\cdot) = 0\} \qquad (6)$$

for $\forall\, \lambda \in [0, \lambda_0]$

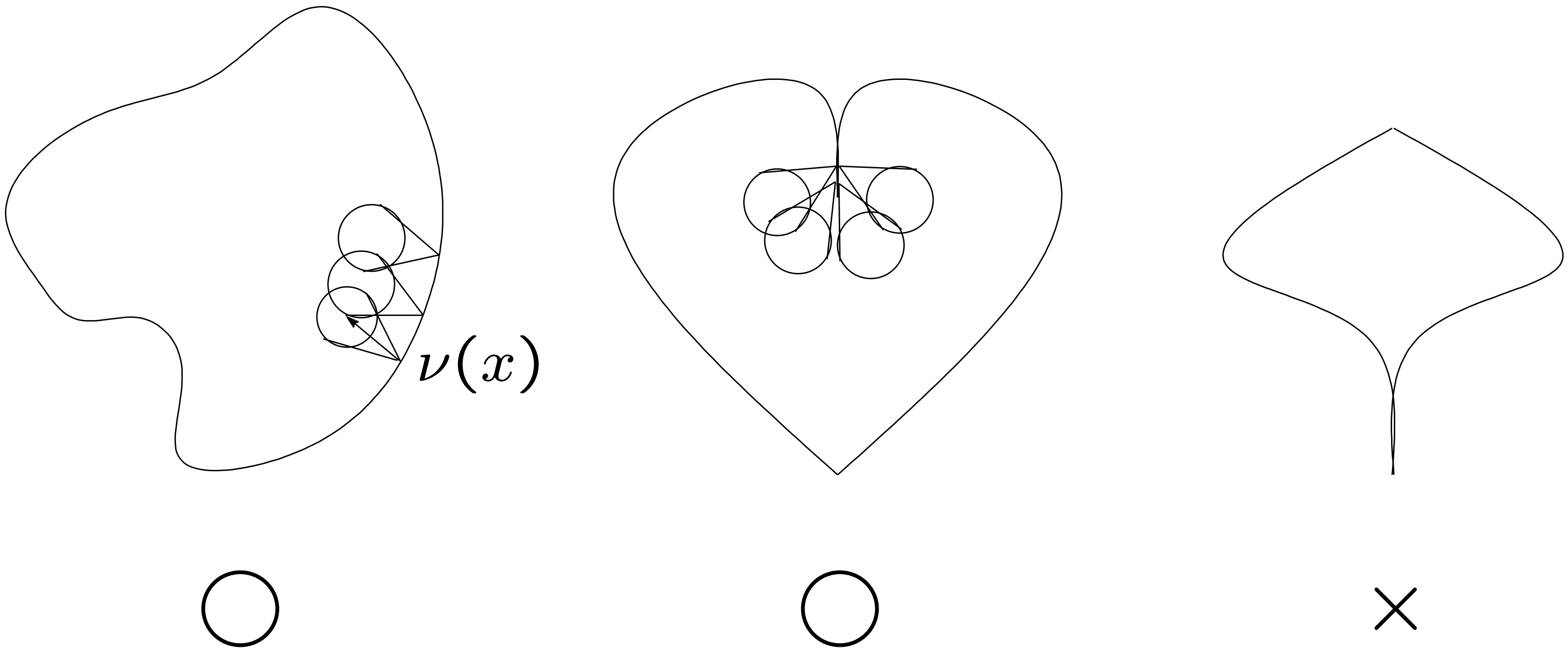
Then we prove that such a property is preserved for the solution of (4), at least for short time, i.e.,

Lemma 1 (Barles-Ley-M.).

$$u(x,t) - u(x - \lambda x, t) \leq -\eta(t)\lambda \textit{ in a nbd of } \{u(\cdot,t) = 0\} \qquad (7)$$

for $\forall\, \lambda \in [0, \overline{\lambda}]$, $\forall\, t \in [0, \bar{t} \wedge T]$ *and some* $\bar{t} > 0$, $\overline{\lambda} \in [0, \lambda_0]$, *where* $\eta : [0, \bar{t} \wedge T] \to [0, \infty)$ *is a non-increasing continuous function such that*

$$\eta(t) > 0 \textit{ for } \forall\, t \in [0, \bar{t} \wedge T), \quad \eta(\bar{t}) = 0.$$



Let $d^s_{\Gamma_0}$ be the signed distance function to Γ_0.

○: $u_0 = d^s_{\Gamma_0}$ satisfies (6).

×: $u_0 = d^s_{\Gamma_0}$ does not satisfy (6).

We derive lower gradient estimate (5) from (7) formally here. We have

$$\begin{aligned}\lambda\eta(t_0) &\le u(x_0+\lambda\nu(x_0),t_0)-u(x_0,t_0)\\ &= \lambda\langle Du(x_0,t_0),\nu(x)\rangle + o(\lambda\|\nu\|_\infty)\\ &\le \lambda|Du(x_0,t_0)|\|\nu\|_\infty + o(\lambda\|\nu\|_\infty)\\ &\text{in a neighborhood of } \{u(\cdot,t)=0\}\end{aligned}$$

for all $t \in [0,\bar{t}\wedge T]$ with $o(r)/r \to 0$ as $r \to 0$. Dividing λ in the above and taking a sufficiently small $\lambda \in (0,\overline{\lambda}]$, we get the lower estimate (5).

Remark.

We remark that if we can get $\eta(t) > 0$ for $\forall\, t \in [0, T]$ on the above, we can get a global uniqueness result. Can we get the estimate in global-time generally?

$\Rightarrow$ NO! (See [8, 10])

$\Rightarrow$ Short-Time Result or

Global-Time Result under suitable assumption on c_0, c_1, u_0.

§5 Short Time Uniqueness

Sketch of Proof.

Let u_1, u_2 be solutions of (I). Set

$$\delta_\tau := \max_{\mathbb{R}^N \times [0,\tau]} |(u_1 - u_2)(x,t)|$$

for $\tau \in (0, T]$. (τ will be fixed later.) We calculate that

$$\begin{aligned}
\delta_\tau &\le \tau \| \big(c[\mathbf{1}_{\{u_1 \ge 0\}}] - c[\mathbf{1}_{\{u_2 \ge 0\}}] \big)(x,t) \|_{L^\infty(\mathbb{R}^N \times [0,\tau])} \\
&\le C\tau \max_{t \in [0,\tau]} \int_{\mathbb{R}^N} |(\mathbf{1}_{\{u_1(\cdot,t) \ge 0\}} - \mathbf{1}_{\{u_2(\cdot,t) \ge 0\}})(y)|\, dy \\
&\le C\tau \sum_{i=1}^{2} \max_{t \in [0,\tau]} \int_{\mathbb{R}^N} \mathbf{1}_{\{-\delta_\tau \le u_i(\cdot,t) < 0\}}(y)\, dy.
\end{aligned}$$

1. Lower Gradient Estimate (LGE): $|Du| \ge \eta(t) \ge \overline{\eta} > 0$.
2. Perimeter Estimate (PE): $\mathrm{Per}\,(\{u_i(\cdot,t) = r\}) \le C$

for $\forall\, r \in [-\delta_\tau, 0]$.

$$
\text{(I)} \quad \begin{cases} u_t = \left(c[\mathbf{1}_{\{u \geq 0\}}](x,t) + \mathrm{div}\left(\dfrac{Du}{|Du|}\right)\right)|Du| & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 \in W^{1,\infty}(\mathbb{R}^N) & \text{in } \mathbb{R}^N. \end{cases}
$$

$$
\begin{aligned}
&\int_{R^N} \mathbf{1}_{\{-\delta_\tau \le u_i(\cdot,t)<0\}}(y)\,dy \\
(\textit{Coarea Formula}) \ &= \int_{-\delta_\tau}^{0} \int_{\{u_i(\cdot,t)=r\}} \frac{1}{|Du_i(x,t)|}\, d\mathcal{H}^{N-1}\, dr \\
(\mathrm{LGE}) \ &\le C \int_{-\delta_\tau}^{0} \int_{\{u_i(\cdot,t)=r\}} \frac{1}{\eta(t)}\, d\mathcal{H}^{N-1}\, dr \\
&\le \frac{C\delta_\tau}{\overline{\eta}} \sup_{-\delta_\tau \le r \le 0} \mathrm{Per}\,(\{u_i(\cdot,t)=r\}) \\
(\mathrm{PE}) \ &\le \frac{C\delta_\tau}{\overline{\eta}}.
\end{aligned}
$$

Therefore, $\delta_\tau \le C\delta_\tau \tau$, which implies that if τ is small enough, δ_τ must be 0.

Remark.

It is worth explaining the technicalities in the proof. In this talk, for easy explanation, we have used “Perimeter Estimate (PE)”. But it is not necessary for our proof. See Section 5 in the abstract and the preprint [6] for details.

Thank you for
your kind attention!