

Stability analysis of steady states for surface diffusion equation in a bounded domain

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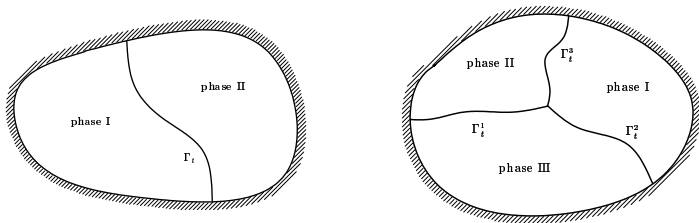
(Muroran Institute of Technology)

joint work with H. Garcke and K. Ito

Mathematical Aspects of Crystal Growth

30 July, 2010

Model



Along the phase boundary Γ_t

$$V = -m(\gamma\kappa)_{ss} \quad (\text{surface diffusion eq.})$$

where

V : normal velocity of Γ_t , κ : curvature of Γ_t ,

s : arc-length parameter,

m : mobility (constant), γ : surface tension (constant)

The surface diffusion equation is proposed by

[W. W. Mullins](#) (J. Appl. Phys., '57)

- In works concerned with thermal grooving.

[F. Davi and M. Gurtin](#) (Z. Angew. Math. Phys., '90)

- Within rational thermodynamics.

[J. E. Taylor and J. W. Cahn](#) (J. Statist. Phys., '94)

- As the H^{-1} -gradient flow for the area (energy) functional of a surface.

[J. W. Cahn, C. M. Elliott, and A. Novick-Cohen](#) (European J. Appl. Math., '96)

- As a singular limit of Cahn-Hilliard equation with degenerate mobility.

⇒ • The energy of a surface is decreasing and the volume of the enclosed domain is preserved.

- A 4th order parabolic P.D.E., so that a comparison principle does not work.

2 phases

At $\Gamma_t^i \cap \partial\Omega$,

- angle condition: $\angle(\Gamma_t^i, \partial\Omega) = \pi/2$
- no-flux: $(\gamma^i \kappa^i)_s = 0$

3 phases

At $\Gamma_t^i \cap \partial\Omega$,

- angle condition: $\angle(\Gamma_t^i, \partial\Omega) = \pi/2$
- no-flux: $(\gamma^i \kappa^i)_s = 0$

At triple-junction,

- angle condition: $\angle(\Gamma_t^i, \Gamma_t^j) = \theta^k$ where θ^i satisfies

$$\frac{\sin \theta^1}{\gamma^1} = \frac{\sin \theta^2}{\gamma^2} = \frac{\sin \theta^3}{\gamma^3} \quad (\text{Young's law})$$

- the continuity of chemical potential: $\gamma^1 \kappa^1 + \gamma^2 \kappa^2 + \gamma^3 \kappa^3 = 0$
- a balance of fluxes: $m^1(\gamma^1 \kappa^1)_s = m^2(\gamma^2 \kappa^2)_s = m^3(\gamma^3 \kappa^3)_s$

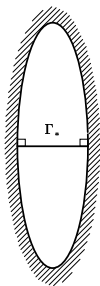
Question

A criterion of stability of stationary curves?

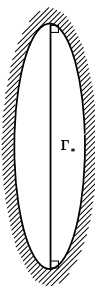


Investigate the sign of eigenvalues for the linearized system.

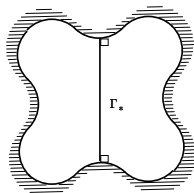
[Observation & Expectation]



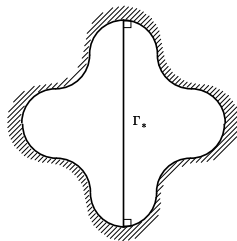
stable



unstable



stable



unstable

We expect that a criterion of stability of stationary curves depends on the curvature of an exterior boundary.

2 phases - Setting

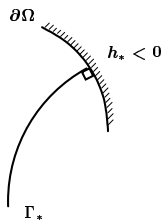
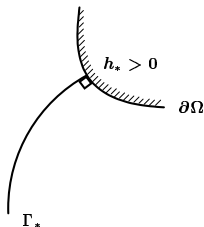
Let Γ_* be a part of circle or a line segment denoted by

$$\Gamma_* = \{\Phi_*(\sigma) \mid \sigma \in [\ell_*^-, \ell_*^+]\}$$

with $\Phi_*(\ell_*^\pm) \in \partial\Omega$, where σ is an arc-length parameter along Γ_* and $L_* = \ell_*^+ - \ell_*^-$ is the length of Γ_* . Also, set

κ_* : curvature of Γ_* , h_*^\pm : curvature of $\partial\Omega$ at $\Gamma_* \cap \partial\Omega$.

[The sign convention of h_*^\pm]



2 phases - Linearized system

The linearized system around a stationary curve Γ_* is

$$\begin{cases} \rho_t = -m[\gamma\{\rho_{\sigma\sigma} + (\kappa_*)^2\rho\}]_{\sigma\sigma} & (\sigma \in (\ell_*^-, \ell_*^+)), \\ \rho_\sigma + h_*^\pm \rho = 0 & (\sigma = \ell_*^\pm), \\ \{\rho_{\sigma\sigma} + (\kappa_*)^2\rho\}_\sigma = 0 & (\sigma = \ell_*^\pm) \end{cases}$$

for ρ with $\int_{\ell_*^-}^{\ell_*^+} \rho d\sigma = 0$. Then, we introduce the bilinear form corresponding to the above system:

$$\begin{aligned} I[\rho_1, \rho_2] &= \int_{\ell_*^-}^{\ell_*^+} \{\rho_{1,\sigma}\rho_{2,\sigma} - (\kappa_*)^2\rho_1\rho_2\} d\sigma \\ &\quad + h_*^- \rho_1\rho_2|_{\sigma=\ell_*^-} + h_*^+ \rho_1\rho_2|_{\sigma=\ell_*^+}. \end{aligned}$$

A linearized system also has the H^{-1} -gradient flow structure, so that we can derive the following properties for the eigenvalues to this linearized system.

2 phases - Eigenvalue problem

- $\{\lambda_n\} \subset \mathbb{R}$. (i.e. Linearized operator is self-adjoint in the suitable space.)
- The eigenvalues are characterized by $I[\rho, \rho]$ and $(\rho, \rho)_{-1}$. In particular,

$$\lambda_1 = - \inf_{\rho \in \mathcal{E} \setminus \{0\}} \frac{I[\rho, \rho]}{(\rho, \rho)_{-1}}.$$

- The eigenvalues depend continuously on κ_* , h_*^\pm , and ℓ_*^\pm , and are monotone decreasing with respect to h_*^\pm .

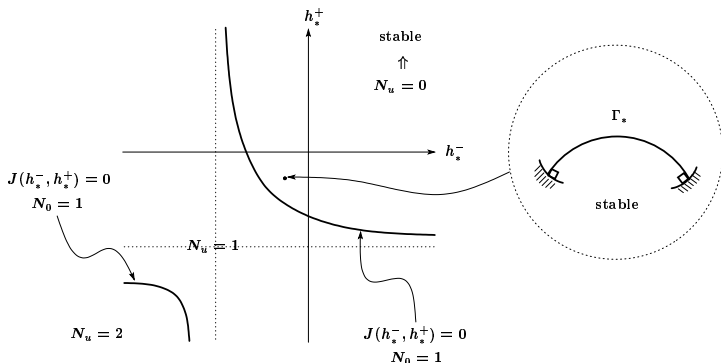
Moreover, we have the following properties.

$I[\rho, \rho] > 0$ ($\rho \in \mathcal{E} \setminus \{0\}$) provided that h_*^\pm are large enough.

0 is eigenvalue if and only if $J(h_*^-, h_*^+) = 0$ (see the figure).

2 phases - A criterion of stability

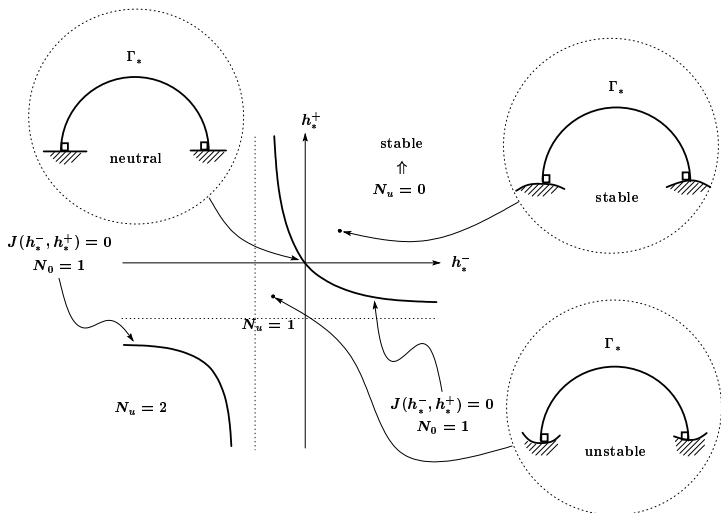
Case: $0 < \kappa_* L_* < \pi$



Remark In the case $\kappa_* = 0$, we have a similar figure.

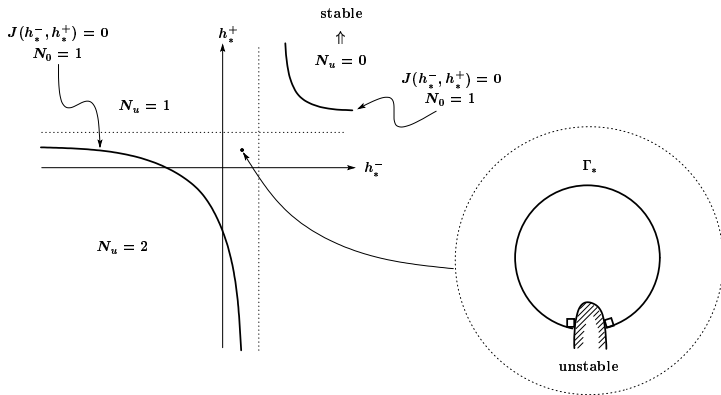
2 phases - A criterion of stability

Case: $\kappa_* L_* = \pi$



2 phases - A criterion of stability

$$\text{Case: } \pi < \kappa_* L_* < 2\pi$$



3 phases - Bilinear form

We have the following bilinear form:

$$\begin{aligned} I[\rho_1, \rho_2] &= \sum_{i=1}^3 \gamma^i \left[\int_0^{\ell^i} \left\{ \partial_\sigma \rho_1^i \partial_\sigma \rho_2^i - (\kappa_*^i)^2 \rho_1^i \rho_2^i \right\} d\sigma + h_*^i \rho_1^i \rho_2^i \Big|_{\sigma=\ell^i} \right] \\ &\quad - \frac{\gamma^1}{s^1} (c^2 \kappa_*^2 - c^3 \kappa_*^3) \rho_1^1 \rho_2^1 \Big|_{\sigma=0} - \frac{\gamma^2}{s^2} (c^3 \kappa_*^3 - c^1 \kappa_*^1) \rho_1^2 \rho_2^2 \Big|_{\sigma=0} \\ &\quad - \frac{\gamma^3}{s^3} (c^1 \kappa_*^1 - c^2 \kappa_*^2) \rho_1^3 \rho_2^3 \Big|_{\sigma=0}, \end{aligned}$$

where $s^i := \sin \theta^i$ and $c^i := \cos \theta^i$.

Remark

A similar bilinear form is used in the double bubble analysis (see F. Morgan and W. Wichiramala (Proc. Amer. Math. Soc., '02)).

3 phases - Eigenvalue problem

Also, we have the following :

- $\{\lambda_n\} \subset \mathbb{R}$. (i.e. Linearized operator is self-adjoint in the suitable space.)
- The eigenvalues are characterized by $I[\rho, \rho]$ and $(\rho, \rho)_{-1}$. In particular,

$$\lambda_1 = - \inf_{\rho \in \mathcal{E} \setminus \{0\}} \frac{I[\rho, \rho]}{(\rho, \rho)_{-1}}.$$

- The eigenvalues depend continuously on κ_*^i , h_*^i , and ℓ_*^i , and are monotone decreasing with respect to h_*^i .

3 phases - A criterion of stability

Case: $\kappa_*^i = 0$ ($i = 1, 2, 3$)

For simplicity, set

$$\begin{cases} \gamma^1 = \gamma^2 = \gamma^3 = 1 & (\Rightarrow \theta^1 = \theta^2 = \theta^3 = 120^\circ), \\ \ell_*^1 = d, \ell_*^2 = \ell_*^3 = 1. \end{cases}$$

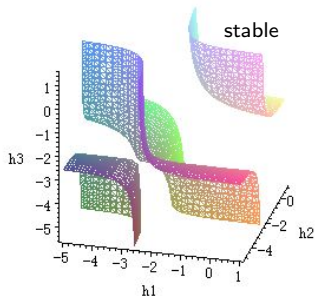
Then we have for $u^i(\eta) = \rho^i(\ell_*^i \eta)$

$$I[\rho, \rho] = \frac{1}{d} \int_0^1 (u_\eta^1)^2 d\eta + \sum_{i=2}^3 \int_0^1 (u_\eta^i)^2 d\eta + \sum_{i=1}^3 h_*^i (u^i)^2 \Big|_{\eta=1}$$

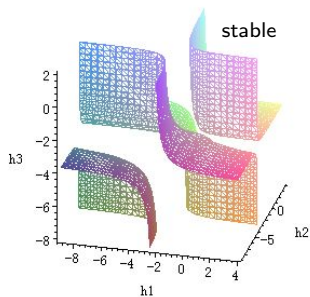
$I[\rho, \rho] > 0$ ($\rho \in \mathcal{E} \setminus \{0\}$) provided that $h_*^i > 0$ ($i = 1, 2, 3$).

0 is eigenvalue if and only if $J(h_*^1, h_*^2, h_*^3) = 0$ (see the figure).

3 phases - A criterion of stability



$$d = 1$$



$$d = 0.001$$

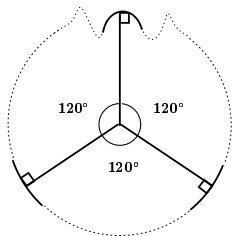
$$(-1 + 6d^2 + 16d^3 + 3d^4 > 0) \quad (-1 + 6d^2 + 16d^3 + 3d^4 < 0)$$

0 is eigenvalue if and only if (h_*^1, h_*^2, h_*^3) is on the surface.

(The multiplicity is 1 except one point $(h_{*,c}^1, h_{*,c}^2, h_{*,c}^3)$.)

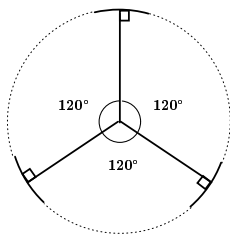
3 phases - A criterion of stability

- Case: $d = 1$



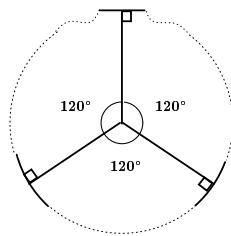
unstable

$$h_*^1 < -1, \\ h_*^2 = h_*^3 = -1$$



neutral

$$h_*^1 = -1, \\ h_*^2 = h_*^3 = -1$$



stable

$$h_*^1 = 0, \\ h_*^2 = h_*^3 = -1$$

Eigenvalue is monotone decreasing with respect to h_*^i .

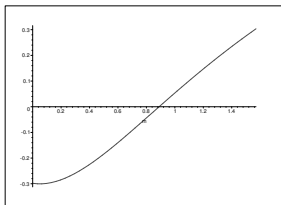
3 phases - A criterion of stability

Case : $\kappa_*^1 = 0$ and $\kappa_*^2 = -\kappa_*^3 (\neq 0)$

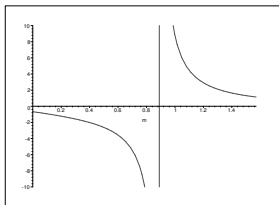
For simplicity, set

$$\begin{cases} \gamma^1 = \gamma^2 = \gamma^3 = 1 & (\Rightarrow \theta^1 = \theta^2 = \theta^3 = 120^\circ) \\ \ell_*^1 = \sqrt{3}, \ell_*^2 = d, \ell_*^3 = 4\pi/3, \kappa_*^2 = -\kappa_*^3 = -1/2. \end{cases}$$

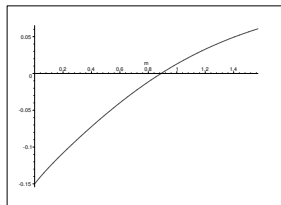
Then we have the following graph for $h_{*,a}^i - h_{*,c}^i$.



$h_{*,a}^1 - h_{*,c}^1$

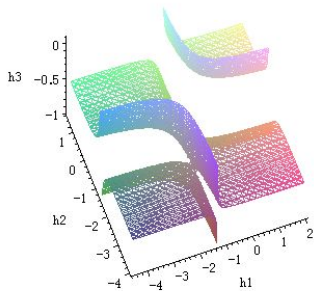


$h_{*,a}^2 - h_{*,c}^2$

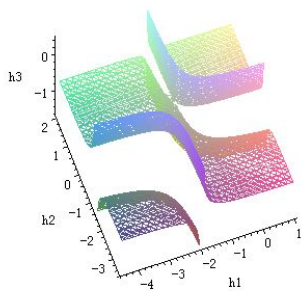


$h_{*,a}^3 - h_{*,c}^3$

3 phases - A criterion of stability



$$d = 0.5\pi$$



$$d = 0.05\pi$$

- Anisotropic case?
- 3D case?
 - This is concerned with the analysis of the surface with constant mean curvature.
- Dynamics far away from stable stationary curves?
 - Singularity will appear in finite time and pinch-off will happen.
How do we analyze it mathematically.