A Hamilton-Jacobi-Bellman equation in the space of probability measures

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Controlled 2D vorticity equation

$$\partial_t \rho + \operatorname{div}(\rho u) = \nu \Delta \rho + m, \quad (1)$$

$$u := u_\rho = -K^\perp * \rho. \quad (2)$$

In the above, $x = (x_1, x_2) \in \mathbb{R}^2$ and $\nu > 0$,

$$N(x) := -\frac{1}{2\pi} \log |x|, \quad K(x) := \nabla N(x) = -\frac{1}{2\pi} \frac{x}{|x|^2}, \quad x \in \mathbb{R}^2 \setminus \{0\},$$

and

$$K^\perp(x) := \nabla^\perp N(x), \text{ where } \nabla^\perp = (\partial_{x_2}, -\partial_{x_1}).$$

m is the control variable. We will look at this problem in the space of probability measures in $\mathbb{R}^2$ with finite 2-nd moments.

Wasserstein space

Let $\rho, \gamma \in \mathcal{P}_2(\mathbb{R}^2)$.

$$\Gamma(\rho,\gamma) := \{\pi(dx,dy) \in \mathcal{P}(\mathbb{R}^4) : \pi(dx,\mathbb{R}^2) = \rho(dx), \pi(\mathbb{R}^2,dy) = \gamma(dy)\}.$$

The Wasserstein 2-metric d on $\mathcal{P}_2(\mathbb{R}^2)$ is defined as

$$d^2(\rho,\gamma) := \inf\Big\{ \int_{\mathbb{R}^4} |x-y|^2 \pi(dx,dy) : \pi \in \Gamma(\rho,\gamma)\Big\}.$$

$(\mathcal{P}_2(\mathbb{R}^2), d)$ is a non-locally compact, complete separable metric space. If ρ, γ have Lebesgue densities then the minimizing measure exists, is unique and is given by

$$\pi_{\rho,\gamma}(dx,dy) = \rho(dx)\delta_{T_{\rho,\gamma}(x)}(dy),$$

where

$$T_{\rho,\gamma}(x) = x - \nabla p_{\rho,\gamma}(x) = \nabla \varphi_{\rho,\gamma}(x)$$

is Brenier's optimal map for some convex function $\varphi_{\rho,\gamma}$.

Wasserstein space

In particular γ is the push forward of ρ by $T_{\rho,\gamma}$ and we have

$$d^2(\rho,\gamma) = \int_{\mathbb{R}^2} |\nabla p_{\rho,\gamma}(x)|^2 \rho(dx).$$

We want $\rho(t) \in \mathcal{P}_2(\mathbb{R}^2)$, so the control variable m should push only in the "tangent direction" to $P_2(\mathbb{R}^2)$ at $\rho(t)$ for every t.

$\mathcal{P}_2(\mathbb{R}^2)$ has a differential structure.

Tangent space: $H_{-1,\rho}(\mathbb{R}^2)$

Tangent space to $P_2(\mathbb{R}^2)$ at ρ can be identified with

$$H_{-1,\rho}(\mathbb{R}^2) := \{m \in \mathcal{D}'(\mathbb{R}^2) : \|m\|_{-1,\rho} < \infty\},$$

$$\|m\|^2_{-1,\rho} := \sup_{\varphi \in C_c^\infty(\mathbb{R}^2)} \{2\langle \varphi, m\rangle - \int_{\mathbb{R}^2} |\nabla\varphi|^2 d\rho\}, \quad \forall m \in \mathcal{D}'(\mathbb{R}^2).$$

It can be shown that every $m \in H_{-1,\rho}(\mathbb{R}^2)$ has a representation

$$m = -\mathrm{div}(\rho v)$$

where

$$v \in L^2_\nabla(\rho) \equiv \overline{\{\nabla\varphi : \varphi \in C_c^\infty(\mathbb{R}^2)\}}^{L^2(\rho)}.$$

If P_ρ is the projection in $L^2(\rho)$ onto $L^2_\nabla(\rho)$ and $v_1, v_2 \in L^2(\rho)$ we define

$$\langle -\mathrm{div}(\rho v_1), -\mathrm{div}(\rho v_2)\rangle_{-1,\rho} = \int_{\mathbb{R}^2} (P_\rho v_1)(P_\rho v_2) d\rho.$$

$(H_{-1,\rho}(\mathbb{R}^2), \langle\cdot,\cdot\rangle_{-1,\rho})$ is a Hilbert space.

Control problem

Infinitesimal running cost

$$h(\rho) - L(\rho, m) := h(\rho) - \frac{1}{4\nu}\|m\|^2_{-1,\rho},$$

$h \in C_b(P_2(\mathbb{R}^2))$. We Identify m with $\dot\rho$ through (1). We want to maximize the discounted running cost over certain set of controls.

Value function for control (or variational) problem:

$$f(\rho_0) = \sup\{\int_0^\infty e^{-\alpha s}\Big(h(\rho(s)) - L(\rho(s), \dot\rho(s))\Big)ds : \rho \in \mathcal{K}_{\rho_0}\},$$

where $\mathcal{K}_{\rho_0}$ is a certain class of paths such that $\rho(0) = \rho_0$.

Particle system

(1) with $m = 0$ is the mean-field (i.e. law of large number) limit for a collection of stochastic vortex particles interacting in a particular manner. Let particles $(X_1, \ldots, X_n)$ model vortices rotating, say, counter-clockwise. As we move to a macroscopic level, we lose track of individual particles and only see collective effects given by number density of the vortices, which is represented by the probability measure

$$\rho_n(t, dx) = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(t)}(dx).$$

The dynamic of particles is defined through a system of SDE

$$dX_i = u_{\rho_n}(X_i)dt + \sqrt{2\nu}dB_i,$$

$$u_{\rho_n}(z) = -\nabla^{\perp} N_n * \rho_n(z) = -\frac{1}{n} \sum_{i=1}^{n} \nabla^{\perp} N_n(z - X_i),$$

$(B_1, \ldots, B_n)$ is a standard Brownian motion in $\mathbb{R}^{2n}$ and N_n is some Lipschitz smooth potential approximating the Newtonian potential N.

Connection with large deviations

One wants to identify a function $\mathbb{S}$ which takes values in $[0, +\infty]$ and is defined over Borel sets A in an appropriately defined path space, such that

$$-\mathbb{S}(A^\circ) \le \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\rho_n(\cdot) \in A^\circ)$$
$$\le \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\rho_n(\cdot) \in \bar{A}) \le -\mathbb{S}(\bar{A})$$

$\mathbb{S}$ should have a "density" $\mathbb{I}$, known as the action functional (or rate function):

$$\mathbb{S}(A) = \inf_{\rho(\cdot)\in A} \mathbb{I}(\rho(\cdot)),$$

$$\mathbb{I}(\rho(\cdot)) = -\lim_{\epsilon\to 0+} \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\rho_n(\cdot) \in B_\epsilon(\rho(\cdot))) = \int_0^T L(\rho(t), \dot{\rho}(t))dt.$$

$\mathbb{S}$ is really a quantity arising from the nonlinear scaling behavior of the stochastic processes $\rho_n(\cdot)$, it cannot be obtained from the deterministic 2-D incompressible Navier-Stokes equation or its vorticity formulation.

Connection with large deviations

Justification of the above and identification of $\mathbb{I}$ (hence $\mathbb{S}$) is related to variational problem (of optimal control nature) (1-2). Its key component is establishing well posedness of Hamilton-Jacobi-Bellman equations

$$(\alpha I - H)f = h$$

for sufficiently large class of functions h, which is the dynamic programming equation associated with our control problem. (H arises as the limit of $H_n g = n^{-1}e^{-ng}A_n e^{ng}$, where A_n is the generator of ρ_n.) Such a general program is developed in the book of J. Feng and T. Kurtz.

Here we only study the HJB equation and develop techniques for dealing with PDE in the Wasserstein space.

Functionals

- Second moment:
$$M_2(\rho) := \int_{\mathbb{R}^2} |x|^2 d\rho$$
- Entropy:
$$s(\rho) := \int_{\mathbb{R}^2} \rho \log \rho dx$$
- Internal energy:
$$e(\rho) := \frac{1}{2}\langle N * \rho, \rho\rangle = \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} N(x-y)\rho(dx)\rho(dy).$$
- Fisher information:
$$I(\rho) := \int_{\mathbb{R}^2} \frac{|\nabla\rho|^2}{\rho} dx = 4\int_{\mathbb{R}^2} |\nabla\sqrt{\rho}|^2 dx.$$

Functionals

Basic inequalities:

- $$M_2(\rho) + s(\rho) \geq C > -\infty$$
- $$|e(\rho)| \leq C_1 + C_2(M_2(\rho) + s(\rho))$$
- $$\|u\|_p \leq C\|\rho\|_2^{1-2/p}, \;\; p > 2$$
- $$\int_{\mathbb{R}^2} |u|^2 d\rho \leq C\|\rho\|_2^2 \leq C(1 + I(\rho))$$
  $$\int_{\mathbb{R}^2} |u| d\rho \leq C\|\rho\|_2$$

Weak solution

Definition

(ρ, m), where $\rho : [0,\infty) \to \mathcal{P}_2(\mathbb{R}^2), m \in H_{-1,\rho}$ is said to be a (weak) solution to (1) provided that

1. $\rho \in C([0,\infty),\mathcal{P}(\mathbb{R}^2)) \cap L^2([0,t],L^2(\mathbb{R}^2))$ for each $0 < t < \infty$;
2. for all $0 \le s < t < \infty$ and $\varphi = \varphi(t,x) \in C_c^\infty([0,\infty) \times \mathbb{R}^2)$,

$$\begin{aligned}&\langle \varphi(t), \rho(t)\rangle - \langle \varphi(s), \rho(s)\rangle \\ &= \int_s^t \Big(\int_{x\in\mathbb{R}^2} (\partial_r \varphi(r,x) + \nabla\varphi(r,x)u(r,x) + \nu\Delta\varphi(t,x))\rho(r,dx) \\ &\qquad + \langle m(r), \varphi(r)\rangle \Big) dr.\end{aligned}$$

If $\rho \in L^2([0,t],L^2(\mathbb{R}^2))$ then $u\rho \in L^1([0,t] \times \mathbb{R}^2)$ so the integrals make sense.

Controlled state equation

QUESTION: Can we solve the controlled state equation (1-2), i.e. is the set of admissible paths $\mathcal{K}_{\rho_0}$ non-empty?

Let $J_n(z) = n^2 J(nz)$ be standard mollifiers and $G_n = J_n * J_n$. Set

$$u_{G_n * \rho_n(t)}(x) = -(K^{\perp} * G_n) * \rho_n(x).$$

Take $p \in C_c^{\infty}([0,\infty) \times \mathbb{R}^2)$ and set

$$m := -\nabla \cdot (\rho \nabla p), \quad v(t,x) := \nabla_x p(t,x).$$

Lemma

Let $M_2(\rho_0) < +\infty, s(\rho_0) < +\infty$, and $T > 0$. Then there exists $\rho(\cdot) \in C([0,T]; \mathcal{P}(\mathbb{R}^2))$ such that $(\rho, -\nabla(\rho\nabla p))$ is a weak solution of (1) on $[0,T]$ and $\rho(0) = \rho_0$.

Controlled state equation

- Solve system of SDE

$$dX_i = u_{G_n * \rho_n(t)}(X_i)dt + v(t, X_i)dt + \sqrt{2\nu}dB_i, \ \ X_i(0) = X_{0i}, \ 1 \le i \le n$$

where $X_{0i}, i = 1, 2, ...,$ are i.i.d. random variables with law ρ_0.
- Obtain estimates for $\mathbb{E}M_2(\rho_n(t))$ and $\mathbb{E}\int_0^T \|J_n * \rho_n(t)\|_2^2 dt$.
- Show that $\{\rho_n(\cdot) : n = 1, ...\}$ is tight in $C([0, T]; \mathcal{P}(\mathbb{R}^2))$.
    - Compact containment

$$\lim_{C\to\infty} \sup_n \mathbb{P}(\exists t \in [0, T], \rho_n(t) \notin \{M_2(\rho) \le C\}) = 0.$$

    - For every $\varphi \in C_c^\infty(\mathbb{R}^2)$, $\{\langle \varphi, \rho_n(t)\rangle : n = 1, 2 \ldots\}$ is tight in $C([0, T]; \mathbb{R})$. It is enough to show that

$$\mathbb{E}[|\langle \varphi, \rho_n(t + h) - \rho_n(t)\rangle|] \le Ch^{1/4}.$$

Controlled state equation

- Use Skorohod representation theorem i.e. that there exist random variables $\rho, \tilde{\rho}_n, n = 1, 2, ...$ defined on some probability space such that $\tilde{\rho}_n$ has the same law as ρ_n for $n = 1, 2, ...$ and such that

$$\tilde{\rho}_n \to \rho \quad a.s. \text{ in } C([0,T]; \mathcal{P}(\mathbb{R}^2)) \text{ as } n \to +\infty$$

  to pass in the limit in the estimates and show that limit is deterministic and get estimates on $M_2(\rho(t))$ and $\int_0^T \|\rho(t)\|_2^2 dt$. (technical)

- Pass to the limit in the Ito formula

$$\begin{aligned}&\mathbb{E}\langle \varphi, \rho_n(t)\rangle - \mathbb{E}\langle \varphi, \rho_n(s)\rangle \\ &\quad = \mathbb{E}\int_s^t \left[\langle \varphi_t + \nabla\varphi \cdot (u_{G_n * \rho_n} + v) + \nu\Delta\varphi, \rho_n(r)\rangle\right] dr\end{aligned}$$

  to show that the limit $\rho(\cdot)$ solves the state equation (1-2).

Controlled state equation

Lemma

Every weak solution $(\rho, -\nabla(\rho v))$ *of (1) on* $[0,T]$ *with* $v = \nabla p$ *as before is such that* $\rho \in AC((0,T);\mathcal{P}_2(\mathbb{R}^2))$ *and satisfies*

$$M_2(\rho(t)) - M_2(\rho(s)) = \int_s^t \Big(4\nu + 2\int_{\mathbb{R}^2} xv\rho dx\Big) dr,$$

$$s(\rho(t)) - s(\rho(s)) = -\nu \int_s^t I(\rho(r))dr + \int_s^t \Big(\int_{\mathbb{R}^2} v \cdot \frac{\nabla \rho}{\rho} d\rho\Big) dr,$$

$$d^2(\rho(t),\sigma) - d^2(\rho(s),\sigma) = 2\int_s^t \int_{\mathbb{R}^2} -\nabla p_{\rho(r),\sigma}(x)(\nu\frac{\nabla\rho}{\rho} + u + v)\rho(r,dx)dr$$

for all $0 \le s < t \le T$ *and* $\sigma \in \mathcal{P}_2(\mathbb{R}^2)$ *with* $s(\sigma) < \infty$*.*

Controlled state equation

Corollary

Let $\rho_0 \in \mathcal{P}_2(\mathbb{R}^2)$, be such that $s(\rho_0) < +\infty$. Let $p : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ be such that there exists a sequence $0 = t_0 < t_1 < ... < t_n < ..., t_n \to +\infty$ such that $p \in C_c^\infty([t_n, t_{n+1}] \times \mathbb{R}^2)$ for $n = 0, 1, ...$ Then there exists $\rho(\cdot) \in C([0,T]; \mathcal{P}_2(\mathbb{R}^2))$ such that $(\rho, -\nabla \cdot (\rho \nabla p))$ is a weak solution of (1) on $[0,\infty)$ and $\rho(0) = \rho_0$. Moreover every weak solution like this satisfies the conclusions of the previous lemma and

$$d(\rho(t), \rho_0) \leq C_1 \left(C_2 + s(\rho_0) + M_2(\rho_0) + \int_0^t \int_{\mathbb{R}^2} |\nabla p|^2 d\rho dr \right)^{\frac{1}{2}} \sqrt{t}$$

for $0 < t \leq 1$.

The set of $\rho(\cdot)$ as above will constitute the set of admissible paths (or controls). We will denote it $\mathcal{K}_{\rho_0}$.

Differential calculus on $\mathcal{P}_2(\mathbb{R}^2)$

We formally view $(\mathcal{P}_2(\mathbb{R}^2), d)$ as an infinite dimensional Riemannian manifold with tangent space $T_\rho \mathcal{P}_2(\mathbb{R}^2)$ at ρ modeled using $H_{-1,\rho}(\mathbb{R}^2)$.

Definition (Gradient)

For each $p \in C_c^\infty(\mathbb{R}^2)$, let $\rho^p = \rho^p(t,x), t \geq 0$ be defined by

$$\partial_t \rho^p(t) + \mathrm{div}(\rho^p \nabla p) = 0, \quad \rho^p(0) = \rho_0.$$

The gradient of a function $f : \mathcal{P}_2(\mathbb{R}^2) \mapsto \mathbb{R} \cup \{\pm\infty\}$ at ρ_0, denoted grad $f(\rho_0)$, exists if it can be identified as the unique element in $\mathcal{D}'(\mathbb{R}^2)$ satisfying

$$\lim_{t \to 0+} \frac{f(\rho^p(t)) - f(\rho_0)}{t} = \langle \mathrm{grad} f(\rho_0), p \rangle, \quad \forall p \in C_c^\infty(\mathbb{R}^2).$$

Differential calculus on $\mathcal{P}_2(\mathbb{R}^2)$

Definition (Symplectic Gradient)

The symplectic gradient of f evaluated at ρ_0, denoted J-grad $f(\rho_0)$ is defined the same except that ∇p in the continuity equation is replaced by $\nabla^\perp p$.

GRADIENTS OF OUR FUNCTIONALS:

$$
\begin{aligned}
\mathrm{grad}_\rho M_2(\rho) &= -\mathrm{div}(2x\rho) \\
\mathrm{grad}_\rho s(\rho) &= -\Delta\rho, \quad \text{for } s(\rho) < \infty, \\
\mathrm{grad}_\rho d^2(\rho,\gamma) &= -2\mathrm{div}(\rho\nabla p_{\rho,\gamma}), \quad \rho,\gamma \text{ have Lebesgue densities.} \\
\mathrm{grad}_{\rho_0} e(\rho_0) &= -\mathrm{div}(\rho_0(\nabla N * \rho_0)) \\
J\text{-}\mathrm{grad}_{\rho_0} e(\rho_0) &= -\mathrm{div}(\rho_0 u_{\rho_0}).
\end{aligned}
$$

Differential calculus on $\mathcal{P}_2(\mathbb{R}^2)$: Gradient formulas

If $I(\rho) < \infty$ then:

$$
\begin{aligned}
\|\mathrm{grad}_\rho s\|^2_{-1,\rho} &= I(\rho) \\
\|J\text{-}\mathrm{grad}_\rho e\|^2_{-1,\rho} &= \int_{\mathbb{R}^2} |P_\rho u_\rho|^2 \rho(dx) \leq C(1 + I(\rho)), \\
\|\mathrm{grad}_\rho M_2(\rho)\|^2_{-1,\rho} &= 4M_2(\rho) \\
\|\mathrm{grad}_\rho d^2(\rho,\gamma)\|^2_{-1,\rho} &= 4\|\nabla\cdot(\rho_i \nabla p_{\rho,\gamma})\|^2_{-1,\rho} = 4d^2(\rho,\gamma) \\
\langle J\text{-}\mathrm{grad}_\rho e(\rho), \mathrm{grad}_\rho s(\rho)\rangle_{-1,\rho} &= -\int_{\mathbb{R}^2} \nabla\rho \cdot K^\perp * \rho\, dx = 0 \\
\langle J\text{-}\mathrm{grad}_\rho e(\rho), \mathrm{grad}_\rho M_2(\rho)\rangle_{-1,\rho} &= -2\int_{\mathbb{R}^2} K^\perp * \rho \cdot x\rho\, dx = 0
\end{aligned}
$$

Differential calculus on $\mathcal{P}_2(\mathbb{R}^2)$: Gradient formulas

$$
\begin{aligned}
\langle \mathrm{grad}_\rho s(\rho), \mathrm{grad}_\rho d^2(\rho,\gamma)\rangle_{-1,\rho} &= 2\int_{\mathbb{R}^2} \nabla\rho \cdot \nabla p_{\rho,\gamma} dx \\
\langle \mathrm{grad}_\rho s(\rho), \mathrm{grad}_\rho M_2(\rho)\rangle_{-1,\rho} &= 2\int_{\mathbb{R}^2} \nabla\rho \cdot x\, dx = -4 \\
\langle J\text{-}\mathrm{grad}_\rho e(\rho), \mathrm{grad}_\rho d^2(\rho,\gamma)\rangle_{-1,\rho} &= -2\int_{\mathbb{R}^2} K^\perp * \rho \cdot \nabla p_{\rho,\gamma} \rho\, dx \\
\langle \mathrm{grad}_\rho M_2(\rho), \mathrm{grad}_\rho d^2(\rho,\gamma)\rangle_{-1,\rho} &= 2\int_{\mathbb{R}^2} x \cdot \nabla p_{\rho,\gamma} \rho\, dx
\end{aligned}
$$

Controlled gradient-Hamiltonian flow

Corollary

Let $\rho(\cdot) \in \mathcal{K}_{\rho_0}$. Then the weak solution of the controlled PDE (1) can be re-written as an abstract evolution which is a mixture of Hamiltonian flow, negative gradient flow, and control variable in the form

$$\partial_t \rho = J\text{-grad}_\rho e - \nu \text{grad}_\rho s + m,$$

where $m = -\nabla \cdot (\rho \nabla p)$.

Viscosity solutions: Equation

$E = \mathcal{P}_2(\mathbb{R}^2)$. For $p \in C_c^\infty(\mathbb{R}^2)$ we denote $m_p := -\nabla \cdot (\rho \nabla p) \in H_{-1,\rho}$. If $I(\rho) < +\infty$ and $\tilde{m} \in H_{-1,\rho}$ we define

$$
\begin{aligned}
H(\rho, \tilde{m}) = \sup_{m \in H_{-1,\rho}} \Big[& \langle J\text{-grad}_\rho e - \nu \, \text{grad}_\rho s, \tilde{m} \rangle_{-1,\rho} \\
& + \langle m, \tilde{m} \rangle_{-1,\rho} - \frac{1}{4\nu} \| m_i \|_{-1,\rho}^2 \Big] \\
= \sup_{p \in C_c^\infty(\mathbb{R}^2)} \Big[& \langle J\text{-grad}_\rho e - \nu \, \text{grad}_\rho s, \tilde{m} \rangle_{-1,\rho} \\
& + \langle m_p, \tilde{m} \rangle_{-1,\rho} - \frac{1}{4\nu} \| m_p \|_{-1,\rho}^2 \Big] \\
= \langle J\text{-grad}_\rho e & - \nu \, \text{grad}_\rho s, m \rangle + \nu \| m \|_{-1,\rho}^2 .
\end{aligned}
$$

HJB equation:

$$\alpha f - H(\rho, \text{grad}_\rho f) = h(\rho).$$

Viscosity solutions: Test functions

Definition

$\psi : E \to \mathbb{R} \cup \{+\infty\}$ is a test function if

$$\psi(\rho) = \theta s(\rho) + \delta M_2(\rho) + \sum_{k=1}^{\infty} \beta_k d^2(\rho, \gamma^k) + c,$$

where $0 < \theta < 1, \delta > 0, c \in \mathbb{R}$, $\beta_k \geq 0$ for $k \geq 1$, $\sum_{k=1}^{\infty} \beta_k < +\infty$, the set $\{\gamma^k : k \geq 1\}$ is bounded and every γ^k has Lebesgue density.

- $\theta s(\rho) + \delta M_2(\rho)$ plays the role of a cut-off function and a function which produces coercive term $I(\rho)$ in the equation. $0 < \theta < 1$ is needed for this.
- $\sum_{k=1}^{\infty} \beta_k d^2(\rho, \gamma^k)$ plays the role of a doubling function and a function needed to do perturbed optimization.

Viscosity solutions: Test functions

$\sum_{k=1}^{\infty} \beta_k d^2(\rho, \gamma^k)$ is continuous in E.
Since convergence in E implies convergence of second moments also $M_2(\rho)$ is continuous in E.
$s(\rho)$ is lower semicontinuous in E, since relative entropy is lsc (in the narrow topology of E) and $M_2(\rho)$ is continuous. ($s(\rho) + M_2(\rho)$ is the relative entropy of ρ with respect to a Gaussian measure, shifted by a constant.)
If $I(\rho) = \infty$ we set

$$H(\rho, \mathrm{grad}_\rho \psi(\rho)) = -\infty, \quad H(\rho, -\mathrm{grad}_\rho \psi(\rho)) = +\infty.$$

Lemma

Let ψ be a test function. Then:
(i) $\rho \to H(\rho, \mathrm{grad}_\rho \psi(\rho))$ is upper semicontinuous on E.
(ii) $\rho \to H(\rho, -\mathrm{grad}_\rho \psi(\rho))$ is lower semicontinuous on E.

Viscosity solutions: Definition

Definition

$g : E \to \mathbb{R}$ is a viscosity subsolution of $\alpha f - H(\rho, \mathrm{grad}_\rho f) = h(\rho)$ if whenever $(g - \psi)^*$ has a local maximum at ρ_0 for a test function ψ, we have

$$\alpha(g - \psi)^*(\rho_0) + \psi(\rho_0) - H(\rho_0, \mathrm{grad}_{\rho_0} \psi(\rho_0)) \leq h(\rho_0).$$

$g : E \to \mathbb{R}$ is a viscosity supersolution of $\alpha f - H(\rho, \mathrm{grad}_\rho f) = h(\rho)$ if whenever $(g + \psi)_*$ has a local minimum at ρ_0 for a test function ψ, we have

$$\alpha(g + \psi)_*(\rho_0) - \psi(\rho_0) - H(\rho_0, -\mathrm{grad}_{\rho_0} \psi(\rho_0)) \geq h(\rho_0).$$

A function $g : E \to \mathbb{R}$ is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

In fact the definition implies that ρ_0 in the subsolution/supersolution part necessarily satisfies $I(\rho_0) < +\infty$ so $H(\rho_0, \pm\mathrm{grad}_{\rho_0} \psi(\rho_0))$ is always finite.

Viscosity solutions: Perturbed optimization

Lemma (version of Borwein-Preiss variational principle)

Let $g : E \times E \to [-\infty, +\infty)$ *be upper semicontinuous and such that* $g(\rho,\gamma) = -\infty$ *if either* ρ *or* γ *does not have Legesgue density. Let for* $n \geq 1$ (ρ^0, γ^0) *be such that*

$$g(\rho^0, \gamma^0) > \sup_{E\times E} g - \frac{1}{n}.$$

Then there exist sequences $(\tilde{\rho}^k), (\tilde{\gamma}^k)$ *of measures that have Lebesgue densities such that* $d(\tilde{\rho}^k, \rho^0) \leq 1, d(\tilde{\gamma}^k, \gamma^0) \leq 1, k \geq 1,$ $\tilde{\rho}^k \to \rho^n, \tilde{\gamma}^k \to \gamma^n$ *for some* $\rho^n, \gamma^n \in E$*, and sequences of nonnegative numbers* $(\beta^1_k), (\beta^2_k)$ *such that* $\sum_{k=1}^{+\infty} \beta^i_k = 1, i = 1, 2$*, such that*

$$g(\rho^n, \gamma^n) > \sup_{E\times E} g - \frac{1}{n}$$

Viscosity solutions: Perturbed optimization

Lemma (version of Borwein-Preiss variational principle)

and

$$g(\rho^n, \gamma^n) - \frac{1}{n}\sum_{k=1}^{+\infty} \beta_k^1 d^2(\rho^n, \tilde{\rho}^k) - \frac{1}{n}\sum_{k=1}^{+\infty} \beta_k^2 d^2(\gamma^n, \tilde{\gamma}^k)$$
$$\geq g(\rho, \gamma) - \frac{1}{n}\sum_{k=1}^{+\infty} \beta_k^1 d^2(\rho, \tilde{\rho}^k) - \frac{1}{n}\sum_{k=1}^{+\infty} \beta_k^2 d^2(\gamma, \tilde{\gamma}^k).$$

for all $(\rho, \gamma) \in E \times E$.

Viscosity solutions: Comparison

Theorem (Comparison)

Let $h_1, h_2 \in C_b(E)$ *and be uniformly continuous on level sets of* $s + M_2$*. Let* $\overline{f}$ *be a viscosity subsolution of* $\alpha f - H(\rho, \mathrm{grad}_\rho f) = h_1(\rho)$ *and* $\underline{f}$ *be a viscosity supersolution of* $\alpha f - H(\rho, \mathrm{grad}_\rho f) = h_2(\rho)$*. Then*

$$\overline{f} - \underline{f} \leq \frac{1}{\alpha}(h_1 - h_2).$$

Therefore, if $h \in C_b(E)$ *and is uniformly continuous on level sets of* $s + M_2$ *then the equation* $\alpha f - H(\rho, \mathrm{grad}_\rho f) = h(\rho)$ *has at most one bounded viscosity solution. Moreover, such a solution is uniformly continuous on level sets of* $s + M_2$*.*

IDEA OF PROOF:

- Take $\beta > 1$. Look at the function

$$(\beta\overline{f} - \theta(s + M_2))^*(\rho) - (\underline{f} + \theta(s + M_2))_*(\gamma) - \frac{1}{2\epsilon}d^2(\rho,\gamma)$$

- Use perturbed optimization technique to produce global maximum.
- Plug stuff in, use differential calculus in E and estimate. The term $\langle \mathrm{grad}_\rho s(\rho), \mathrm{grad}_\rho s(\rho)\rangle_{-1,\rho}$ produces a coercive term $I(\rho)$. The most difficult part is to control the terms involving $\mathrm{grad}\, d^2(\rho,\gamma)$.
- Key estimates: (i) If $I(\gamma) + I(\rho) < +\infty$,

$$\langle -\mathrm{grad}_\rho s, \mathrm{grad}_\rho d^2\rangle_{-1,\rho} + \langle -\mathrm{grad}_\gamma s, \mathrm{grad}_\gamma d^2\rangle_{-1,\gamma} \leq C d^2(\rho,\gamma).$$

Viscosity solutions: Comparison

(ii) (difficult) If $I(\gamma) + I(\rho) < +\infty$, then for each $\delta > 0$ there exists a constant $C_\delta \geq 0$ such that

$$\begin{aligned} &\langle J\text{-grad}_\rho e, \text{grad}_\rho d^2 \rangle_{-1,\rho} + \langle \mathsf{J}\text{-grad}_\gamma e, \text{grad}_\gamma d^2 \rangle_{-1,\gamma} \\ &\quad \leq \delta d(\rho,\gamma) \sqrt{I(\rho) + I(\gamma)} C(s(\rho) + M_2(\rho), s(\gamma) + M_2(\gamma)) \\ &\qquad + C_\delta d^2(\rho,\gamma). \end{aligned}$$

The proof of this inequality is based on a result about approximation of the velocity field u whose proof in turn requires rather sophisticated harmonic analysis techniques.

Viscosity solutions: Comparison

Lemma (Approximation of the velocity field u)

For every $\delta > 0$ there exists a bounded, Lipschitz continuous and antisymmetric function function $J_\delta : \mathbb{R}^2 \mapsto \mathbb{R}$ such that for every $\rho \in \mathcal{P}_2(\mathbb{R}^2)$, the function $u_\delta = J_\delta * \rho$ satisfies

$$\int_{\mathbb{R}^2} |u(x) - u_\delta(x)|^2 \rho(x) dx \leq \delta I(\rho) \Big(C + s(\rho) + M_2(\rho) \Big)^3$$

for some absolute constant C.

Viscosity solutions: Existence

Theorem

Let $\alpha > 0$ and $h \in C_b(E)$. Then the value function f

$$f(\rho) = \sup\{\int_0^\infty e^{-\alpha s}\Big(h(\rho(s)) - L(\rho(s), \dot{\rho}(s))\Big)ds : \rho \in \mathcal{K}_\rho\},$$

where $\mathcal{K}_\rho$ is the class of admissible paths defined before is a viscosity solution of $\alpha f - H(\rho, \mathrm{grad}_\rho f) = h(\rho)$. If in addition h is uniformly continuous on level sets of $s + M_2$ then f is continuous and is the unique bounded viscosity solution.

MAIN POINTS OF PROOF: (a bit tricky since f may be discontinuous)

- Use dynamic programming principle.
- Use integral representation for $\psi(\rho(t)) - \psi(\rho(0))$ for a test function ψ (chain rule).
- Use uniform Hölder continuity of near optimal trajectories.
- Use upper/lower semicontinuity of $\rho \to H(\rho, \pm\mathrm{grad}_\rho\psi(\rho))$.