Viscosity methods and nonlinear PDE
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A Hamilton-Jacobi equations with discontinuous Hamiltonian arising from weighted mean curvature flow

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Our goal is to present an application of the viscosity theory to the wmc flow

$$\beta V = \sigma + \kappa_\gamma \tag{1}$$

for "admissible graphs with facets", where κ_γ is a singular curvature.

It turns out that equation (1) written in the local coordinates takes the form of a Hamilton-Jacobi equations

$$u_t + H(t, x, u, u_x) = 0 \quad t > 0, x \in \mathbb{R}, \qquad u(0, x) = u_0(x)\ x \in \mathbb{R}. \tag{2}$$

In interesting cases the Hamiltonian H is discontinuous.

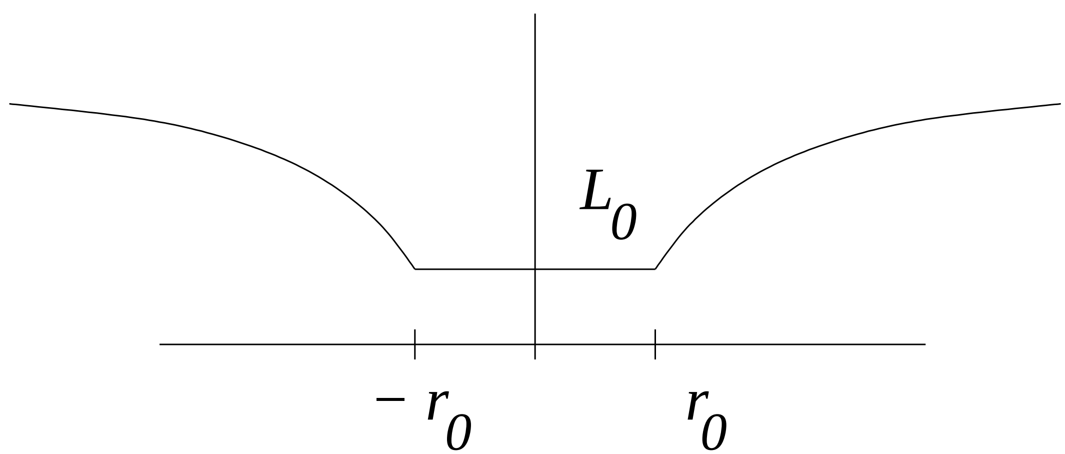
Our goals:
1) establish existence of viscosity solutions;
2) establish a Comparison Principle.

1. Motivation.

In many free boundary problems the modified Gibbs-Thomson law appears, an example is the ice crystal growth. The law takes the form of a driven weighted mean curvature (wmc) flow

$$\beta V = \sigma + \kappa_\gamma \qquad \text{on } \Gamma(t). \tag{3}$$

Here κ_γ is the wmc and it has be to carefully interpreted. Formally, the wmc flow is a 2nd order parabolic equation. It is interesting to look at the evolution of a graph of the function over $\mathbb{R}$.



The driving term σ appearing in (3) will satisfy a set of physical restriction. Additionally, we will impose conditions on its behavior at space infinity for the sake of studying the Comparison Principle.

The wmc flow (3) is very interesting also if $\sigma \equiv 0$, provided that the initial data are general. A variational solution was constructed by G.Bellettini, M.Novaga, M.Paolini, [2-4] and A.Chambolle [6]. Y.Giga–T.Fukui [12] used the theory of nonlinear semigroups. A different approach implemented is by P.Mucha and P.R., [19]. Our present problem is closely related to the work by M.-H.Giga and Y.Giga [13], [14] on viscosity solutions.

2. Evolution of graphs by WMC

2.1. Ingredients of $\beta V = \sigma + \kappa_\gamma$. They are:

• the positive driving term satisfying the symmetries $\sigma(x_1, x_2) = \sigma(\pm x_1, \pm x_2)$ and the *Berg's effect*, i.e.

$$x_i \frac{\partial \sigma}{\partial x_i}(x_1, x_2) > 0 \quad x_i \neq 0,\ i = 1, 2;$$

• the energy density function $\gamma(p_1, p_2) = |p_1|\gamma_\Lambda + |p_2|\gamma_R$,
• κ_γ, the wmc of Γ: $\kappa_\gamma = -\mathrm{div}_S \left(\nabla_\xi \gamma\right)(\mathbf{n})$, ($\mathrm{div}_S$ means the surface divergence). The main difficulty is to give meaning the terms looking like $\delta_{\mathbf{n}_\Lambda}(\mathbf{n})\frac{d\mathbf{n}}{dx}$, where the normal $\mathbf{n}$ may have jumps.
• the kinetic coefficient β is consistent with γ, it is a function over $\mathbb{R}^2$.

2.2. The evolution in the local coordinates

We used here an approach based on a variational principle to derive a tractable set of equations, it is similar to that used by G.Bellettini, M.Novaga, M.Paolini [2-4] and A.Chambolle [6].

We restrict our attention to graphs of *admissible* functions $d : \mathbb{R} \to \mathbb{R}_+$, i.e. d is even Lipschitz continuous; d_x vanishes on an interval containing 0 and $d|_{[0,+\infty)}$ is non-decreasing.

Proposition 2.1. (see [15], [16]) *The wmc equation (3) takes the following form*

$$\begin{aligned} \beta(0,1)\dot{L}_0 &= \fint_0^{r_0} \sigma(t,s,L_0)\,ds + \frac{\gamma_\Lambda}{r_0} \quad \textit{on} \qquad [0,r_0] \\ d_t &= \sigma(t,s,d)m(d_x) \quad \textit{for} \qquad s \in (r_0(t),\infty) \end{aligned} \tag{4}$$

where m *is* $1/\beta(\mathbf{n})$ *written in the local coordinates and* m *is Lipschitz continuous such that it is convex for* $|p| \leqslant 1$, $m(p) = m(-p)$, *and*

$$\frac{1}{\beta(0,1)} = m(0) \leqslant m(p), \quad m \in C^2(\mathbb{R} \setminus \{0\}), \quad m(p) \leqslant C(1+|p|).$$

Note: The point r_0 is a genuine zero-dimensional free boundary. For a generic σ, $\dot{r}_0(0)$ is different from zero, the sign of $\dot{r}_0(0)$ may be determined from $r_{00}, L_0, d_0, \sigma, m$. We are interested in a more difficult case of $\dot{r}_0(0) > 0$.

In order to write (4) as a single equation we set

$$\bar{d}(t,x) = \begin{cases} L_0(t), & \text{if } x < r_0(t) \\ d(t,x), & \text{if } x \geqslant r_0(t), \end{cases} \tag{5}$$

$$H(t,x,u,p) = \begin{cases} -\sigma(t, r_0^*(t), u) m(p), & \text{if } x < r_0(t) \\ -\sigma(t,x,u) m(p), & \text{if } x \geqslant r_0(t). \end{cases}$$

Hence (4) takes the form

$$\bar{d}_t + H(t,x,d,d_x) = 0 \text{ in } (0,T_0) \times \mathbb{R}, \quad \bar{d}(0,x) = \bar{d}_0(x), x \in \mathbb{R}. \tag{6}$$

Here r_0^* is uniquely defined by $\sigma(t, r_0^*(t), u) = \fint_0^{r_0} \sigma(t,s,L_0)\, ds + \frac{\gamma_\Lambda}{r_0}$. We note

Proposition 2.2 *If* $\dot{r}_0(0) > 0$*, then* $r_0^*(t) > r_0(t)$*, hence* H *is discontinuous*.

Here are our main results.

Theorem 2.3 (see [13]) *Let us suppose that* σ *satisfies the above constraints,* $\bar{d}_0$ *is an admissible initial condition, and* $\dot{r}_0(0) > 0$. *Then, there exists a unique properly defined interfacial curve* r_0 *and a BUC-viscosity solution* $\bar{d}$ *to (6) (and (2)). Moreover,* $\bar{d}$ *piecewise* C^1*.*

Theorem 2.4 (see [17]) *Let us suppose that* u *is BUC-subsolution* v *is a piecewise* C^1*-supersolution to (6). Moreover,* σ *and* v *satisfy additional technical conditions at* ∞*. If* $u(0,x) \leqslant v(0,x)$*, then* $u(t,x) \leqslant v(t,x)$ *for* $t \geqslant 0$*.*

3. Classical Comparison Principle

3.1. Classical Comparison Principle

We recall the standard definitions for continuous Hamiltonians.

Definition 3.1. (a) We shall say that u (resp. v) belonging to $BUC((0,T)\times\mathbb{R})$ is a *viscosity subsolution* (resp. a *viscosity supersolution*) of

$$u_t + F(t,x,u,u_x) = 0.$$

provided that for all C^1 functions $\varphi : (0,T)\times\mathbb{R}\to\mathbb{R}$ such that $u-\varphi$ (resp. $v-\varphi$) has a local maximum (resp. minimum) at (t_0,x_0), then

$$\varphi_t(t_0,x_0) + F(t_0,x_0,u(t_0,x_0),\varphi_x(t_0,x_0)) \leqslant 0.$$

$$(\text{resp. } \varphi_t(t_0,x_0) + F(t_0,x_0,u(t_0,x_0),\varphi_x(t_0,x_0)) \geqslant 0)\,.$$

(b) We shall say that $d\in BUC((0,T)\times\mathbb{R})$ is a *viscosity solution* of (2) provided that it is its viscosity subsolution as well as a viscosity supersolution.

Theorem 3.1. (classical Comparison Principle)
Assume that H is Lipschitz continuous and strictly increasing with respect to the third argument. If u is a subsolution and v is a supersolution to

$$u_t + H(t, x, u, u_x) = 0 \quad t > 0, x \in \mathbb{R}, \qquad u(0, x) = u_0(x) \; x \in \mathbb{R}$$

with $u(0, x) \leqslant v(0, x)$, then for all $t > 0$ we have $u(t, x) \leqslant v(t, x)$.

Much less is know when H is discontinuous, most of the papers deal with $H = H(t, x, \nabla u)$, i.e. for non-homogeneous eikonal equation, e.g. Camilli, Siconofli [5], Chen, Hu [7], Deckelnik, Elliott [9], De Zan, Soravia [10], Tourin 1992. Our effort is on extending them to the case of H depending on u.

3.2. Our discontinuous H.

In the case of interest H takes the form

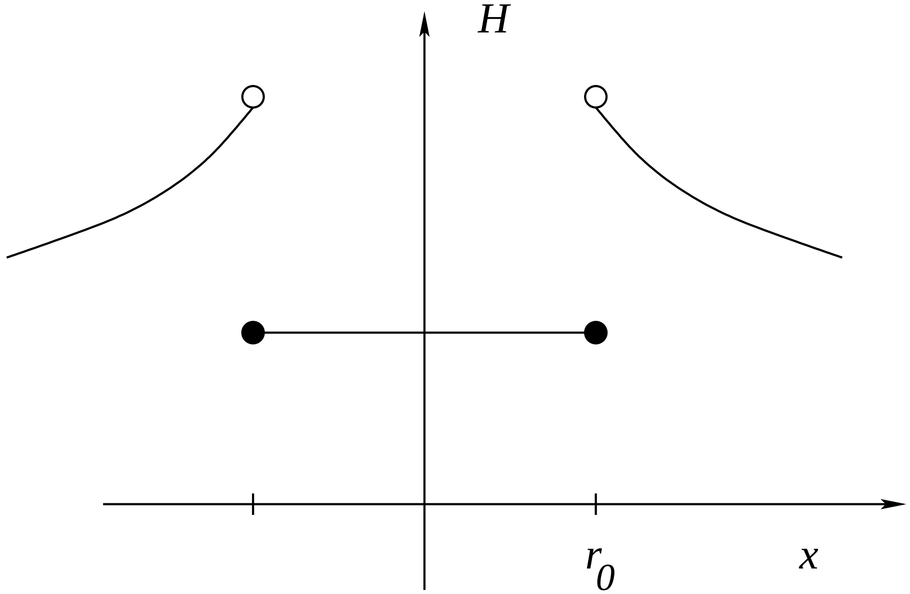
$$H(t, x, u, p) = \begin{cases} -\sigma(t, r_0^*(t), u)m(p), & \text{if } |x| < r_0(t) \\ -\sigma(t, x, u)m(p), & \text{if } |x| \geqslant r_0(t). \end{cases} \tag{7}$$

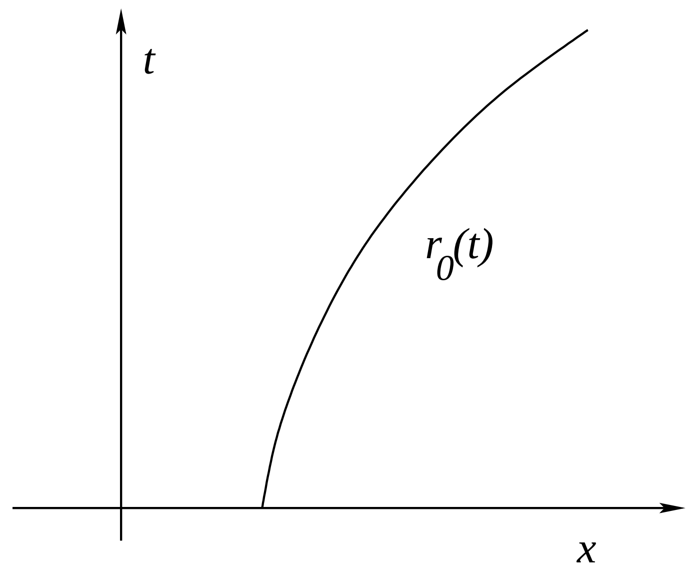
In addition to the presented properties of σ we need,

$$0 < \frac{\partial \sigma}{\partial u}(t, x, u) \leqslant M. \tag{8}$$

The assumptions on r_0, r_0^* shall be presented below.

Here is the graph of H at any t and a sketch of r_0, r_0^*,





3.3. Assumptions on H

We shall list the assumption reflecting the main features of the pictures.

(R1) $r_0, r_0^* \in C^0([0,T])$, for all $t \in [0,T]$ $r_0^*(t) > r_0(t)$ and $\{(t, r_0(t) : \ t \in [0,T]\}$ is a Lipschitz curve.

This reflects our need to localize the discontinuity, this is in line with most of the literature.

(H1) Hamiltonian H is lower semicontinuous in $[0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$;

(H2) H is continuous away from $\Gamma = \{(t, \pm r_0(t)) : \ t \in [0,T]\}$ and it has a jump discontinuity at Γ.

(H3) H^* is continuous in $G = \{(t,x) : \ |x| \geqslant r_0(t)\}$, while H_* is continuous on the closure of $([0,T] \times \mathbb{R}) \setminus G$.

Here H^* (resp. H_*) is the *upper* (resp. *lower*) *semicontinuous envelope* of H. These are formally defined by:

$$\begin{aligned} H^*(x) &= \lim_{\epsilon \longrightarrow 0^+} \sup\{H(z) : \; z \in B(x,\epsilon)\}, \\ H_*(x) &= \lim_{\epsilon \longrightarrow 0^+} \inf\{H(z) : \; z \in B(x,\epsilon)\}. \end{aligned} \tag{9}$$

These conditions implying discontinuity of H require a proper adjustment of the definition of viscosity solutions, it is in line with of the notion of sub-(super-)solution introduced by Barles-Perthame [1], Ishii [18], and more recently by Coclite, Risebro, [8], for discontinuous Hamiltonians.

Definition 3.2. (a) We shall say that u (resp. v) belonging to $BUC((0,T) \times \mathbb{R})$ is a *viscosity subsolution* (resp. *viscosity supersolution*) of (13) provided that for all C^1 functions $\varphi : (0,T) \times \mathbb{R} \to \mathbb{R}$ such that $u - \varphi$ (resp. $v - \varphi$) has a local maximum (resp. minimum) at (t_0, x_0), then

$$\varphi_t(t_0,x_0) + (H)^*(t_0,x_0,u(t_0,x_0),\varphi_x(t_0,x_0)) \leqslant 0.$$

$$(\text{resp. } \varphi_t(t_0,x_0) + (H)_*(t_0,x_0,v(t_0,x_0),\varphi_x(t_0,x_0)) \geqslant 0)\,.$$

(b) We shall say that a bounded, uniformly continuous function $d : (0,T_1) \times \mathbb{R} \to \mathbb{R}$ is a *viscosity solution* of (13) provided that it is a viscosity subsolution as well as a viscosity supersolution of (13).

Symmetry of H is just for the sake of simplicity. That is, we impose,
(H4) For any ϵ_1, ϵ_2 in $\{-1, 1\}$ we have $H(t, \epsilon_1 x, u, \epsilon_2 p) = H(t, x, u, p)$.

Monotonicity of H is crucial for our argument, thus we assume,
(H5) Hamiltonian H is strictly increasing with respect to u, i.e. there is a positive h_0, such that the following inequality holds for all u_2, u_1, x, t and p,

$$H(t, x, u_2, p) - H(t, x, u_1, p) \geqslant h_0(u_2 - u_1). \tag{10}$$

Proposition 3.2. *If H satisfies (H5) so do H^* and H_*.*

Remark. It is possible to convert H given by (7) into one satisfying (10), by means of the following change of variables $v_{new} = e^{t\lambda} u_{old}$, where $\lambda = -2M$ and M is the constant appearing in (8).

(H6) For all t, u and p function $x \mapsto H(t, x, u, p)$ is decreasing for $x > r_0(t)$, moreover $H(t, x, u, p) = H(t, r_0^*(t), u, p)$ for $x \in [-r_0(t), r_0(t)]$.

This reflects the special conditions we consider.

(H7) $\lim\limits_{\substack{|x| \to \infty \\ p \to 0}} H(t, x, u, p) = H^\infty \in C([0, T] \times \mathbb{R})$ locally uniformly with respect to $(t, u) \in [0, T] \times \mathbb{R}$, i.e. H^∞ does not depend upon p.

We do not impose explicit boundary conditions, but we have to control the behavior of supersolution at infinity.

Definition 3.3. For H satisfying (H7) we shall say that a piecewise C^1-function w is a *supersolution at infinity* provided that w is a supersolution, the following limits exist and are uniform with respect to $t \in [0, T]$,

$$w_t \to w_t^\infty, \quad w \to w^\infty, \quad w_x \to 0 \qquad \text{as } |x| \to \infty$$

and

$$w_t^\infty(t) + H^\infty(t, w^\infty(t)) \geqslant 0. \tag{11}$$

We shall call w a *strict supersolution at infinity* if it is a supersolution at infinity and the inequality in (11) is strict.

Theorem 3.3. (see [17]) *Let us assume that a measurable function H satisfies (R1) and (H1–H7) and for $u, v \in BUC([0,T] \times \mathbb{R})$ following conditions are valid:*
(a) v is a supersolution to (6), u is a subsolution to (6).
(b) v is a piecewise C^1-function.
(c) v is a supersolution of (6) at infinity.
If $u(0,x) \leqslant v(0,x)$, then for all $t > 0$

$$u(t,x) \leqslant v(t,x).$$

We impose conditions permitting us to control the behavior of H and supersolutions at infinity. We assume that the set of non-differentiability points of v is small and sets of discontinuities of H and v_x are 'aligned'.

The proof proceeds in several stages: we move the problem away from the jump discontinuity of H by considering a "shifted supersolution". We modify H to make it a continuous function, so that the classical CP becomes applicable.

In order to state our next observation it is convenient to introduce the notion of a strict supersolution. It is known in the literature, see e.g. Tourin, [20], for C^1 sub-, supersolution, here however, we have to relax the regularity assumptions.

Definition 3.4. We shall say that a supersolution v is *a strict supersolution* of (6), if for any test function $\varphi \in C^1$ such that $v - \varphi$ has a minimum at (t_0, x_0), then

$$\varphi_t(t_0, x_0) + H^*(t_0, x_0, v(t_0, x_0), \varphi_x(t_0, x_0)) > 0.$$

We define *a strict subsolution* of (6) in a similar way.

Proposition 3.4. (see [17]) *Let us suppose that the assumptions (R1) and (H1)–(H7) hold. If v is a supersolution of (6) so is $v + \epsilon$ for any positive ϵ. Moreover, $v + \epsilon$ is a strict supersolution.*

We define the regularized Hamiltonian. For $\delta > 0$ we set

$$H^\delta(t,x,u,p) = \begin{cases} H(t,x,u,p) & |x| \geqslant r_0(t)+\delta, \\ (1-\frac{\lambda}{\delta})H(t,r_0^*,u,p) + \frac{\lambda}{\delta}H(t,r_0+\delta,u,p) & |x| = r_0(t)+\lambda,\ \lambda \in (0,\delta), \\ H(t,r_0^*(t),u,p) & |x| \leqslant r_0(t). \end{cases}$$

Lemma 3.5. (see [17]) *If u is a subsolution to (6), then it is also a subsolution to (12) below,*

$$d_t + H^\delta(t,x,d,d_x) = 0. \tag{12}$$

We define a shifted supersolution v^δ by

$$v^\delta(t,x) = \begin{cases} v(t,x-\delta) & \text{for } x > \delta, \\ v(t,0) & \text{for } x \in [-\delta,\delta], \\ v(t,x+\delta) & \text{for } x < -\delta. \end{cases}$$

We have to show that for a given ϵ and sufficiently small δ, function $v^\delta + \epsilon$ is indeed a supersolution to (12).

Lemma 3.6. *Let us suppose that assumptions (R1) and (H1)–(H7) hold and:*
(a) w *is piecewise a* C^1 *function;*
(b) w *is a supersolution of (6);*
(c) w *is a supersolution at infinity of (6).*
Then, for any $\epsilon > 0$ *there is such* $\delta_0(\epsilon) > 0$ *that for any* $\delta \in (0, \delta_0(\epsilon))$*, function* $w^\delta + \epsilon$ *is a supersolution of (12).*

We notice that due to (10) the Hamiltonian at infinity H^∞ is also strictly increasing with respect to u. It is just sufficient to pass to the limit in (10) to deduce that

$$H^\infty(t, u_2) - H^\infty(t, u_1) \geqslant h_0(u_2 - u_1).$$

This inequality combined with (11) shows that $w + \epsilon$ is a strict supersolution at infinity.

Proof of the CP.

We notice that if we have a supersolution v, then for any $\epsilon > 0$ $v + \epsilon$ is not only a strict supersolution but also a strict supersolution at infinity. Thus, the is $\delta(\epsilon) >$ such that $v^\delta(0, x) + \epsilon > u(0, x)$. We notice that $v^\delta + \epsilon$ is a supersolution, while u is a subsolution to

$$u_t + H^\delta(t, x, u, u_x) = 0 \quad t > 0, x \in \mathbb{R}, \qquad u(0, x) = u_0(x)\ x \in \mathbb{R}.$$

We may now apply the classical Comparison Principle to deduce that

$$u(t, x) \leqslant v^\delta(t, x) + \epsilon, \qquad t \in [0, T].$$

but $\delta(\epsilon) \to 0$ while $\epsilon \to 0$. Continuity of v implies

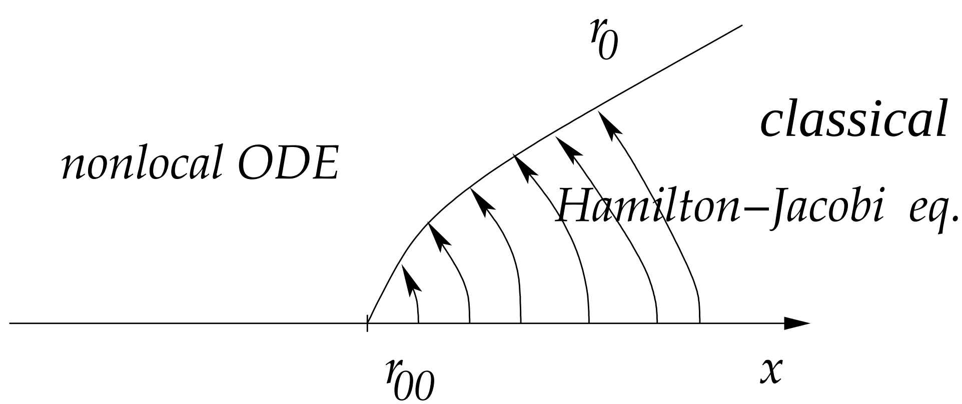
$$u(t, x) \leqslant v(t, x).$$

4. Existence of viscosity solutions.

Here is our main result.

Theorem 4.1 (see [15])*Let us suppose that σ satisfies the symmetry conditions and Berg's effect hold, $\bar{d}_0$ is an admissible initial condition defined by d_0 and L_0 by (5) and $\dot{r}_0(0) > 0$. Then, there exists a unique properly defined interfacial curve r_0 and a BUC-viscosity solution $\bar{d}$ to (6). Thus, functions $d(t,x) := \bar{d}(t,x)$ for $|x| \geqslant r_0(t)$ and $L_0(t) := \bar{d}(t, r_0(t))$ form a solution to (4). Moreover, if the assumptions of the Comparison Principle are satisfied, then the solution is unique.*

The picture explains the difficulty, when $\dot{r}_0(0) > 0$.



4.1 Classical results

We gather information about the eq. with a locally Lipschitz Hamiltonian F

$$d_t + F(t, x, d, d_x) = 0 \text{ in } (0, T) \times \mathbb{R} \quad d(0, x) = \bar{d}_0(x), x \in \mathbb{R}. \tag{13}$$

We use the standard definition of viscosity solutions.

Proposition 4.2. (see [18], [11]) *Let us suppose that all the assumption on* σ*,* γ *and* m *hold. If* $0 \leqslant d_{0x} \leqslant p_0 < 1$*, then there exist* $T > 0$ *and a unique Lipschitz continuous viscosity solution to*

$$d_t - \sigma(t, s, d)m(d_x) = 0 \text{ in } (0, T) \times \mathbb{R}, \quad d(0, x) = d_0(x), x \in \mathbb{R}, \tag{14}$$

such that $|d_x| \leqslant 1$*.*

After regularizing σ, m and the initial data d_0 we may construct unique solutions to the regularized problem by the method of characteristics. That is it suffices to look at

$$\begin{aligned} \dot{x} &= -\frac{dm^\delta}{dp}(p)\sigma^\delta(t, x, z) \\ \dot{p} &= m^\delta(p)\frac{\partial \sigma^\delta}{\partial x}(t, x, z) \\ \dot{z} &= m^\delta(p)\sigma^\delta(t, x, z) - \frac{dm^\delta}{dp}(p)\sigma^\delta(t, x, z)p. \end{aligned}$$

Here, δ is the regularizing parameter.

We have also strict monotonicity of the viscosity solution on (r_{00}, ∞).

Proposition 4.3 *Let us suppose that all the assumption on* σ*,* γ *and* m *hold. If* $d^+_{0,x}(r_{00}) \geqslant \delta > 0$ *and* $\lambda_1 > r_{00}$*, then there is* $\delta_0 = \delta_0(\lambda_1) > 0$ *such that for all* $t \in [0, T]$ *and all* x*,* y *satisfying* $r_{00} \leqslant x \leqslant y \leqslant \lambda_1$ *we have* $d(t, y) - d(t, x) \geqslant \delta_0(y - x)$*.*

The monotonicity is easily proved for the regularized system. This property survives the pointwise limit.

The solution may be even localized.

Proposition 4.4 *Let us suppose that all the assumption on* σ*,* γ *and* m *hold. Let* d_{01}*,* d_{02} *satisfy* $(d_{01} - d_{02})|_{[\lambda_0,\lambda_1]} \equiv 0$ *and* d_1*,* d_2 *are the corresponding solutions to (14). Then, for any* $t < T_1 := (\lambda_1 - \lambda_0)/\sup \sigma \cdot \sup m_p$ *we have*

$$(d_1 - d_2)|_{[0,t]\times[\lambda_0,\lambda_1-\mu t]} \equiv 0.$$

This follows from the fact the characteristics emanating from $\{t = 0\}$ turn left and the finite speed of signal propagation along the characteristics.

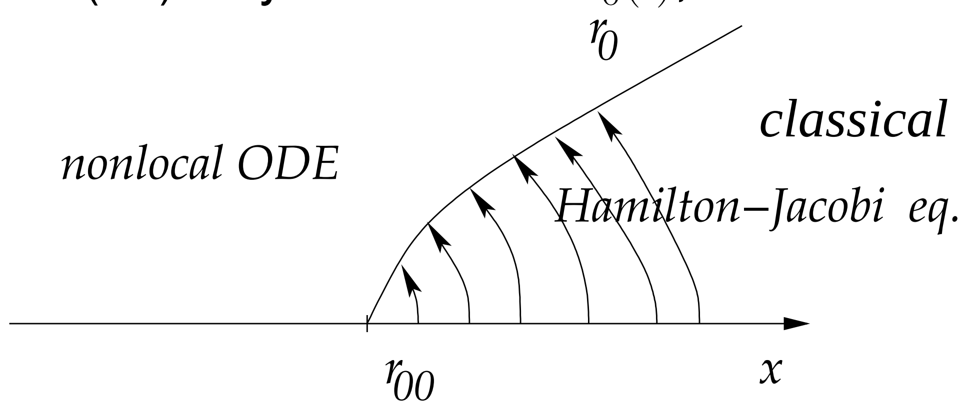
4.2. The interfacial curve

An obvious condition guaranteeing that $\bar{d}$ is continuous is

$$L_0(t) = d(t, r_0). \tag{15}$$

The identity (15) will be called **the matching condition**.

If we draw characteristics of the HJ and the interfacial curve with $\dot{r}_0(t) > 0$, then we see that (15) fully determines $r_0(\cdot)$, because it is a 'shock wave':



These observations help us rewrite (15). If $\tilde{r}_0(\cdot)$ is given, then it is easy to solve eq. for $L_0 = L_0(\tilde{r}_0)$ because it is an ODE (cf. (4)). Moreover, $d(t,\cdot)$ is strictly increasing, thus

$$r_0 = (d(t,\cdot))^{-1} L_0(r_0) =: \mathcal{K}(r_0). \tag{16}$$

Theorem 4.5. (see [15]) *Under our assumptions there exists at least one continuous solution r_0 to (16). Moreover, if $d^+_{0x}(r_{00}) > 0$, then the solution is unique and Lipschitz continuous.*

The proof is based on Schauder theorem, which yields a fixed point of $\mathcal{K}$. If $d_{0x}(r_{00}) > 0$, then uniqueness follows from the Banach contraction Theorem.

Once we constructed a unique r_0 when $d^+_{0,x}(r_{00}) > 0$ we are done, for:
(a) we have already constructed d as a viscosity solution to HJ eq. $(4)_2$;
(b) the fixed point argument yields, as a by-product L_0, thus automatically we have $\bar{d}$.

What happens when $(d_0)^+_x(r_{00}) = 0$**?** In this case the method used to show existence of the matching curve does not yield uniqueness. We want to develop a general approach. The theory of viscosity solutions of Hamilton-Jacobi equation is the tool of choice.

Our program is quite natural:
We will 'regularize' the data, so that $(d_0^\epsilon)_x^+(r_{00}) = \epsilon > 0$. Then we have to pass to the limit with ϵ. The limiting process involves:
(a) Proving existence of the limit of the matching curves r_0^ϵ; we'll call them *proper matching curves*. This is technical.
(b) Showing that $(d^\epsilon, L_0^\epsilon)$ forms a viscosity solution $\bar{d}^\epsilon$ to a HJ equation with the *discontinuous* Hamiltonian $\bar{H}^\epsilon$. This is standard for the usual definition of viscosity solutions with discontinuous Hamiltonians.
(c) proving existence of $\bar{d}$ a uniform limit of $\bar{d}^\epsilon$. This is not difficult for we can easily show that $\bar{d}^\epsilon$ is monotone decreasing with respect to ϵ.
(d) finding a proper notion of convergence of discontinuous Hamiltonians H^ϵ to H. This is one of the key points.
(e) showing that $\bar{d}$ is a viscosity solution to the HJ eq. with Hamiltonian $\bar{H}$. Same as (c).
(f) showing that $\bar{d}$ is unique a viscosity solution. This is another key point, the Comparison Principle we have shown comes into play.

4.3. Viscosity solutions to a HJ eq. with a discontinuous Hamiltonian

We begin with the 'regularization' of the data. We set

$$d_0^\epsilon(x) = d_0(x) + \eta^\epsilon(x),$$

where

$$\eta^\epsilon(x) = \begin{cases} 0 & x \leqslant r_{00} \\ \epsilon(x - r_{00}) & r_{00} \leqslant x < \rho_0 \\ \epsilon(\rho_0 - r_{00}) & x \geqslant \rho_0. \end{cases}$$

If $(d_0^\epsilon, L_0^\epsilon)$ is a solution to (4) with the regularized data, then

$$\bar{d}^\epsilon(t,x) = \begin{cases} L_0^\epsilon(t) & \text{if } x < r_0^\epsilon(t) \\ d^\epsilon(t,x) & \text{if } x \geqslant r_0^\epsilon(t). \end{cases} \tag{17}$$

is a solution to the HJ eq. (6) with the 'regularized' Hamiltonian H^ϵ,

$$H^\epsilon(t,x,u,p) = \begin{cases} -\sigma^\epsilon(t,(r_0^\epsilon)^*(t),u)m^\epsilon(p), & \text{if } x < r_0^\epsilon(t) \\ -\sigma^\epsilon(t,x,u)m^\epsilon(p), & \text{if } x \geqslant r_0^\epsilon(t), \end{cases} \tag{18}$$

The Stability Theorem

We want to show that indeed we have at least one solution if $\epsilon = 0$. It is found as a limit of viscosity solutions of (6) when $\epsilon \to 0$. For this purpose we have to define convergence of discontinuous Hamiltonians. We will use the standard notions of *upper relaxed limit* and *lower relaxed limit* to study convergence of sequences of discontinuous functions. If u^ϵ, $\epsilon > 0$ is a sequence of locally bounded measurable functions, then we set

$$\limsup_{\epsilon\longrightarrow 0^+}{}^* u^\epsilon(x) = \lim_{\epsilon\longrightarrow 0^+} \sup\{u^\delta(z) : \ z \in B(x,\epsilon), \ \delta \in (0,\epsilon)\},$$

$$\liminf_{\epsilon\longrightarrow 0^+}{}_* u^\epsilon(x) = \lim_{\epsilon\longrightarrow 0^+} \inf\left\{u^\delta(z) : \ z \in B(x,\epsilon), \ \delta \in (0,\epsilon)\right\}.$$

We recall that $\limsup_{\epsilon\longrightarrow 0^+}{}^* u^\epsilon$ (resp. $\liminf_{\epsilon\longrightarrow 0^+ *} u^\epsilon$) is upper semicontinuous (resp. lower semicontinuous).

Definition 4.1 We shall say that a sequence of discontinuous Hamiltonians H^ϵ, $\epsilon > 0$, *converges* to H provided that:

$$\limsup_{\epsilon \to 0^+}{}^* H^\epsilon = H^* \quad \text{and} \quad \liminf_{\epsilon \to 0^+}{}_* H^\epsilon = H_*.$$

Proposition 4.6 (see [15]) *Let us assume that $\bar{d}_0^\epsilon$, $\epsilon > 0$ is a sequence of admissible regularization of initial data d_0 and $\bar{d}^\epsilon$ are the corresponding viscosity solutions to (6) with H^ϵ given by (18). We have already established uniform convergence of $\bar{d}^\epsilon$ to $\bar{d}^0$. Then, H^ϵ converges to H in the sense defined above and $\bar{d}^0$ is a viscosity solution to*

$$\bar{d}_t + H(t, x, \bar{d}, \bar{d}_x) = 0, \qquad d(0, x) = d_0(x).$$

Under the additional assumptions on σ solution $\bar{d}^0$ is unique.

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