Existence of viscosity solutions for a nonlocal equation modelling polymer crystal growth

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Existence for a coupled system of equations

$$\begin{cases} \dfrac{\partial u}{\partial t} = g(v)|Du| \\ \dfrac{\partial v}{\partial t} - \Delta v + g(v)\mathcal{H}^{N-1}\lfloor_{\{u(\cdot,t)=0\}} = 0 \end{cases} \quad \text{in } \mathbb{R}^N \times (0,+\infty),$$

Initial conditions : $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$ in $\mathbb{R}^N$,

$N \geq 1$, $g : \mathbb{R} \to \mathbb{R}_+$, $u_0, v_0 : \mathbb{R}^N \to \mathbb{R}$ are given

Unknowns : $u, v : \mathbb{R}^N \times (0,+\infty) \to \mathbb{R}$

$\mathcal{H}^{N-1}\lfloor\{u(\cdot,t)=0\}$: $N-1$-Hausdorff measure restricted to

$$\Gamma_t := \{x \in \mathbb{R}^N \ ; \ u(x,t) = 0\}.$$

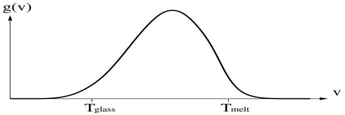
Motivation : growth of polymer crystals in a heterogeneous temperature field

Mathematical model : Eder (97), Burger-Capasso-Salani (02), Friedman-Velazquez (01)

- Each point x of the boundary Γ_t of a single crystallite Ω_t moves according to the law

$$\overrightarrow{\mathcal{V}}(x,t) = g(v(x,t))\,\overrightarrow{n}(x,t)$$

$\overrightarrow{\mathcal{V}}(x,t)$: normal velocity, $\overrightarrow{n}(x,t)$: outer unit normal
g : material function of the specific polymer



- Thermal energy required for crystallization
➪ heat equation for $v(x,t)$ with a negative heat source

$$-\frac{\partial}{\partial t}\mathbf{1}_{\Omega_t} = -g(v(x,t))\delta_{\Gamma_t}$$

Nonlocal front propagation problem

It leads to the equations :

$$\overrightarrow{\mathcal{V}}(x,t) = g(v(x,t))\overrightarrow{n}(x,t)$$

with

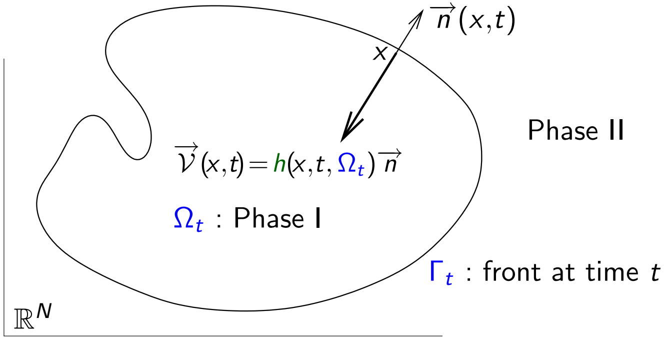
$$\frac{\partial v}{\partial t} - \Delta v = -g(v(x,t))\mathcal{H}^{N-1}{}_{\lfloor \Gamma_t}$$

(up to positive constants)

➪ **Problem :** Given an initial front (Γ_0, Ω_0), study (Γ_t, Ω_t) for all $t \geq 0$.

Level-set approach

Osher-Sethian 88/Chen-Giga-Goto 91/Evans-Spruck 91



Idea : To represent Γ_t as the 0-level-set of an auxiliary function $u : \mathbb{R}^N \times [0, T] \to \mathbb{R}$

Level-set approach

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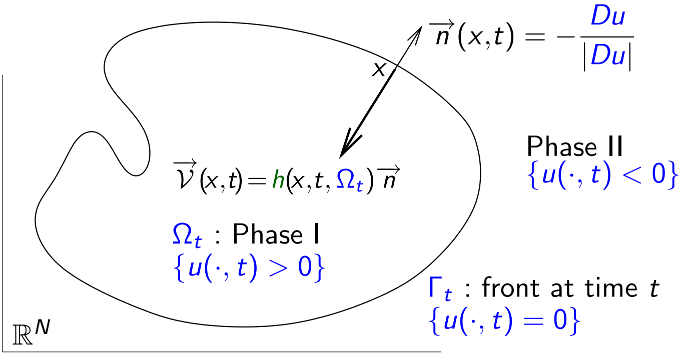


Idea : To represent Γ_t as the 0-level-set of an auxiliary function $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$

$$\mathcal{V} = -\frac{\frac{\partial u}{\partial t}}{|Du|} \quad \Longrightarrow \quad \boxed{\frac{\partial u}{\partial t} = h(x,t,\{u(\cdot,t) \geq 0\})|Du|}$$

Nonlocal H-J equation

Our case

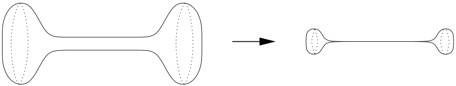
$$\begin{cases} \dfrac{\partial u}{\partial t} = g(v)|Du| \\ \dfrac{\partial v}{\partial t} - \Delta v + g(v)\mathcal{H}^{N-1}\lfloor_{\{u(\cdot,t)=0\}} = 0 \end{cases} \quad \text{in } \mathbb{R}^N \times (0,+\infty),$$

Initial conditions : $u(x,0) = u_0(x), \; v(x,0) = v_0(x) \text{ in } \mathbb{R}^N,$

$h(x,t,\{u(\cdot,t) \geq 0\}) = g(v)$
u_0 represents the initial crystal or front : $\{u_0 = 0\} = \Gamma_0$
v_0 represents the intial temperature.

Difficulties

- Nonlinear problems, singularities occurs in finite time



Mean curvature flow of a smooth set in $\mathbb{R}^3$

- Nonlocal velocity

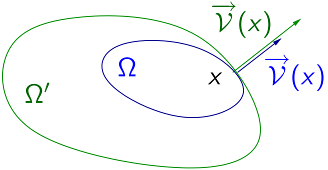
- Nonmonotone evolution

- The singular term $g(v)\mathcal{H}^{N-1}\lfloor_{\{u(\cdot,t)=0\}}$

Monotone and nonmonotone evolutions

Monotonicity of the velocity with respect to the nonlocal term

$$\Omega \subset \Omega' \text{ et } x \in \partial\Omega \cap \partial\Omega' \quad \Longrightarrow \quad h(x,\Omega) \leq h(x,\Omega')$$



➪ Inclusion or avoidance principles :
$\Omega_0 \subset \Omega'_0 \Longrightarrow \forall t \geq 0,\ \Omega_t \subset \Omega'_t$
➪ Maximum principle, viscosity solutions

Nonmonotone cases : preservation of inclusion fails

Here : crystallization does not take place at a prescribed temperature

The singular term

Representation for the heat equation

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v + g(v)\mathcal{H}^{N-1}\lfloor_{\Gamma_t} = 0 \\ v_0 \equiv 0 \end{cases}$$

$$\boxed{v(x,t) = -\int_0^t \int_{\Gamma_s} G(x-y, t-s) g(v(y,s)) d\mathcal{H}^{N-1}(y) ds}$$

- Existence/regularity of v strongly depends on the regularity of $(\Gamma_t)_{t\geq 0}$
- Existence/regularity of $(\Gamma_t)_{t\geq 0}$ strongly depends on the regularity of v

Known results

- [Burger-Capasso-Salani 2002] : Construction of the model
- [Friedman-Velásquez 2001] : Short time existence and uniqueness for **smooth initial positions** close to a sphere in $\mathbb{R}^3$.
- [Su-Burger Preprint 2007] Long time existence and estimates **in** $\mathbb{R}^2$**.**

Main result : existence for the system

Assumptions :

- g Lipschitz continuous, $0 < A \leq g(r) \leq B \quad \forall r \in \mathbb{R}$.
- $u_0 = d^s_{\bar{\Omega}_0}$ where $d^s_{\bar{\Omega}_0}$ is the signed distance to a compact subset $\bar{\Omega}_0 \subset \mathbb{R}^N$ satisfying the interior ball condition.
- $v_0 : \mathbb{R}^N \to \mathbb{R}$ is bounded and Lipschitz continuous.

Theorem. [Cardaliaguet-OL-Monteillet 2009] The system

$$\begin{cases} \dfrac{\partial u}{\partial t} = g(v)|Du| \\ \dfrac{\partial v}{\partial t} - \Delta v + g(v)\mathcal{H}^{N-1}\lfloor_{\{u(\cdot,t)=0\}} = 0 \end{cases} \quad \text{in } \mathbb{R}^N \times (0,+\infty),$$

Initial conditions : $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$ in $\mathbb{R}^N$,

has a solution, u is bounded uniformly continuous and v is Hölder continuous in $\mathbb{R}^N \times [0,T]$.

Remarks

- Definition of solutions adapted from [Giga-Goto-Ishii 92, Soravia-Souganidis 96, Barles-Cardaliaguet-OL-Monteillet 09]

  Definition. A solution on $\mathbb{R}^N \times [0, T]$ is a map $(u, v) : \mathbb{R}^N \times [0, T] \to \mathbb{R}^2$ which is bounded, uniformly continuous, such that u satisfies the equation

  $$\frac{\partial u}{\partial t} = g(v)|Du| \text{ in } \mathbb{R}^N \times (0, T), \;\; u(x, 0) = u_0(x) \text{ in } \mathbb{R}^N$$

  in the viscosity sense, with $\int_0^T \mathcal{H}^{N-1}(\{u(\cdot, t) = 0\}) < +\infty$, and such that $v(\cdot, 0) = v_0$ and v satisfies in the sense of distributions

  $$\frac{\partial v}{\partial t} - \Delta v + g(v)\mathcal{H}^{N-1}\lfloor\{u(\cdot, t) = 0\} = 0 \text{ in } \mathbb{R}^N \times (0, T).$$

- **Uniqueness** is open.

Study of the eikonal equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = c(x,t)|Du| \text{ in } \mathbb{R}^N \times [0,T], \\ u(x,0) = u_0(x) \text{ in } \mathbb{R}^N. \end{cases}$$

under the assumptions

- $c : \mathbb{R}^N \times [0,T]$ measurable and satisfies

$$0 < A \leq c(x,t) \leq B, \tag{1}$$

$$|c(x,t) - c(y,t)| \leq C|y-x|\,(1 + |\log|x-y||)\,, \tag{2}$$

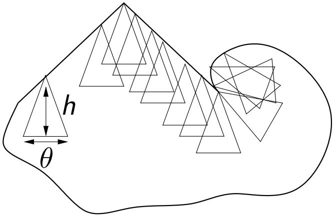
$$|c(x,t) - c(y,t)| \leq \omega|y-x|^\alpha. \tag{3}$$

- $u_0 = d^s_{\Omega_0}$ where $d^s_{\Omega_0}$ is the signed distance to a compact subset $\bar{\Omega}_0 \subset \mathbb{R}^N$ satisfying the interior ball condition.

Remark : Best we can assume since latter $c := g(v)$ and best regularity for v is like (2)-(3).
(even for smooth Γ_t [Friedman-Velásquez 2001]).

Previous results for the eikonal equation

- c positive $C^{1,1}$ in space : uniform interior **ball** property for the front [Cannarsa-Frankowska 06, Alvarez-Cardaliaguet-Monneau 05, Barles-OL 06]
- c positive Lipschitz continuous in space : uniform interior **cone** property for the front [Barles-Cardaliaguet-OL-Monteillet 09]



➩ the front Γ_t has **finite perimeter**

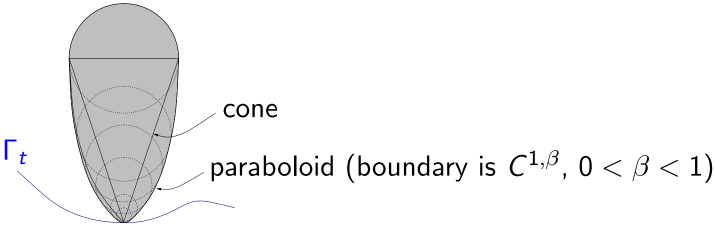
New estimates under our assumptions

Under assumption (1)-(2)-(3) (**Hölder** velocity) :

- Eikonal equation has a unique viscosity solution given by

$$u(x,t) = \sup\left\{u_0(x(0))\ ,|x'(s)| \leq c(x(s),s) \text{ a.e., } x(t) = x\right\}$$

- Γ_t has still the unform interior **cone** property.
- Moreover, Γ_t has a uniform interior **paraboloid** property

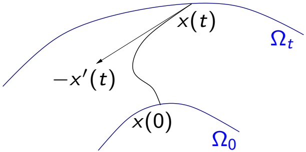


Some ideas

$$\begin{cases} \dfrac{\partial u}{\partial t} = c(x,t)|Du| \\ u(x,0) = u_0(x) \end{cases} \quad \text{governs} \quad \begin{cases} \Gamma_t = \{u(\cdot,t) = 0\} \\ \Omega_t = \{u(\cdot,t) > 0\} \end{cases}$$

$\overline{\Omega}_t$ is the reachable set of a control system :

$$\{x(t) \ : \ x'(s) = c(x(s),s)a(s), \ 0 \le s \le t, \ x(0) \in \overline{\Omega}_0, \ a \in L^\infty, |a| \le 1, \}$$



Pontryagine Maximum principle : there exists an adjoint p s.t.

$$\begin{cases} x'(s) = c(x(s),s)\frac{p(s)}{|p(s)|} \\ -p'(s) = Dc(x(s),s)|p(s)| \end{cases}$$

Application : kind of stability for fronts having a uniform **paraboloid** property

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = c_n(x,t)|Du_n| \text{ in } \mathbb{R}^N \times [0,T], \\ u_n(x,0) = u_0(x) \text{ in } \mathbb{R}^N. \end{cases}$$

under the assumptions

- c_n satisfies (1)-(2)-(3) (with same constants)
- $c_n \longrightarrow c_\infty$ a.e.
- $|Du_0(x)| > 0$ on $\{u_0 = 0\}$ (viscosity sense)

Minimal time function : $z_n(x) = \inf\{t \geq 0 : u_n(x,t) \geq 0\}$

Proposition.

- $\{u_n(\cdot,t) = 0\} = \{z_n = t\}$
- $\frac{1}{B} \leq |Dz_n(x)| \leq \frac{1}{A}$ (viscosity sense)
- $\frac{Dz_n}{|Dz_n|} \longrightarrow \frac{Dz_\infty}{|Dz_\infty|}$ a.e. in $\{0 < z_\infty < T\}$
- $|Dz_n| \longrightarrow |Dz_\infty|$ in L^∞-weak* in $\{0 < z_\infty < T\}$

Estimates for the heat equation with a singular term such as Γ_t has uniform **cone** property

$$\frac{\partial v}{\partial t} - \Delta v + g(v)\mathcal{H}^{N-1}{}_{\lfloor \Gamma_t} = 0, \quad v(x,0) = v_0(x) \quad \text{in } \mathbb{R}^N \times [0,T].$$

- v_0, g Lipschitz continuous and bounded
- $(\Gamma_t)_{t\in[0,T]}$ given with interior **cone** property with opening ρ and heigth 2ρ

Proposition.

There is a unique solution v and it satisfies :

$$|v| \le C(1 + |\log(\rho)|)$$
$$|v(x,t) - v(y,t)| \le \frac{C}{\rho}(1 + |\log|x-y|\,|)|x-y|$$
$$|v(x,t) - v(y,t)| \le \frac{C}{\rho^{1/4}}(1 + |\log(\rho)||x-y|^{1/2}$$
$$|v(x,t) - v(x,s)| \le \frac{C}{\rho}(1 + |\log|t-s|\,|)|t-s|^{1/2}$$

Proof of the existence result : Schauder fixed point theorem

$$\mathcal{X} := \left\{ v : \mathbb{R}^N \times [0,T] \to \mathbb{R} : \begin{array}{l} v \text{ measurable} \\ |v| \leq L_1 \\ |v(x,t) - v(y,t)|| \leq L_2 |x-y|^{1/2} \\ |v(x,t) - v(y,t)|| \leq L_3 (1 + |\log|x-y||)|x-y| \end{array} \right\}$$

closed bounded convex subset of $L^\infty(\mathbb{R}^N \times [0,T])$

$\Phi : v \in \mathcal{X} \mapsto \tilde{v}$ defined as :

- u solution of $\frac{\partial u}{\partial t} = g(v)|Du|, \quad u(x,0) = u_0(x)$
- $\Omega_t := \{u(\cdot,t) > 0\}$, $\Gamma_t := \{u(\cdot,t) = 0\}$, has interior **cone** property with opening $\boxed{\rho = \text{constant} \times (L_2)^{-2}}$
- $\tilde{v}$ solution of $\frac{\partial \tilde{v}}{\partial t} - \Delta \tilde{v} + g(\tilde{v}) \mathcal{H}^{N-1}\lfloor_{\Gamma_t} = 0$, $\tilde{v}(x,0) = v_0(x)$

Then :

➪ Possible to choose L_1, L_2, L_3 such that $\Phi : \mathcal{X} \to \mathcal{X}$

➪ Φ continuous : comes from the stability of the front because of **paraboloid** property

➪ Φ compact because of the estimates for the heat equation.