On the weak Harnack inequality
for
fully nonlinear PDEs
with
unbounded ingredient

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1. Introduction

Fully nonlinear elliptic PDEs:

$$F(x, u, Du, D^2u) = f(x) \quad \text{in } \Omega$$

$x \to F(x, r, \xi, X)$: **measurable**
$\forall$fixed $r \in \mathrm{R}, \xi \in \mathrm{R}^n, X \in S^n$

$f \in L^p$
$\Downarrow$
L^p-viscosity solution
by
Caffarelli-Crandall-Kocan-Święch
(1996)
$\Uparrow$
Caffarelli (1989)
L^p- , Schauder regularity theory

Assumption $\quad p > \frac{n}{2}$

Notation: $\|\cdot\|_r = \|\cdot\|_{L^r}$ $(r > 0)$

$$g \in L^p_+ \Longleftrightarrow g \in L^p \ \& \ g \geq 0$$

Fixed ellipticity constants:

$$0 < \lambda \leq \Lambda$$

$$S^n_{\lambda,\Lambda} := \{A \in S^n \mid \lambda I \leq A \leq \Lambda I\}$$

Pucci extremal operators

$$\mathcal{P}^+(X) := \max_{A \in S^n_{\lambda,\Lambda}} \{-\mathrm{tr}(AX)\}$$

$$\mathcal{P}^-(X) := \min_{A \in S^n_{\lambda,\Lambda}} \{-\mathrm{tr}(AX)\}$$

Example 1

$$-a(x) \triangle u + \mu(x)|Du|^{m} = f(x)$$

$$a(x) = \begin{cases} \lambda \ \ (x \in \Omega_0 \subset \Omega), \\ \Lambda \ \ (x \in \Omega \setminus \Omega_0) \end{cases}$$

$$F(x, \xi, X) := -a(x)\mathrm{tr}X + \mu(x)|\xi|^{m}$$

$m = 1$: linear case
$m > 1$: superslinear case

Example 2

$$\max_a \min_b \{-\mathrm{tr}(A^{a,b}(x)D^2u) + \langle \mu^{a,b}(x)Du, Du\rangle^{m/2}\} = f(x)$$

$$A^{a,b}(x) \in S^n_{\lambda,\Lambda} \ (x \in \Omega)$$

$$\mu^{a,b}(x) \in S^n \ (x \in \Omega), \ \mu^{a,b}_{ij} \in L^q$$

$$x \to A^{a,b}(x), \mu^{a,b}(x): \text{ measurable}$$

$$F(x,\xi,X) := \max_a \min_b \{-\mathrm{tr}(A^{a,b}(x)X) + \langle \mu^{a,b}(x)\xi, \xi\rangle^{m/2}\}$$

$$F(x, u, Du, D^2u) = f(x) \quad \text{in } \Omega \qquad (E)$$

Definition

$u \in C(\Omega)$: L^p-viscosity subsolution of (E)

$\Longleftrightarrow$

$\forall \phi \in W^{2,p}_{loc}(\Omega)$, $u - \phi$: maximum at $x \in \Omega$

$\Downarrow$

$$ess \liminf_{y \to x} \left\{ \begin{array}{c} F(y, u(y), D\phi(y), D^2\phi(y)) \\ -f(y) \end{array} \right\} \le 0$$

Definition

u: L^p-strong subsolution (supersolution) of (E)

$\Longleftrightarrow$

$u \in W^{2,p}_{loc}(\Omega)$:

$$F(x, u(x), Du(x), D^2u(x)) \le (\ge)\ f(x)$$

a.e. in Ω

Assumptions

No dependence on u for F !

(A1) $F(x, 0, O) = 0$

(A2) $f \in L^p_+ \quad (p > \hat{p})$

$\hat{p} \in [n/2, n)$: Escauriaza constant

(A3) uniformly elliptic

$$\mathcal{P}^-(X - Y) \le F(x, \xi, X) - F(x, \xi, Y) \le \mathcal{P}^+(X - Y)$$

(A4) $|F(x, \xi, O)| \le \mu(x)|\xi|^m$

$\exists \mu \in L^q_+$

$m = 1 \Leftrightarrow$ linear order

$m > 1 \Leftrightarrow$ superlinear order

Under (A1)–(A4)

If u: L^p-viscosity subsolution

of

$$F(x, Du, D^2u) = f(x)$$

$$\Downarrow$$

- $u + C$: **L^p-viscosity subsolution**

of

$$\mathcal{P}^-(D^2u) - \mu(x)|Du|^m \leq f^+(x)$$

- $-u + C$: **L^p-viscosity supersolution**

of

$$\mathcal{P}^+(D^2u) + \mu(x)|Du|^m \geq -f^-(x)$$

2. ABP maximum principle

Aleksandrov-Bakelman-Pucci maximum principle

$\Omega \subset \mathrm{R}^n$: bounded

$u \in C(\overline{\Omega})$: L^n-strong subsolution of

$$\mathcal{P}^-(D^2u) - \mu(x)|Du| \leq f(x)$$

$$\mu, f \in L^n_+(\Omega)$$

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C_1 e^{C_2\|\mu\|_n^n} \|f\|_{L^n(\Omega[u])}$$

$$\Omega[u] = \{x \in \Omega : u(x) > \sup_{\partial\Omega} u^+\}$$

C_k: universal constants

(or upper contact set)

cf. Gilbarg-Trudinger etc.

For L^p-viscosity solution:

Escauriaza constant: $\hat{p} \in [n/2, n)$

$$\forall p > \hat{p}$$

"L^p-strong solvability for the simplest PDE"

$$\text{i.e. } \forall f \in L^p(\Omega),\ g \in C(\partial\Omega)$$

$$\Downarrow$$

$$\exists u \in C(\overline{\Omega}) \cap W^{2,p}_{loc}(\Omega):\ L^p\text{-strong solution}$$

of

$$\mathcal{P}^+(D^2 u) = f(x) \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega$$

s.t.

$$-\sup_{\partial\Omega} g^- - \hat{C}\|f^-\|_p \le u \le \sup_{\partial\Omega} g^+ + \hat{C}\|f^+\|_p$$

$$\|u\|_{W^{2,p}(\Omega')} \le C(\Omega')\left(\|f\|_p + \|g\|_\infty\right)$$

$$\Omega' \Subset \Omega$$

ABP maximum principle
for
L^p-viscosity solutions

Caffarelli-Crandall-Kocan-Święch

$\mu \in L^\infty$, $f \in L^p$ $(p > \hat{p})$

Fok (1996)

$\mu \in L^q(\Omega) \cap L^{\text{“}2n\text{”}}(\text{near } \partial\Omega)$ $(q > n)$

K-Święch(2007)

$\mu \in L^q$ $(q > n)$, $f \in L^p$ $(q \geq p > \hat{p})$

ABP maximum principle

(K-Święch, 2007, linear; $m = 1$)

$$q \geq p > \hat{p} \quad \text{and} \quad q > n$$

$$f \in L^p_+(\Omega),\ \mu \in L^q_+(\Omega)$$

$u \in C_+(\overline{\Omega})$: L^p-viscosity subsolution

of

$$\mathcal{P}^-(D^2u) - \mu(x)|Du| \leq f(x)$$

$$\Downarrow$$

$$(a)\ q \geq p \geq n,\ q > n$$

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C_1 e^{C_2\|\mu\|_n^n}\|f\|_{L^n(\Omega[u])}$$

$(b)\ q > n > p > \hat{p}$

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u^+ + C_3 e^{C_4\|\mu\|_n^n} \|\mu\|_q^{\exists N} \|f\|_p$$

$$+ C_3 \sum_{k=0}^{N-1} \|\mu\|_q^k \|f\|_p$$

New even for L^p-strong solutions

when $n > p > \hat{p}$.

by

iterated comparison function method

Idea **strong solvability**

$$\mathcal{P}^+(D^2 v) = -f(x)$$

$$w := u + v$$

$$\mathcal{P}^-(D^2 w) - \mu(x)|Dw| \le \mu(x)|Dv| \in L^{p_1}$$

$$p_1 > p$$

then, by iteration

Superlinear case $m > 1$

$\exists$ counter examples

(K-Święch, 2004, 2007)

even among classical solutions

ABP maximum principle $\quad m > 1$

(a) $q \geq p > n$, $f \in L^p_+(\Omega)$, $\mu \in L^q_+(\Omega)$

$\|f\|_p^{m-1}\|\mu\|_q < \delta$ for some $\delta > 0$

$u \in C(\overline{\Omega})$: L^p-viscosity subsolution
of

$$\mathcal{P}^-(D^2u) - \mu(x)|Du|^m \leq f(x)$$

$$\Downarrow$$

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C_5\|f\|_p$$

$$C_5 = C_5(\|\mu\|_q)$$

(b) $q > n \geq p > \hat{p}$, $f \in L^p_+, \mu \in L^q_+$

$$n(q-p) > mq(n-p)$$

(i.e. $p = n \Longrightarrow \forall m > 1$)

(also, $p \uparrow n \Longrightarrow$ large $m > 1$)

$\|f\|_p^{m-1}\|\mu\|_q < \delta$ for some $\delta > 0$

$u \in C(\overline{\Omega})$: L^p-viscosity subsolution

of

$$\mathcal{P}^-(D^2u) - \mu(x)|Du|^m \leq f(x)$$

$$\Downarrow$$

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C_6\|f\|_p$$

$$C_6 = C_6(\|\mu\|_q)$$

precise dependence $\Leftarrow$ K. Nakagawa

Relation between L^p-viscosity and L^p-strong

(A5) $$|F(x,\xi,X) - F(x,\eta,X)| \le \mu(x)(|\xi|^{m-1} + |\eta|^{m-1} + 1)|\xi - \eta|$$

Theorem $$(A1) - (A5)$$
$$q > n, \quad q \ge p > \hat{p}$$
$$mq(n-p) < n(q-p)$$
$$\Downarrow$$

- L^p-strong (sub/super)solution $\Longrightarrow$ L^p-viscosity (sub/super)solution

- L^p-viscosity (sub/super)solution $\in W^{2,p}_{loc}$ $\Longrightarrow$ L^p-strong (sub/super)solution

3. Main results

$m = 1$ (K-Święch, 2009)

$q > n,\ q \geq p > \hat{p},\ \mu \in L^q_+,\ f \in L^p_+$

$u \geq 0$: L^p-viscosity supersolution

of

$$\mathcal{P}^+(D^2u) + \mu(x)|Du| \geq -f(x)$$

$$\Downarrow$$

(wH) $$\|u\|_{L^r(Q_1)} \leq C\left(\inf_{Q_1} u + \|f\|_{L^p(Q_4)}\right)$$

for some $r > 0,\ C > 0$

$$Q_r := \left(-\frac{r}{2}, \frac{r}{2}\right)^n$$

Known results

L^n-strong supersolution $u \geq 0$

of

$$\mathcal{P}^+(D^2u) + \mu(x)|Du| \geq -f(x) \quad a.e.$$

$$f \in L^n_+, \ \mu \in L^{2n}_+$$

$$\Downarrow$$

(wH) holds

e.g. Gilbarg-Trudinger

Known results for viscosity solutions

u : L^p-viscosity supersolution

of

$$\mathcal{P}^+(D^2u) + \mu(x)|Du| \geq -f(x)$$

$$\Downarrow$$

$\mu \equiv 0$: Caffarelli (1989)

$\mu \in L^\infty$: K (2004): Beginner's guide

$\mu \in L^q$ $(q > 2n)$: Fok (1996)

$$\mu(x)|Du| \leq \mu(x)^2 + 4^{-1}|Du|^2$$

$+$ Hopf-Cole transformation

$m > 1$ (K-Święch, 2009)

(a0) $q = \infty$, $p > \hat{p}$,
$n > m(n-p)$

(a1) $n < p \leq q < \infty$

(a2) $\hat{p} < p \leq n < q < \infty$,
$mq(n-p) < n(q-p)$

$\mu \in L^q_+$, $f \in L^p_+$

(b) $1 < m < 2 - \frac{n}{q}$

$u \geq 0$: L^p-viscosity supersolution

of

$$\mathcal{P}^+(D^2u) + \mu(x)|Du|^m \geq -f(x)$$

$$\Downarrow$$

(wH) $$\|u\|_{L^r(Q_1)} \leq C\left(\inf_{Q_1} u + \|f\|_{L^p(Q_4)}\right)$$

for some $r > 0$, $C > 0$

4. Applications

Applications

(1) (local) Hölder continuity

(2) Strong maximum principle

(3) Boundary weak Harnack inequality

$$\Downarrow$$

- (a) Global Hölder continuity
- (b) Existence of strong solutions
- (c) ABP type maximum principle
  in unbounded domains etc.

(4) Local maximum principle

5. Idea of Proof

Strategy of proof of (wH) with $m = 1$

$$\mathcal{P}^+(D^2u) + \mu(x)|Du| \geq -f(x)$$

$$\mu \in L^q_+, \quad f \in L^p_+$$

Reductions

(1) $f \equiv 0$

$\Uparrow$

the existence of L^p-strong solution

of

$$\mathcal{P}^-(D^2v) - \mu(x)|Dv| = f(x) \geq 0 \text{ in } Q_4$$

$$v = 0 \text{ on } \partial Q_4$$

assuming spt μ: compact

ABP maximum principle

$$\Downarrow$$

$$0 \leq v \leq C\|f\|_p$$

$$w := u + v \geq 0:$$

L^p-viscosity supersolution

of

$$\begin{aligned} & \mathcal{P}^+(D^2 w) + \mu(x)|Dw| \\ \geq\ & \mathcal{P}^+(D^2 u) + \mathcal{P}^-(D^2 v) + \mu(x)|Dw| \\ \geq\ & \mu(x)(-|Du| + |Dv| + |Dw|) \\ & -f(x) + f(x) \\ \geq\ & 0 \end{aligned}$$

i.e. w: L^p-viscosity supersolution

of

$$\mathcal{P}^+(D^2 w) + \mu(x)|Dw| \geq 0$$

If weak Harnack holds for $f \equiv 0$

$$(0 \le u \le w,\ v \le \hat{C}\|f\|_p)$$

$$\Downarrow$$

$$\begin{aligned} \|u\|_{L^r(Q_1)} &\le \|w\|_{L^r(Q_1)} \\ &\le C \inf_{Q_1} w \\ &= C \inf_{Q_1} (u+v) \\ &\le C \inf_{Q_1} u + C'\|f\|_p \end{aligned}$$

Reduction (2)

$$\inf_{Q_1} u \leq 1 \Longrightarrow \|u\|_{L^r(Q_1)} \leq \exists C_0$$

(More precisely,

$$\inf_{Q_3} u \leq 1 \Longrightarrow \|u\|_{L^r(Q_1)} \leq C_0)$$

$$\Uparrow$$

$$U := \frac{u}{\inf_{Q_1} u + \delta} \quad (\delta > 0)$$

$$\Downarrow$$

$$\mathcal{P}^+(D^2U) + \mu(x)|DU| \geq 0$$

Lemma (cf. Caffarelli, K-Święch)

$$\exists \phi \in C(\overline{Q}_4) \cap W^{2,p}_{loc}(Q_4),\ \xi \in C(Q_4)$$

such that

$$\begin{cases} (i) & \phi \le 0 & \text{in } Q_4 \\ (ii) & \mathcal{P}^-(D^2\phi) & \\ & -\mu(x)|D\phi| = \xi(x) & \text{in } Q_4 \\ (iii) & \phi \le -2 & \text{in } Q_3 \\ (iv) & \phi = 0 & \text{on } \partial Q_4 \\ (v) & \xi = 0 & \text{in } Q_4 \setminus Q_1 \end{cases}$$

$\Uparrow$

Assume tempolarlly $\|\mu\|_q << 1$

+

Cabré's covering argument

cf. Caffarelli's idea; when $\mu \equiv 0$

$$\phi(x) := c_1 - c_2|x|^{\text{“}2-n\text{”}}$$

$$(x \in Q_4,\ x \neq 0)\ c_k > 0$$

$$\Downarrow$$

$$\mathcal{P}^-(D^2\phi(x)) = 0$$

s.t.

Properties $(i) - (iv)$: OK

In Q_1, mollify ϕ s.t. $\phi \in C^2(Q_4)$;

$$\mathcal{P}^-(D^2\phi) \geq \xi \quad \text{in } Q_1$$

$$\Downarrow$$

ξ appears ($supp\ \xi \subset Q_1$ i.e. (v))

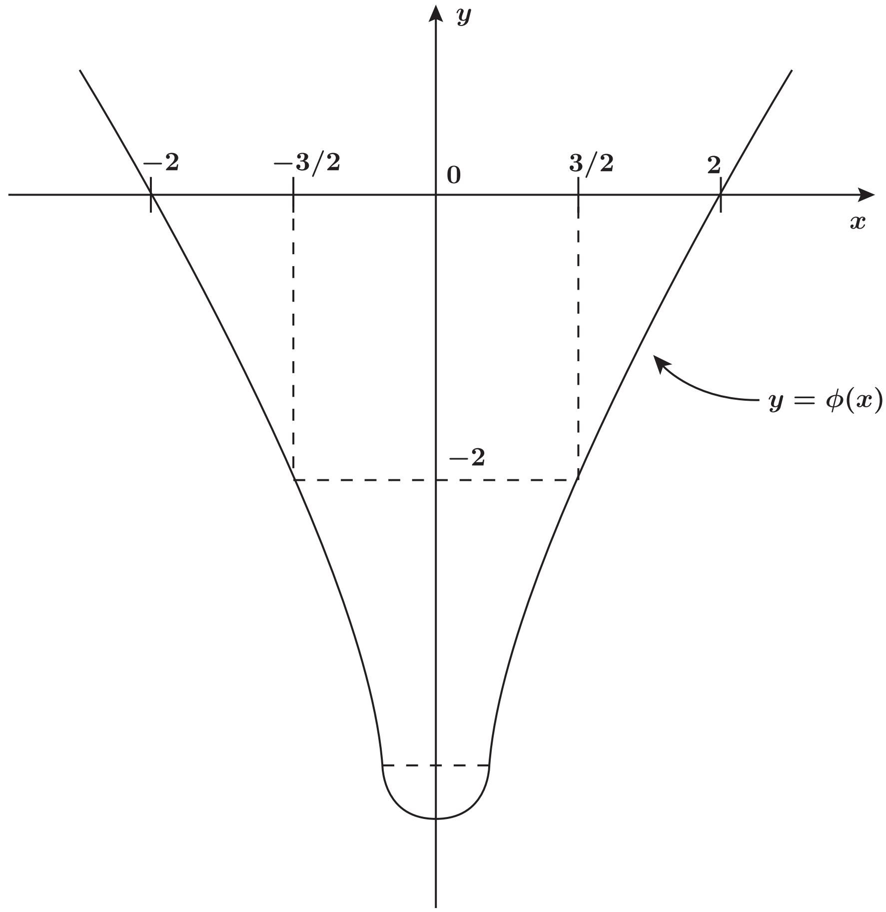

図 1: $n = 1$

But ! in case when $\mu \neq 0$

No "fundamental solution like" function ϕ

Existence of L^p-strong solution ϕ

i.e. strong solvability

(K-Święch, 2009) $\cdots$ a bit complicated

$$\mathcal{P}^-(D^2\phi) - \mu(x)|D\phi| = 0 \quad \text{in } Q_4 \setminus Q_{1/2}$$

$$\phi = 0 \quad \text{on } \partial Q_4, \qquad \phi = -\text{``100''} \quad \text{on } \partial Q_{1/2}$$

Multiply $\alpha > 1$ s.t. $\alpha\phi \le -2$ in Q_3.

Mollify ϕ in $Q_{1/2}$.

$$\text{Under } f \equiv 0,\ \inf_{Q_1} u \leq 1$$

$$\Downarrow$$

$$\exists r, C_0 > 0 \text{ s.t. } \|u\|_{L^r(Q_1)} \leq C_0$$

$$\Longleftrightarrow\ |\{x \in Q_1\ | u(x) > t\}| \leq t^{-\sigma}$$

$$\Longleftrightarrow\ |\{x \in Q_1\ |\ u(x) > M^k\}| \leq \theta^k$$

for some $\theta \in (0,1)$, $M > 1$ ($\forall k \in \mathrm{N}$)

Aim : $k = 1$

$$|\{x \in Q_1\ |\ u(x) > M\}| \leq \theta \in (0,1)$$

+ scaling + cube decomposition

$\boldsymbol{w := u + \phi}$:

L^p-viscosity supersolution

of

$$\mathcal{P}^+(D^2w) + \mu(x)|Dw|$$

$$\geq \mathcal{P}^+(D^2u) + \mathcal{P}^-(D^2\phi) + \mu(x)|Dw|$$

$$\geq \mu(x)(|Dw| - |Du| + |D\phi|)$$

$$\geq \xi(x)$$

i.e. $\mathcal{P}^+(D^2w) + \mu(x)|Dw| \geq \xi(x)$

ABP maximum principle

+

$$(-\phi \geq 2 \text{ in } Q_3, \sup_{Q_3}(-u) \geq -1)$$

$$\Downarrow$$

$$1 \leq \sup_{Q_3}(-w) \leq \sup_{Q_4}(-w)$$

$$\leq C\|\xi\|_{L^n(Q_1 \cap \{w<0\})}$$

$$|\{x \in Q_1 \mid u(x) < -\phi(x)\}| \geq \exists \tau \in (0,1)$$

$$\theta := 1 - \tau, \quad \min_{Q_4} \phi =: -M < -2$$

$$|\{x \in Q_1 \mid u(x) > M\}| \leq \theta$$

Strategy of proof of $m > 1$ case

Need strong solvability !

i.e. Existence of L^p-strong solution

of

$$\mathcal{P}^-(D^2u) - \mu(x)|Du|^m = f(x) \quad \text{in } \Omega$$

$$u = g \quad \text{on } \partial\Omega$$

under (a0) or (a1) or (a2)

$+$

"finer" cube-decomposition lemma

$\Leftarrow$ hypo. (b)

(cf. K-Takahashi, 2002)

$\Downarrow$

weak Harnack inequality

Existence theorem: $\mu \in L^q_+$, $f \in L^p_+$

One of the following holds:

(a0) $q = \infty$, $p > \hat{p}$,
$n > m(n-p)$

(a1) $n < p \le q < \infty$

(a2) $\hat{p} < p < n < q < \infty$,
$mq(n-p) < n(q-p)$

$$\|\mu\|_q \|f\|_p^{m-1} < \exists\delta$$

(cf. ABP maximum principle)

$$\Downarrow$$

$$\exists u \in W^{2,p}(\Omega)$$

s.t.

$$\mathcal{P}^-(D^2u) - \mu(x)|Du|^m = f(x) \quad a.e. \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega.$$

Idea of proof: (when $n < p < q < \infty$)

$$r := \frac{mpq}{q-p}$$

Fixed $\forall v \in W^{1,r}(\Omega)$

Note $\|\mu|Dv|^m\|_p \leq \|\mu\|_q \|\|Dv\|_r^m$

i.e. $f + \mu|Dv|^m \in L^p(\Omega)$

Define $T: W^{1,r}(\Omega) \to W^{2,p}(\Omega)$

by

$$u := Tv \in C(\overline{\Omega}) \cap W^{2,p}(\Omega)$$

s.t.

$$\mathcal{P}^-(D^2u) = f(x) + \mu(x)|Dv|^m$$

$$u = 0 \quad \text{on } \partial\Omega$$

Caffarelli: $u \in W^{2,p}_{loc}(\Omega)$ (1989)

Winter: $u \in W^{2,p}(\Omega)$ (2009)

$$W^{2,p}(\Omega) \subset W^{1,r}(\Omega) : \text{ compact}$$

$$\exists R > 0 \text{ s.t. } T : \mathcal{B}_R \to \mathcal{B}_R$$

$$\mathcal{B}_R := \{v \in W^{1,r}(\Omega) \mid \|v\|_p + \|Dv\|_p \leq R\}$$

Scauhder's fixed point theorem

$$\Downarrow$$

$$\exists u = Tu$$

Existence of L^p-strong solutions.

Local maximum principle

Linear growth case: $m = 1$

cf. Caffarelli's argument in the book

1 Reduction to $f \equiv 0$

2 supersolutions u

$$\Downarrow$$

$$(\star) \quad |\{x \in Q_1 \mid u(x) \geq t\}| \leq A_1 t^{-r_1}$$

for some $A_1, r_1 > 0$

3 subsolutions satisfying $(\star)$,
$\exists M_1 > 1$ such that if $u(x_1) \geq M_1$
then $u(x_2) \geq M_1^2$ near x_2

$$\Downarrow$$

contradiction to $u \in C(Q_4)$.

i.e. u is bounded.

$$w(x) := A_1^{1/r_1} \frac{u(x)}{\|u\|_{L^{r_1}(Q_1)}} \text{ satisfies } (\star)$$

$$\Downarrow$$

w: bounded

$$\Downarrow$$

Local maximum principle

Superlinear growth case: $m > 1$

Under the assumptions for weak Harnack

+

$\|\mu\|_q$: small

Differences from linear growth case

1. **Existence of L^p-strong solutions**

   of

   $$\mathcal{P}^+(D^2 v) + \mu(x)|Dv|^m = g(x) \text{ in } \Omega$$

   $$v = h(x) \text{ on } \partial\Omega$$

   $$\Uparrow$$

   K-Święch 2009, JFPTA

2. **finer cube decomposition**

   K-Takahashi (2002), K-Święch (2009)

Thank you very much !