Large time behavior of solutions of Hamilton-Jacobi-Bellman equations with quadratic nonlinearity in gradients[1]

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[1] A part of this talk is based on a joint work with S.-J. Sheu (Academia Sinica, Taiwan).

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Cauchy problem for HJB equation

$$\begin{cases} \partial_t u - \dfrac{1}{2}\Delta u + H(x, Du) = 0 & \text{in } (0,\infty)\times\mathbb{R}^N \\ u(0,\cdot) = u_0 & \text{in } \mathbb{R}^N \end{cases} \tag{CP}$$

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(H3) $u_0 \in \Phi_0 := \{v \in C(\mathbb{R}^N) \mid \inf_{\mathbb{R}^N} (v - \phi_0) > -\infty\}$

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(H2) and **(H3)** hold if

- $\gamma_1 > 0$ and $\inf_{\mathbb{R}^N} u_0 > -\infty$ $(\phi_0 \equiv 0)$
- $\beta > 0$ and $\inf_{\mathbb{R}^N} u_0 > -\infty$ ($\phi_0(x) := -\varepsilon |x|^2$, $\varepsilon > 0$ small)

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- $\dfrac{u(T,x)}{T} \longrightarrow -\lambda$: growth rate (independent of u_0)
- $u(T,x) + \lambda T \longrightarrow \phi(x)$: steady state (depending on u_0)

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Note. Nagai ('03,'09): quadratic $H(x,p)$, $u_0 \equiv 0$

$$\frac{u(T,x)}{T} \longrightarrow -\lambda, \quad u(T,x) - u(T,0) \longrightarrow \phi(x) \quad \text{as } T \to \infty$$

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$\Longrightarrow \quad \exists\, T^* \in (0,\infty)$ s.t. $\displaystyle\lim_{T\to T^*} \frac{u(T,x)}{T} = -\infty$.

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$$\Longrightarrow \qquad \lim_{T\to\infty} \frac{u(T,x)}{T} = 0 \quad \text{and} \quad \lim_{T\to\infty} u(T,x) = \infty$$

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Theorem (Ich.'09, Ich.-Sheu'10). $Q := (0,\infty)\times\mathbb{R}^N$.

(a) $\forall\, u_0 \in \Phi_0,\ \exists!\, u \in C^{1,2}(Q)\cap C(\overline{Q})$: sol. of

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(b) $\exists\, \lambda^* \in \mathbb{R}$ s.t.

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has a solution ϕ if and only if $\lambda \ge \lambda^*$. Moreover, in case $\lambda = \lambda^*$, solution is unique up to an additive constant.

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$$(\phi - \phi_0)(x) \le \alpha(\phi - \psi_0)(x) + C, \quad x \in \mathbb{R}^N,$$

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Theorem (Ich.-Sheu'10). Assume either $\kappa_1 = \kappa_2$ or (H4).

Then, $\forall\, u_0 \in \Phi_0$, $\exists\, \phi \in \Phi_0$: sol. of (EP) with $\lambda = \lambda^*$ s.t.

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Notice. Solutions of (EP) contain ambiguity of constants.

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Proposition (Ich.-Sheu'10). Condition (H4) holds if

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Note. V: sub-quadratic $\Longrightarrow$ (H4) automatically holds.

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Dynamical approach.

Stochastic control interpretation for $u(T,x)$:

$$u(T,x) = \inf_{\xi} E^x \left[\int_0^T L(X_t^\xi, \xi_t)\, dt + u_0(X_T^\xi) \right] \qquad (\phi_0 \equiv 0)$$

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- $\xi = (\xi_t)_{t \geq 0}$: (admissible) control
- $X_t^\xi = X_0 - \int_0^t \xi_s\, ds + W_t$: controlled process

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Intuitively, $\xi_t := D_pH(X_t, D\phi(X_t))$ is the optimal control for

$$\phi(x) - \lambda T = \inf_{\xi} E^x\left[\int_0^T L(X_t^{\xi}, \xi_t)\,dt + \phi(X_T^{\xi})\right]$$

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Remark. X is ergodic $\iff$ $\exists!\,\mu(dy)$ with $\mu(\mathbb{R}^N) = 1$ s.t.

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(b) Let μ be the invariant probability measure for X. Then,

$$\int_{\mathbb{R}^N} e^{\kappa_1(\phi-\phi_0)(y)}\mu(dy) < \infty.$$

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$v(T,x) := e^{\kappa_1 w(T,x)}$ solves linear equation

$$\begin{cases} \partial_t v - \dfrac{1}{2}\Delta v + \kappa_1 D\phi(x)\cdot Dv = 0 & \text{in } (0,\infty)\times\mathbb{R}^N, \\ v(0,\,\cdot\,) = e^{\kappa_1(\phi - u_0)} & \text{in } \mathbb{R}^N. \end{cases}$$

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Probabilistic representation of $v(T, x)$:

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Remark. $E^x[f(X_T)] \underset{T\to\infty}{\longrightarrow} \int_{\mathbb{R}^N} f(y)\mu(dy)$ for all $f \in L^1(\mu)$.

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(2) $\Omega(u_0) = \{a\}$ for some $a \in \mathbb{R}$.

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In view of condition $\kappa_1 I \le D^2_{pp}H \le \kappa_2 I$, we have

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$|Du(T,\cdot)|$ can be estimated by $|u(T,\cdot)|$. (equi-continuity)

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$$\int_{\mathbb{R}^N} (e^{-\kappa_1 w_\infty(y)} - \inf_{x\in\mathbb{R}^N} e^{-\kappa_1 w_\infty(x)}) \mu(dy) \le 0 \qquad (\Longrightarrow\ w_\infty = \text{const.})$$

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Changing the role of a and b, we also have $a \ge b$, so that $a = b$.

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