Vector fields on differentiable schemes and derivations on differentiable rings

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1 Introduction

Let M, N be C^∞-manifolds and $f : N \to M$ a C^∞-map. Write an $\mathbb{R}$-algebra $C^\infty(M)$ as a set of C^∞-functions on M, and a homomorphism $f^* : C^\infty(M) \ni h \mapsto h \circ f \in C^\infty(N)$.

We can regard vector fields $V : N \to TM$ along f as an $\mathbb{R}$-**derivation** $V : C^\infty(M) \to C^\infty(N)$ by f^* i.e. V is an $\mathbb{R}$-linear map such that

$$V(h_1h_2) = f^*(h_1) \cdot V(h_2) + f^*(h_2) \cdot V(h_1) \text{ for any } h_1, h_2 \in C^\infty(M).$$

Note that in this case, V turns to be a C^∞-**derivation**, i.e. V satisfies that

$$V\big(g \circ (h_1, \dots, h_l)\big) = \sum_{i=1}^{l} f^*\Big(\frac{\partial g}{\partial x_i} \circ (h_1, \dots, h_l)\Big) \cdot V(h_i)$$
$$\text{for any } l \in \mathbb{N},\ h_1, \dots, h_l \in C^\infty(M), \text{and } g \in C^\infty(\mathbb{R}^l).$$

$C^\infty(M)$ is a kind of "C^∞-**ring**" with the property: for any $l \in \mathbb{N}$ and $g \in C^\infty(\mathbb{R}^l)$, there exists an operation $\Phi_f : C^\infty(M)^l \ni (h_1, \dots, h_l) \mapsto g \circ (h_1, \dots, h_l) \in C^\infty(M)$. For a C^∞-ring $\mathfrak{C}, \mathfrak{D}$ and a homomorphism $\phi : \mathfrak{C} \to \mathfrak{D}$, when does an $\mathbb{R}$-derivation $v : \mathfrak{C} \to \mathfrak{D}$ over ϕ become a C^∞-derivation?

1.1 Motivations for manifolds and C^∞-rings

C^∞-ringed spaces are sheaves with C^∞-rings. There exists a functor Spec : $\mathbf{C^\infty Rings}^{\mathrm{op}} \to \mathbf{LC^\infty RS}$ such that C^∞-manifolds are regarded as "C^∞-**schemes**" $M = Spec\big(C^\infty(M)\big)$. We can regard a C^∞-manifold M as a "space" associated with $C^\infty(M)$ and a vector field over M as a derivation $C^\infty(M) \to C^\infty(M)$ by the functor *Spec*.

Then, what should we regard as a vector field on C^∞-scheme? To define and study of singular points and vector fields on C^∞-schemes, we study properties of derivations $V : \mathfrak{C} \to \mathfrak{C}$ of C^∞-rings.

2 Differentiable rings and their derivations

2.1 The definition of C^∞-rings

We define C^∞-ring with the following definition.

Definition 2.1 (E. J. Dubuc, c.f. D. Joyce) 1. A C^∞-**ring** (**differentiable ring**) is a set $\mathfrak{C}$ which satisfies that: for any $l \in \{0\} \cup \mathbb{N}$ and any C^∞-map $f : \mathbb{R}^l \to \mathbb{R}$, there exists an operation $\Phi_f : \mathfrak{C}^l \to \mathfrak{C}$ such that

- for any $k \in \{0\} \cup \mathbb{N}$, any C^∞-maps $g : \mathbb{R}^k \to \mathbb{R}$ and $f_i : \mathbb{R}^l \to \mathbb{R} (i = 1, \cdots, k)$,

$$\Phi_g(\Phi_{f_1}(c_1, \dots, c_l), \dots, \Phi_{f_k}(c_1, \dots, c_l)) = \Phi_{g\circ(f_1,\dots,f_k)}(c_1, \dots, c_l) \text{for any } c_1, \cdots, c_l \in \mathfrak{C}.$$

- for all projections $\pi_i(x_1, \dots, x_l) = x_i (i = 1, \cdots, l)$, $\Phi_{\pi_i}(c_1, \dots, c_l) = c_i$ for any $c_1, \cdots, c_l \in \mathfrak{C}$.

2. Let $\mathfrak{C}$ and $\mathfrak{D}$ be C^∞-rings. A morphism between C^∞-rings is a map $\phi : \mathfrak{C} \to \mathfrak{D}$ such that $\phi\big(\Phi_f(c_1, \ldots, c_n)\big) = \Psi_f\big(\phi(c_1), \ldots, \phi(c_n)\big)$.
3. We will write $\mathbf{C^\infty Rings}$ for the category of C^∞-rings.

Any C^∞-ring $\mathfrak{C}$ has a structure of the commutative $\mathbb{R}$-algebra. Define addtion on $\mathfrak{C}$ by $c + c' := \Phi_{(x,y)\mapsto x+y}(c, c')$. Define multiplication on $\mathfrak{C}$ by $c \cdot c' := \Phi_{(x,y)\mapsto xy}(c, c')$. Define scalar multiplication by $\lambda \in \mathbb{R}$ by $\lambda c := \Phi_{x\mapsto \lambda x}(c)$. Define elements 0 and 1 in $\mathfrak{C}$ by $0_{\mathfrak{C}} := \Phi_{\emptyset\mapsto 0}(\emptyset)$ and $1_{\mathfrak{C}} := \Phi_{\emptyset\mapsto 1}(\emptyset)$.

Example 2.1 1. Suppose that M is a C^∞-manifold.

(a) The set $C^\infty(M)$ has a structure of C^∞-ring by $(c_1, \ldots, c_n) \mapsto f \circ (c_1, \ldots, c_n)$.

(b) Let $I \subset C^\infty(M)$ be an ideal of an $\mathbb{R}$-algebra. We can define a quotient $\mathbb{R}$-algebra $C^\infty(M)/I$.
For any natural number $l \in \mathbb{N}$ and a C^∞-map $f \in C^\infty(\mathbb{R}^l)$,
$f(x_1 + y_1, \ldots, x_l + y_l) - f(x_1, \ldots, x_l) = \sum_{i=1}^{l} y_i g_i(x, y)$ by Hadamard's lemma.
Then $f \circ (c_1 + i_1, \ldots, c_n + i_n) - f \circ (c_1, \ldots, c_n) = \sum_{k=1}^{n} i_k \cdot g_k \circ (c_1, \ldots, c_n, i_1, \ldots, i_n)$
for any $c_1, \ldots, c_n \in \mathfrak{C}$ and $i_1, \ldots, i_n \in I$. Therefore the $\mathbb{R}$-algebra $C^\infty(M)/I$ has a structure of C^∞-ring.

(c) The set $C^\infty_p(M)/{m_p}^{k+1}$ of k-jet functions on a point $p \in M$ has a structure of C^∞-ring.

2. The set of real numbers $\mathbb{R}$ has a structure of C^∞-ring by $(r_1, \ldots, r_n) \mapsto f(r_1, \ldots, r_n)$.

We define two derivations on C^∞-rings as followings.

Definition 2.2 (R. Hartshorne, D. Joyce) Let $\mathfrak{C}$ be a C^∞-ring and $\mathfrak{M}$ be a $\mathfrak{C}$-module.

1. An $\mathbb{R}$**-derivation** is an $\mathbb{R}$-linear map $d : \mathfrak{C} \to \mathfrak{M}$ such that

$$d(c_1 c_2) = c_2 \cdot d(c_1) + c_1 \cdot d(c_2) \text{ for any } c_1, c_2 \in \mathfrak{C}.$$

2. A C^∞**-derivation** is an $\mathbb{R}$-linear map $d : \mathfrak{C} \to \mathfrak{M}$ such that

$$d\big(\Phi_f(c_1, \ldots, c_n)\big) = \sum_{i=1}^{n} \big(\Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n)\big) \cdot d(c_i) \text{ for any } n \in \mathbb{N}, f \in C^\infty(\mathbb{R}^n) \text{ and } c_1, \ldots, c_n \in \mathfrak{C}.$$

By definition, we have that any C^∞-derivation is an $\mathbb{R}$-derivation.

Example 2.2 Let M be a C^∞-manifold and $C^\infty(T^*M)$ the set of C^∞-sections to the cotangent bundle T^*M on M.

1. Define $\mathbb{R}$-mapping $d : C^\infty(M) \to C^\infty(T^*M)$ as $\big(d(f)\big)(x) : T_x M \ni v \mapsto v(f) \in \mathbb{R}$ for any $f \in C^\infty(M)$ and $x \in M$. This $\mathbb{R}$-mapping d is the C^∞-derivation.
2. Let $V : M \to TM$ be a C^∞-vector field of M. Define a smooth function $V(f)$ as $V(f) : M \ni x \mapsto V_x(f) \in \mathbb{R}$. We can regard $V : C^\infty(M) \to C^\infty(M)$ as the $\mathbb{R}$-derivation.

2.2 k-jet projections of C^∞-ring

Definition 2.3 (D. Joyce) Let $\mathfrak{C}$ be a C^∞-ring.

1. An $\mathbb{R}$-point of $\mathfrak{C}$ is a homomorphism $p : \mathfrak{C} \to \mathbb{R}$ of C^∞-rings.
The set of $\mathbb{R}$-points $p : \mathfrak{C} \to \mathbb{R}$ is a base space of the C^∞-scheme $Spec\mathfrak{C}$.
2. For any $\mathbb{R}$-point $p : \mathfrak{C} \to \mathbb{R}$, the localization $\mathfrak{C}_p := \mathfrak{C}[s^{-1} | s \in \mathfrak{C}, p(s) \neq 0]$ by $\{s \in \mathfrak{C} | p(s) \neq 0\}$ always exists with the unique maximal ideal $m_p \subset \mathfrak{C}_p (\mathfrak{C}_p/m_p \tilde{=} \mathbb{R})$.
3. For any nonnegative number $k \in \{0\} \cup \mathbb{N}$, define natural projections as

$$j^k_p : \mathfrak{C} \to \mathfrak{C}_p/{m_p}^{k+1},\ j^\infty_p : \mathfrak{C} \to \mathfrak{C}_p/{m_p}^\infty ({m_p}^\infty := \cap_{k\in\mathbb{N}} {m_p}^k),$$
$$j^k := (j^k_p)_{p:\mathfrak{C}\to\mathbb{R}} : \mathfrak{C} \to \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/m_p^{k+1},\ j^\infty := (j^\infty_p)_{p:\mathfrak{C}\to\mathbb{R}} : \mathfrak{C} \to \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/m_p^\infty.$$

Example 2.3 Let M be a C^∞-manifold and $p \in M$. For the $\mathbb{R}$-point $e_p : C^\infty(M) \ni f \mapsto f(p) \in \mathbb{R}$, a localization $\big(C^\infty(M)\big)_{e_p}$ is isomorphic to the set $C_p^\infty(M)$ of germs of C^∞-functions at p. Its unique maximal ideal is $m_{e_p} = \{[f,U]_p \in C_p^\infty(M) | f(p) = 0\}$.

2.3 k-jet determined C^∞-rings

Definition 2.4（1,I. Moerdijk and G.E. Reyes, 2,3,Yamashita） Let $\mathfrak{C}$ be a C^∞-ring.

1. $\mathfrak{C}$ is **point determined** if for each $c \in \mathfrak{C}$, $c = 0$ if and only if $p(c) = 0$ for all $\mathbb{R}$-point $p : \mathfrak{C} \to \mathbb{R}$.
2. Let $k \in \mathbb{N}$. $\mathfrak{C}$ is **k-jet determined** if $j^k : \mathfrak{C} \to \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/m_p^{k+1}$ is injective.
3. $\mathfrak{C}$ is **∞-jet determined** if $j^\infty : \mathfrak{C} \to \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/m_p^\infty$ is injective.

Example 2.4 Suppose that M is a C^∞-manifiold.

1. $C^\infty(M)$ is a point determined C^∞-ring.
2. $C_p^\infty(M)/{m_p}^{k+1}$ is not a point determined C^∞-ring, but a k-jet determined C^∞-ring.

For two C^∞-rings $\mathfrak{C}$ and $\mathfrak{D}$ with operations $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ and $\Psi_f : \mathfrak{D}^n \to \mathfrak{D}$ for $f \in C^\infty(\mathbb{R}^n)$, we can define a direct product $\mathfrak{C} \times \mathfrak{D}$. This product has a structure of C^∞-ring by $\Xi_f : (\mathfrak{C} \times \mathfrak{D})^n \to \mathfrak{C} \times \mathfrak{D}$ as

$$\Xi_f : (\mathfrak{C} \times \mathfrak{D})^n \ni \big((c_1,d_1),\dots,(c_n,d_n)\big) \mapsto \big(\Phi_f(c_1,\dots,c_n), \Psi_f(d_1,\dots,d_n)\big) \in \mathfrak{C} \times \mathfrak{D}.$$

For direct product of k-jet determined C^∞-rings, we have a following lemma.

Lemma 2.1（Yamashita） Let $\mathfrak{C}$ and $\mathfrak{D}$ be k,l-jet determined C^∞-rings and $k' := \min(k,l)$.
The direct product $\mathfrak{C} \times \mathfrak{D}$ is a k'-jet determined C^∞-ring.

Example 2.5 Let M and M' be m-dimensional C^∞-manifolds. Write $M \sqcup M'$ as a disjoint union of C^∞-manifolds M and M'. $C^\infty(M)$ and $C^\infty(M')$ are point determined C^∞-rings. Furthermore, $C^\infty(M) \times C^\infty(M') = C^\infty(M \sqcup M')$ is a point determined C^∞-ring, too.

Proposition 2.1（Yamashita） Let $\mathfrak{C}$ be a C^∞-ring and $k,l = \{0\} \cup \mathbb{N} \cup \{\infty\} (k \le l)$.
If $\mathfrak{C}$ is a k-jet determined C^∞-ring, then $\mathfrak{C}$ is also l-jet determined.

3 Algebraic viewpoints

3.1 The universality of cotangent bundles

Proposition 3.1（Yamashita） Let $\mathfrak{C}$ be a C^∞-ring and $\mathfrak{F}_\mathfrak{C}$ a free $\mathfrak{C}$-module generated by $d(c) (c \in \mathfrak{C})$.
Define $\mathfrak{C}$-submodules of $\mathfrak{F}_\mathfrak{C}$ as

$$\mathfrak{M}_{\mathfrak{C},\mathbb{R}} := \big\langle d(c_1c_2) - c_2 d(c_1) + c_1 d(c_2) \big\rangle_\mathfrak{C} \text{ and}$$
$$\mathfrak{M}_{\mathfrak{C},C^\infty} := \big\langle d\big(\Phi_f(c_1,\dots,c_n)\big) - \sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(c_1,\dots,c_n) d(c_i) \big\rangle_\mathfrak{C}.$$

If $\mathfrak{M}_{\mathfrak{C},\mathbb{R}} = \mathfrak{M}_{\mathfrak{C},C^\infty}$, any $\mathbb{R}$-derivation $d : \mathfrak{C} \to \mathfrak{M}$ is C^∞-derivation.

Example 3.1 Let $\mathfrak{C}$ $\mathfrak{D}$ be C^∞-rings and $\phi : \mathfrak{C} \to \mathfrak{D}$ a homomorphism of C^∞-rings. Suppose that $\mathfrak{C}$ is a local C^∞-ring which has a maximal ideal m with $m^{k+1} = 0 (k \in \{0\} \cup \mathbb{N})$.
$\mathfrak{C}$ has a property that $\mathfrak{M}_{\mathfrak{C},\mathbb{R}} = \mathfrak{M}_{\mathfrak{C},C^\infty}$ because $\Phi_f(c_1,\dots,c_n)$ is the sum of $\Phi_{\frac{\partial^\alpha f}{\partial x^\alpha}}(c_1,\dots,c_n)$.
Therefore, any $\mathbb{R}$-derivation $V : \mathfrak{C} \to \mathfrak{D}$ is a C^∞-derivation.

3.2 The relation between k-jet determined C^∞-rings and derivations

Theorem 3.1 (Yamashita) Let $\mathfrak{C}, \mathfrak{D}$ be C^∞-rings, $\phi : \mathfrak{C} \to \mathfrak{D}$ a homomorphism of C^∞-rings and $k \in \mathbb{N} \cup \{\infty\}$. Suppose that $\mathfrak{D}$ is point determined or k-jet determined.

Then any $\mathbb{R}$-derivation $V : \mathfrak{C} \to \mathfrak{D}$ over ϕ is a C^∞-derivation.

Example 3.2

1. Let V be an $\mathbb{R}$-derivation $V : C^\infty(M) \to C^\infty(N)$ over the pull-back $f^* : C^\infty(M) \to C^\infty(N)$. $C^\infty(N)$ is a point determined C^∞-ring. From the previous theorem, this $\mathbb{R}$-derivation is a C^∞-derivation.
2. $C^\infty(\mathbb{R})/\langle x^{k+1}\rangle_{C^\infty(\mathbb{R})}$ is not point determined but k-jet determined C^∞-ring.
   Any $\mathbb{R}$-derivation $V : C^\infty(\mathbb{R})/\langle x^{k+1}\rangle_{C^\infty(\mathbb{R})} \to C^\infty(\mathbb{R})/\langle x^{k+1}\rangle_{C^\infty(\mathbb{R})}$ is C^∞-derivation such that
   $V(f(x) + \langle x^{k+1}\rangle) = \frac{\partial f}{\partial x}(x)v(x) + \langle x^{k+1}\rangle$ by $v(x) + \langle x^{k+1}\rangle := V(x + \langle x^{k+1}\rangle)$.

For the previous example , we have a following corollary by generalizing $C^\infty(\mathbb{R})/\langle x^{k+1}\rangle_{C^\infty(\mathbb{R})}$.

Corollary 3.1 (Yamashita) Let $\mathfrak{C}$ be a k-jet determined C^∞-ring with the form $C^\infty(\mathbb{R}^n)/I$.

For any $\mathbb{R}$-derivation $V : \mathfrak{C} \to \mathfrak{C}$, V is a C^∞-derivation.

Moreover, there exists smooth functions $a_i(x) \in C^\infty(\mathbb{R}^n)$ such that

$$V(f(x) + I) = \sum_{i=1}^{n} a_i(x)\frac{\partial f}{\partial x_i}(x) + I \text{ for any } f(x) + I \in C^\infty(\mathbb{R}^n)/I.$$

4 Applications

Let $\mathfrak{C}$ be a C^∞-ring and $\phi : \mathfrak{C} \to C^\infty(\mathbb{R})$ a homomorphism of C^∞-rings. This homomorphism is regarded as a C^∞-curve $\mathbb{R} \to Spec\mathfrak{C}$.

Suppose that $V : \mathfrak{C} \to C^\infty(\mathbb{R})$ is an $\mathbb{R}$-derivation over ϕ. For the previous theorem, this derivation V is a C^∞-derivation. Furthermore, C^∞-derivation V is regarded as a tangent vector at $Spec\mathfrak{C}$.

For any element $c' \in \mathfrak{C}$, we can define a homomorphism $\psi : \mathfrak{C} \ni c \mapsto \Phi_{\phi(c)}(c') \in \mathfrak{C}$ of C^∞-rings, and a C^∞-derivation $V' : \mathfrak{C} \ni c \mapsto \Phi_{V(c)}(c') \in \mathfrak{C}$ over ψ.

4.1 Applications to C^∞-vector field along C^∞-map

Let $\mathfrak{C}$ be a C^∞-ring ,M a C^∞-manifold and $\phi : \mathfrak{C} \to C^\infty(M)$ a homomorphism of C^∞-rings.

Suppose that $V : \mathfrak{C} \to C^\infty(M)$ is an $\mathbb{R}$-derivation by ϕ. For the previous theorem, this derivation V is a C^∞-derivation.

Therefore, we can define a vector field $V : M \to Spec\mathfrak{C}$ over $Spec\phi : M \to Spec\mathfrak{C}$ as the image of derivation $\mathfrak{C} \to C^\infty(M)$ by the functor $Spec$.

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