RIMS workshop
Potential Theory and its Related Fields

Program and Abstracts

RIMS workshop
Potential Theory and its Related Fields

Dates: September 3 – 7, 2012
Venue: Research Building No. 8 Lecture Room 2,
Faculty of Engineering, Kyoto University
Organizers: Kentaro Hirata (Akita, Chair), Hiroaki Aikawa (Sapporo),
Jun Kigami (Kyoto), Masaharu Nishio (Osaka)

Program

Monday, September 3

10:00 – 10:15 Opening

10:15 – 11:15 **John Lewis**
p harmonic measure in simply connected domains revisited

11:30 – 12:30 **Atsushi Kasue**
Quasi-monomorphisms and p-harmonic functons with finite Dirichlet sum

14:00 – 15:00 **Nageswari Shanmugalingam**
Constructing a prime end boundary for non-simply connected domains in Euclidean spaces and metric measure spaces

15:15 – 15:45 **Vadim Kaimanovich**
Electrical network reduction and the finite Dirichlet problem

15:55 – 16:25 **Hiroaki Masaoka**
On harmonic Hardy-Orlicz spaces

16:40 – 17:10 **Ryozi Sakai**
A characterization of entire functions and approximation

17:20 – 17:50 **Yûsuke Okuyama**
Equilibrium measures for uniformly quasiregular dynamics

Tuesday, September 4

9:15 – 10:15 **Masanori Hino**
Geodesic distances and intrinsic distances on some fractal sets

10:30 – 11:30 **Laurent Saloff-Coste**
Heat kernel estimates on inner uniform domains

11:45 – 12:45 **Kazumasa Kuwada**
Applications of Hopf-Lax formulae to analysis of heat distributions

14:00 – 15:00 **Anders Björn**
The Perron method for p-harmonic functions: Resolutivity and invariance results

15:15 – 15:45 **Tsubasa Itoh**
Modulus of continuity of p-Dirichlet solutions in a metric measure space

15:55 – 16:25 **Yoshihiro Mizuta**
Sobolev's inequality for Riesz potentials in Lorentz spaces of variable exponent

16:40 – 17:10 **Tanran Zhang**
A potential theoretic approach to the curvature equation

17:20 – 17:50 **Sachiko Hamano**
Variation for the metrics induced by Schiffer and harmonic spans

Wednesday, September 5

9:15 – 10:15 **Eleutherius Symeonidis**
A concept of harmonicity for families of planar curves

10:30 – 11:30 **Tomas Sjödin**
Two-phase quadrature domains and harmonic balls

11:45 – Excursion

Thursday, September 6

9:15 – 10:15 **John Mackay**
The quasisymmetric geometry of boundaries of relatively hyperbolic groups

10:30 – 11:30 **Bruce Kleiner**
Asymptotic geometry, harmonic functions, and finite generation of isometry groups

11:45 – 12:45 **Eero Saksman**
Rotation of planar quasiconformal maps

14:00 – 15:00 **Mario Bonk**
Non-linear potential theory and the Rickman-Picard theorem

15:15 – 15:45 **Naotaka Kajino**
Weyl's Laplacian eigenvalue asymptotics for the measurable Riemannian structure on the Sierpiński gasket

15:55 – 16:25 **Tetsu Shimomura**
Hardy averaging operator on generalized Banach function spaces

16:40 – 17:10 **Kiyoki Tanaka**
A representation for harmonic Bergman function and its application

17:20 – 17:50 **Fumi-Yuki Maeda**
Mean continuity for potentials of functions in Musielak-Orlicz spaces

18:30 – Dinner

Friday, September 7

9:15 – 10:15 **Jeremy Tyson**
Distortion of dimension by projections and Sobolev mappings

10:30 – 11:30 **Yoshihiro Sawano**
Morrey spaces and fractional integral operators

11:45 – 12:45 **Thomas Ransford**
Computation of capacities

14:00 – 15:00 **Tom Carroll**
Isoperimetric inequalities for a Sobolev Constant

15:15 – 15:45 **Minoru Yanagishita**
The first boundary value problem of the biharmonic equation for the half-space

15:55 – 16:25 **Hiroaki Aikawa**
Extended Harnack inequalities with exceptional sets and a boundary Harnack principle

16:35 – 17:05 **Kentaro Hirata**
Heat kernel estimates and growth estimates of solutions of semilinear heat equations

17:10 – 17:20 Closing

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Abstracts

Extended Harnack inequalities with exceptional sets and a boundary Harnack principle

Hiroaki Aikawa (Hokkaido University)

The Harnack inequality is one of the most fundamental inequalities for positive harmonic functions and, it is extended for positive solutions to general elliptic equations and parabolic equations. This talk gives a different view point of generalization. We generalize Harnack chains rather than equations. More precisely, we allow a small exceptional set; and yet we obtain a similar Harnack inequality. The size of an exceptional set is measured by capacity. Our extended Harnack inequality includes information for the boundary behavior of positive harmonic functions. It yields a boundary Harnack principle for a very nasty domain whose boundary is given locally by the graph of a function with modulus of continuity worse than Hölder continuity.

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The Perron method for p-harmonic functions: Resolutivity and invariance results

Anders Björn (Linköpings University)

In the Dirichlet problem one looks for a p-harmonic function u on some domain $\Omega \subset \mathbf{R}^n$ which takes prescribed boundary values f. A *p-harmonic function* u is a continuous weak solution of the equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0.$$

(And thus for $p = 2$ we obtain the usual harmonic functions.) Here $1 < p < \infty$ is fixed.

If f is not continuous, then there usually is no p-harmonic function u which takes the boundary values as limits (i.e. such that $\lim_{y\to x} u(y) = f(x)$ for all $x \in \partial\Omega$), and even for continuous f this is not always possible. One therefore needs some other precise definition of what is a *solution* to the Dirichlet problem. For p-harmonic functions there are at least 4 different definitions, of which the *Perron method* is the most general.

For any boundary function $f : \partial\Omega \to [-\infty, \infty]$, the Perron method produces an upper and a lower Perron solution. When these coincide it gives a reasonable solution to the Dirichlet problem, called the *Perron solution* Pf, and f is said to be *resolutive*.

In 2003 Björn–Björn–Shanmugalingam showed the following invariance result: If $f \in C(\partial\Omega)$ and $h = f$ outside a set of p-capacity zero, then h is resolutive and $Ph = Pf$.

We will look at recent improvements of this result. Some of these will be related to the prime end boundary, in the sense of the recent definition of prime ends introduced by Adamowicz–Björn–Björn–Shanmugalingam. Note that for our results we *cannot* use Carathéodory's classical definition, not even in simply connected planar domain. Prime ends will only be mentioned briefly in this talk, see however the talk by Nageswari Shanmugalingam on this topic.

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Non-linear potential theory and the Rickman-Picard theorem

Mario Bonk (University of California, Los Angeles)

According to the Rickman-Picard theorem a non-constant K-quasiregular map from $\mathbf{R}^n$ to the n-sphere can only omit finitely many values, where the maximal number of omitted values is bounded above by a constant only depending on n and K. In my talk I will present a new potential-theoretic method to establish this result. In contrast to earlier potential-theoretic proofs, notably by Eremenko-Lewis and Lewis, the approach is rather elementary and works from first principles. For example, Harnack inequalities for the relevant functions are not needed, but instead the proof relies on Caccioppoli inequalities which are much easier to establish.

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Isoperimetric inequalities for a Sobolev constant

Tom Carroll (University College Cork)

The principal frequency and the torsional rigidity of a bounded region in Euclidean space $\mathbb{R}^n$ may both be expressed in terms of Rayleigh quotients. The principal frequency $\lambda(D)$ of a bounded region D is the smallest eigenvalue of the Dirichlet Laplacian $-\Delta$. It is the lowest tone that a drum with shape D can make, in the case of a planar region D. This eigenvalue is positive and the corresponding eigenfunctions have constant sign. The torsional rigidity $P(D)$ of a bounded, simply connected region D in the plane is a measure of the strength under torsion of a beam which has D as its cross section. It is computed as $P(D) = 2\int_D \varphi(x)\,dx$ from the Prandtl stress function (or torsion function) φ, whose partial derivatives give the stresses in the beam under torsion. The torsion function is a solution of $\Delta\varphi = -2$ in D with zero Dirichlet data. The solution of this p.d.e. in a region D in $\mathbb{R}^n$ has a probabilistic interpretation as the expected exit time of Brownian motion from the region.

The Rayleigh quotient expressions for the eigenvalue and the torsional rigidity are

$$\lambda(D) = \inf\left\{ \frac{\int_D |\nabla u(x)|^2\,dx}{\int_D u^2(x)\,dx} : u \in C_0^\infty(D) \right\}$$

and

$$P(D) = 4\,\sup\left\{ \frac{\left(\int_D u(x)\,dx\right)^2}{\int_D |\nabla u(x)|^2\,dx} : u \in C_0^\infty(D) \right\}.$$

The fundamental frequency and the torsional rigidity can be embedded in a range of parameters associated with a region by setting, for each $p \geq 1$,

$$\mathcal{C}_p(D) = \inf\left\{ \frac{\int_D |\nabla u(x)|^2\,dx}{\left(\int_D u(x)^p\,dx\right)^{2/p}} : u \in L^p(D) \cap W_0^{1,2}(D), u \geq 0, u \not\equiv 0 \right\}.$$

Thus

$$\frac{4}{P(D)} = \mathcal{C}_1(D) \quad \text{and} \quad \lambda(D) = \mathcal{C}_2(D).$$

From another perspective, $\mathcal{C}_p(D)$ gives the sharp constant in the Sobolev embedding: if $n = 2$ and $p \geq 1$, or if $n \geq 3$ and $1 \leq p \leq 2n/(n-2)$, then

$$W_0^{1,2}(D) \subset L^p(D), \qquad \|u\|_{L^p(D)} \leq \mathcal{S}_p \|\nabla u\|_{L^2(D)} \quad \forall\, u \in W_0^{1,2}(D),$$

so that

$$\mathcal{S}_p(D) = \frac{1}{\sqrt{\mathcal{C}_p(D)}}.$$

There are been interest, of late, in extending classical results for the eigenvalue and the torsional rigidity to the Sobolev constant $\mathcal{C}_p$. In this talk, I will describe some of these results, including joint work with Jesse Ratzkin, University of Cape Town.

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Variation for the metrics induced by Schiffer and harmonic spans

Sachiko Hamano (Fukushima University)

Let D be a domain in $\mathbb{C}_z$ bounded by C^ω smooth contours $C_1, \ldots, C_\nu$. For a fixed point $\zeta \in D$, let $\mathcal{P}(D)$ be the set of all univalent functions P on D such that

$$P(z,\zeta) = (z-\zeta)^{-1} + 0 + \sum_{n=1}^{\infty} A_n (z-\zeta)^n \qquad \text{at } z = \zeta.$$

Especially, let $P_1(z,\zeta)$ be the vertical slit mapping and $P_0(z,\zeta)$ be the horizontal slit mapping for (D,ζ). The L_i-principal function $p_i(z,\zeta) := \operatorname{Re} P_i$ $(i=1,0)$ for (D,ζ) is harmonic on $D \setminus \{\zeta\}$, has the pole $\operatorname{Re} \frac{1}{z-\zeta}$ at $z=\zeta$, and satisfies L_i-condition on the boundary: for $j = 1, \ldots, \nu$,

$$(L_1) \quad p_1(z,\zeta) = c_j \text{ (constant) on } C_j \quad \text{and} \quad \int_{C_j} \frac{dp_1(z,\zeta)}{dn_z} ds_z = 0;$$

$$(L_0) \quad \frac{dp_0(z,\zeta)}{dn_z} = 0 \quad \text{on } C_j.$$

The Schiffer span $s(\zeta)$ for (D,ζ) is the difference of L_i-constants $\alpha_i(\zeta) := \operatorname{Re} A_1^i$ $(i=1,0)$, exactly, $s := \alpha_0 - \alpha_1$ (> 0) (see [5]).

Proposition 1. *Let $s(\zeta)$ be the Schiffer span for (D,ζ). For any holomorphic mapping $w = f(z)$ on D, it holds that $s(f(\zeta)) = |f'(\zeta)|^{-2} s(\zeta)$.*

Thus the Schiffer span $s(\zeta)$ induces the metric $s(\zeta)|d\zeta|^2$ on D.

Under the same condition as the above D, we assume that $D \ni 0$. For an arbitrarily fixed $\zeta \in D$, let $\mathcal{Q}(D)$ be the set of all univalent functions Q on D such that

$$Q(z,\zeta) = \begin{cases} z^{-1} + \sum_{n=0}^{\infty} b_n z^n & \text{at } z = 0, \\ \sum_{n=1}^{\infty} B_n (z-\zeta)^n & \text{at } z = \zeta. \end{cases}$$

Especially, let $Q_1(z,\zeta)$ be the circular slit mapping and $Q_0(z,\zeta)$ be the radial slit mapping for $(D,0,\zeta)$. The L_i-principal function $q_i(z,\zeta) := \log|Q_i|$ $(i=1,0)$ for $(D,0,\zeta)$ is harmonic on $D \setminus \{0,\zeta\}$, has the logarithmic poles $-\log|z|$ at $z=0$ and $\log|z-\zeta|$ at $z=\zeta$, and satisfies L_i-condition on each boundary component C_j $(j=1,\ldots,\nu)$. The harmonic span $h(\zeta)$ for $(D,0,\zeta)$ is the difference of L_i-constants $\beta_i(\zeta) := \log|\frac{dQ_i}{dz}(\zeta,\zeta)|$ $(i=1,0)$, exactly, $h := \beta_1 - \beta_0$.

Proposition 2. *Let $h(\zeta)$ be the harmonic span for $(D,0,\zeta)$. For any holomorphic mapping $w = f(z)$ on D, it holds that $s(f(\zeta)) = s(\zeta)$.*

From the geometrical meaning of the harmonic span and from the representation of some reproducing kernel, we see that the harmonic span $h(\zeta)$ induces the metric

$$\mathfrak{h}(\zeta)|d\zeta|^2 := \frac{\partial^2 h(\zeta)}{\partial\zeta\partial\bar{\zeta}} |d\zeta|^2 \quad \text{on } D.$$

Theorem 3. *Let the notation be as above.*

(i) *The metric $s(\zeta)|d\zeta|^2$ is identical with $\mathfrak{h}(\zeta)|d\zeta|^2$ on D.*

(ii) *The metrics $s(\zeta)|d\zeta|^2$ and $\mathfrak{h}(\zeta)|d\zeta|^2$ are of negative curvature at each point in D.*

When $D = \{|z| < 1\}$, we computed in [4] the Schiffer span $s(\zeta)$ for (D, ζ): $s(\zeta) = \frac{2}{(1-|\zeta|^2)^2}$, and in [2] the harmonic span $h(\zeta)$ for $(D, 0, \zeta)$: $h(\zeta) = -2\log(1 - |\zeta|^2)$. Thus we exactly have $\mathfrak{h}(\zeta) := \frac{\partial^2 h(\zeta)}{\partial\zeta\partial\bar{\zeta}} = s(\zeta)$.

Here we shall introduce complex parameter $t \in B := \{|t| < \rho\} \subset \mathbb{C}_t$. We consider a variation of domains $\mathcal{D} : t \in B \to D(t) \subset \mathbb{C}_z$, and identify the variation $\mathcal{D}$ with the subset $\cup_{t\in B}(t, D(t))$ of $B \times \mathbb{C}_z$. When each $D(t)$, $t \in B$ is a domain bounded by C^ω smooth contours $C_j(t)$ $(j = 1, \ldots, \nu)$ in $\mathbb{C}_z$ and each $C_j(t)$ varies C^ω smoothly with $t \in B$. (See [3] for non-smooth variations.) Assume that $D(t) \ni 0$ for $t \in B$. For any fixed $\zeta \in D(t)$, each $D(t)$ carries the Schiffer span $s(t, \zeta)$ for $(D(t), \zeta)$ and the harmonic span $h(t, \zeta)$ for $(D(t), 0, \zeta)$. Applying the variation formulas for spans ([1], [2], [4]) we see the property of variation of the metrics $s(t, \zeta)|d\zeta|^2$ and $\mathfrak{h}(t, \zeta)|d\zeta|^2$ on $D(t)$.

Theorem 4. *If the total space $\mathcal{D} = \cup_{t\in B}(t, D(t))$ is a 2-dimensional pseudoconvex domain in $B \times \mathbb{C}_z$, then $\log s(t, \zeta)$ and $\log \mathfrak{h}(t, \zeta)$ is plurisubharmonic on $\mathcal{D}$.*

References

[1] S. Hamano, *Variation formulas for L_1-principal functions and the application to simultaneous uniformization problem,* Michigan Math. J. **60** No.2 (2011), 271–288.

[2] S. Hamano, F. Maitani and H. Yamaguchi, *Variation formulas for principal functions (II) Applications to variation for the harmonic spans,* Nagoya Math. J. **204** No.2 (2011), 19–56.

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[5] M. Nakai and L. Sario, *Classification Theory of Riemann Surfaces,* Grundlehren Math. Wiss. **164**, Springer-Verlag, 1970.

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Geodesic distances and intrinsic distances on some fractal sets

Masanori Hino (Kyoto University)

The off-diagonal Gaussian asymptotics of the heat kernel density associated with local Dirichlet form is often described by using the intrinsic distances (or Carnot–Caratheodory distances; cf. [4, 3] and the references therein). When the underlying space has a Riemannian structure, the geodesic distance is defined as well, and it coincides with the intrinsic distance in good situations.

Then, what if the underlying space is a fractal set? In typical examples, the heat kernel asymptotics is *sub*-Gaussian; accordingly, the intrinsic distance vanishes identically. However, if we take (a sum of) energy measures as the underlying measure, we can define the nontrivial intrinsic distance as well as the geodesic distance, and can pose a problem whether they are identical. For the 2-dimensional standard Sierpinski gasket, the affirmative answer has been obtained ([1, 2]) by using some detailed information on the transition density. In this talk, I will discuss this problem in a more general framework and provide some partial answers based on purely analytic arguments.

Setting: Let (K, d_K) be a compact metric space, and λ, a finite Borel measure on K. Let $(\mathcal{E}, \mathcal{F})$ be a strong local regular Dirichlet form on $L^2(K, \lambda)$. For $f \in \mathcal{F}$, $\mu_{\langle f\rangle}$ denotes the energy measure of f. Let $N \in \mathbb{N}$ and $\boldsymbol{h} = (h_1, \ldots, h_N) \in \mathcal{F}^N \cap C(K \to \mathbb{R}^N)$. Denote $\sum_{j=1}^d \mu_{\langle h_j\rangle}$ by $\mu_{\langle \boldsymbol{h}\rangle}$. Then, the *intrinsic distance* based on $(\mathcal{E}, \mathcal{F})$ and $\mu_{\langle \boldsymbol{h}\rangle}$ is defined as

$$\mathsf{d}_{\boldsymbol{h}}(x, y) := \sup\{f(y) - f(x) \mid f \in \mathcal{F} \cap C(K) \text{ and } \mu_{\langle f\rangle} \le \mu_{\langle \boldsymbol{h}\rangle}\}, \quad x, y \in K.$$

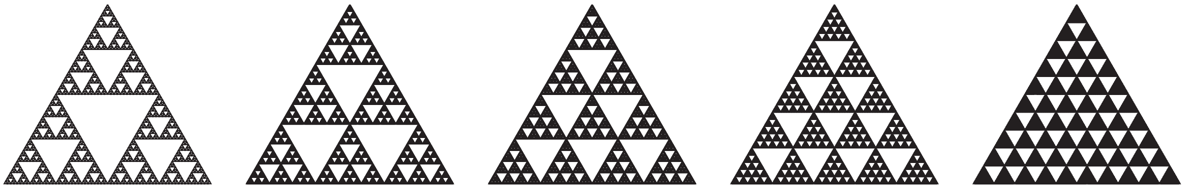

Figure 1: 2-dimensional level l Sierpinski gaskets ($l = 2, 3, 4, 5, 10$)

For a continuous curve $\gamma \in C([0,1] \to K)$, its length based on $\boldsymbol{h}$ is defined as

$$l_{\boldsymbol{h}}(\gamma) := \sup \left\{ \sum_{i=1}^{n} |\boldsymbol{h}(\gamma(t_i)) - \boldsymbol{h}(\gamma(t_{i-1}))|_{\mathbb{R}^N} \,\middle|\, 0 = t_0 < t_1 < \cdots < t_n = 1 \right\}.$$

Then, the *geodesic distance* based on $\boldsymbol{h}$ is defined as

$$\rho_{\boldsymbol{h}}(x,y) := \inf\{l_{\boldsymbol{h}}(\gamma) \mid \gamma \in C([0,1] \to K),\ \gamma(0) = x,\ \text{and}\ \gamma(1) = y\}, \quad x, y \in K.$$

Note that if $\boldsymbol{h} \colon K \to \mathbb{R}^N$ is injective, $\rho_{\boldsymbol{h}}(x,y)$ is equal to the usual geodesic distance between $\boldsymbol{h}(x)$ and $\boldsymbol{h}(y)$ in $\boldsymbol{h}(K) \subset \mathbb{R}^N$.

Results:

1) Suppose further the following:

(A1) (Finitely ramified cell structure) There exists an increasing sequence of finite subsets $\{V_m\}_{m=0}^{\infty}$ of K such that

- (i) $\bigcup_{m=0}^{\infty} V_m$ is dense in K;
- (ii) For each m, $K \setminus V_m$ is decomposed as a finite number of connected components $\{U_\lambda\}_{\lambda \in \Lambda_m}$;
- (iii) $\lim_{m\to\infty} \max_{\lambda \in \Lambda_m} \operatorname{diam}_{d_K} U_\lambda = 0$.

(A2) $\mathcal{F} \subset C(K)$.

(A3) $\mathcal{E}(f,f) = 0$ if and only if f is a constant function.

Then, $\rho_{\boldsymbol{h}}(x,y) \le \mathsf{d}_{\boldsymbol{h}}(x,y)$ for all $x, y \in K$.

2) Consider a 2-dimensional (generalized) Sierpinski gasket (see Figure 1) as K that is also a nested fractal, and take a self-similar Dirichlet form $(\mathcal{E}, \mathcal{F})$ associated with the Brownian motion on K. Suppose also that the harmonic structure associated with it is nondegenerate. Take $\boldsymbol{h} = (h_1, \ldots, h_d)$ such that each h_i is a harmonic function. Then, $\mathsf{d}_{\boldsymbol{h}}(x,y) \le \rho_{\boldsymbol{h}}(x,y)$ for all $x, y \in K$. By combining the result of 1) with this inequlity, $\mathsf{d}_{\boldsymbol{h}}(x,y) = \rho_{\boldsymbol{h}}(x,y)$ holds.

The nondegeneracy condition is verified for level l Sierpinski gaskets with $l \le 50$ by the numerical calculation. The assumptions on K and $(\mathcal{E}, \mathcal{F})$ can be relaxed, which may be explained in the talk.

References

[1] N. Kajino, Heat kernel asymptotics for the measurable Riemannian structure on the Sierpinski gasket, Potential Anal. **36** (2012), 67–115.

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Heat kernel estimates and growth estimates of solutions of semilinear heat equations

Kentaro Hirata (Akita University)

In bounded Lipschitz domains $\Omega \subset \mathbb{R}^n$, we presents a priori estimates near the parabolic boundary of nonnegative solutions of semilinear heat equations

$$\partial_t u - \Delta u = V u^p \quad \text{in } \Omega \times (0, T),$$

where V is a nonnegative locally bounded function satisfying a certain growth condition near the parabolic boundary. This improves an estimate given by Poláčik, Quittner and Souplet [2] when p is not greater than some constant determined by the shape of a domain Ω. Our proof is based on the Riesz decomposition of supertemperatures, two-sided global estimates of heat kernels given in [1] and an iteration argument.

References

[1] K. Hirata, *Upper and lower estimates for parabolic Green functions in Lipschitz domains*, Abstracts in the international workshop on potential theory 2009, http://www.math.sci.hokudai.ac.jp/sympo/iwpt/2009_en.html

[2] P. Poláčik, P. Quittner and P. Souplet, *Singularity and decay estimates in superlinear problems via Liouville-type theorems. II. Parabolic equations*, Indiana Univ. Math. J. **56** (2007), no. 2, 879–908.

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Modulus of continuity of p-Dirichlet solutions in a metric measure space

Tsubasa Itoh (Hokkaido University)

Let $X = (X, d, \mu)$ be a complete connected metric measure space endowed with a metric d and a positive complete Borel measure μ such that $0 < \mu(U) < \infty$ for all non-empty bounded open sets U. Let $1 < p < \infty$. We assume that μ is doubling measure and X admits a $(1, p)$-Poincaré inequality.

For a function f on $\partial\Omega$ we denote by $\mathcal{P}_\Omega f$ the p-Perron solution of f over Ω. A point $\xi \in \partial\Omega$ is said to be a *p-regular point* (with respect to the p-Dirichlet problem) if

$$\lim_{\Omega \ni x \to \xi} \mathcal{P}_\Omega f(x) = f(\xi)$$

for every $f \in C(\partial\Omega)$. If every boundary point is a p-regular point, then Ω is called *p-regular*. It is well known that if Ω is p-regular and $f \in C(\partial\Omega)$, then $\mathcal{P}_\Omega f$ is p-harmonic in Ω and continuous in $\overline{\Omega}$. It is natural to raise the following question:

Question. *Does improved continuity of a boundary function f guarantee improved continuity of $\mathcal{P}_\Omega f$?*

Aikawa and Shanmugalingam [2] studied this question in the context of Hölder continuity. Aikawa [1] investigated this question in the context of general modulus of continuity for the classical setting, i.e., for harmonic functions in a Euclidean domain. The purpose of this talk is to study this question in the context of general modulus of continuity in a metric measure space.

We consider $\psi_{\alpha\beta}$ defined by

$$\psi_{\alpha\beta}(t) = \begin{cases} t^\alpha (-\log t)^{-\beta} & \text{for } 0 < t < t_0, \\ t_0^\alpha (-\log t_0)^{-\beta} & \text{for } t \geq t_0. \end{cases}$$

where either $0 < \alpha < 1$ and $\beta \in \mathbb{R}$ or $\alpha = 0$ and $\beta > 0$; and t_0 is so small that $\psi_{\alpha\beta}$ is concave. We say that f is *$\psi_{\alpha\beta}$-Hölder continuous* if $|f(x) - f(y)| \le C\psi_{\alpha\beta}(d(x, y))$.

Let $E \subset X$. We consider the family $\Lambda_{\psi_{\alpha\beta}}(E)$ of all bounded continuous functions f on E with norm

$$\|f\|_{\psi_{\alpha\beta},E} = \sup_{x\in E} |f(x)| + \sup_{\substack{x,y\in E\\ x\neq y}} \frac{|f(x) - f(y)|}{\psi_{\alpha\beta}(d(x, y))} < \infty.$$

We define the operator norm

$$\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} = \sup_{\substack{f\in\Lambda_{\psi_{\alpha\beta}}(\partial\Omega)\\ \|f\|_{\psi_{\alpha\beta},\partial\Omega}\neq 0}} \frac{\|\mathcal{P}_\Omega f\|_{\psi_{\alpha\beta},\Omega}}{\|f\|_{\psi_{\alpha\beta},\partial\Omega}}.$$

Observe that $\psi_{\alpha\beta}$-Hölder continuity of a boundary function f ensures $\psi_{\alpha\beta}$-Hölder continuity of $\mathcal{P}_\Omega f$ if and only if $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$. Hence we characterize the family of domains Ω such that $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$.

Definition. We say that $E \subset X$ is *uniformly p-fat* or satisfies the *p-capacity density condition* if there are constants $C > 0$ and $r_0 > 0$ such that

$$\frac{\mathrm{Cap}_p(E \cap B(a, r), B(a, 2r))}{\mathrm{Cap}_p(B(a, r), B(a, 2r))} \ge C, \tag{1}$$

whenever $a \in E$ and $0 < r < r_0$.

The uniform p-fatness of the complement of a domain Ω is closely related to the condition $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$. For $\alpha > 0$ we obtain the following theorem.

Theorem 1. *Let Ω be a bounded p-regular domain. If $X \setminus \Omega$ is uniformly p-fat, then there is a constant $0 < \alpha_1 \le \alpha_0$ such that $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$ for $0 < \alpha < \alpha_1$ and $\beta \in \mathbb{R}$. Conversely, if $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$ for some $0 < \alpha < \alpha_0$ and $\beta \in \mathbb{R}$, then $X \setminus \Omega$ is uniformly p-fat, provided that there is a constant $Q \ge p$ such that X is Ahlfors Q-regular, i.e.,*

$$C^{-1}r^Q \le \mu(B(x, r)) \le Cr^Q$$

for every $x \in X$ and $r > 0$.

For $\alpha = 0$ we obtain the following theorem.

Theorem 2. *If $X \setminus \Omega$ is uniformly p-fat, then $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$ for every $\beta > 0$.*

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Electrical network reduction and the finite Dirichlet problem

Vadim Kaimanovich (University of Ottawa)

Electrical networks are studied because of their practical applications, but they are also useful mathematical tools with a wide range of applications. In this talk I will discuss a basic fact concerning the reduction of electrical networks with multiple external nodes that has apparently escaped the attention of both mathematicians and electrical engineers. As a consequence, it leads to a new interpretation of the classical Dirichlet problem for finite networks.

Joint work with A. Georgakopoulos.

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Weyl's Laplacian eigenvalue asymptotics for the measurable Riemannian structure on the Sierpiński gasket

Naotaka Kajino (University of Bielefeld)

On the Sierpiński gasket K, Kigami [3] introduced the notion of the measurable Riemannian structure, with which the "gradient vector field" $\widetilde{\nabla}u$ of a function u, the "Riemannian volume measure" μ and the "geodesic metric" $\rho_{\mathcal{H}}$ are naturally associated. Kigami also proved in [3] the two-sided Gaussian bound for the corresponding heat kernel $p_t^{\mathcal{H}}(x,y)$, and I showed in [1] further several detailed heat kernel asymptotics, such as Varadhan's asymptotic relation

$$\lim_{t\downarrow 0} 4t \log p_t^{\mathcal{H}}(x,y) = -\rho_{\mathcal{H}}(x,y).$$

Furthermore Koskela and Zhou proved in [4] that for any Lipschitz function u on $(K, \rho_{\mathcal{H}})$,

$$|\widetilde{\nabla}u(x)| = \limsup_{y\to x} \frac{|u(y)-u(x)|}{\rho_{\mathcal{H}}(x,y)} =: (\mathrm{Lip}_{\rho_{\mathcal{H}}} u)(x) \qquad \text{for } \mu\text{-a.e. } x \in K,$$

which means that the canonical Dirichlet form $\mathcal{E}(u,u) := \int_K |\widetilde{\nabla}u|^2 d\mu$ associated with the measurable Riemannian structure on K coincides with Cheeger type $H_{1,2}$-seminorm in $(K, \rho_{\mathcal{H}}, \mu)$.

In the talk, Weyl's Laplacian eigenvalue asymptotics is presented for this case. Specifically, let d be the Hausdorff dimension of K and $\mathcal{H}^d$ the d-dimensional Hausdorff measure on K, both with respect to the "geodesic metric" $\rho_{\mathcal{H}}$. Then for some $c_{\mathcal{N}} > 0$ and for any non-empty open subset U of K with $\mathcal{H}^d(\partial U) = 0$,

$$\lim_{\lambda\to\infty} \frac{\mathcal{N}_U(\lambda)}{\lambda^{d/2}} = c_{\mathcal{N}} \mathcal{H}^d(U),$$

where $\mathcal{N}_U(\lambda)$ is the number of the eigenvalues, less than or equal to λ, of the Dirichlet Laplacian on U. Moreover, we will also see that the Hausdorff measure $\mathcal{H}^d$ is Ahlfors regular with respect to $\rho_{\mathcal{H}}$ but that it is singular to the "Riemannian volume measure" μ. A renewal theorem for functionals of Markov chains due to Kesten [2] plays a crucial role in the proof of the above asymptotic behavior of $\mathcal{N}_U(\lambda)$.

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Quasi-monomorphisms and p-harmonic functons with finite Dirichlet sum

Atsushi Kasue (Kanazawa University)

In this talk, infinite, nonlinear resistive networks are considered and Rayleigh's monotonicity law is described. We also discuss a problem on the existence of p-harmonic functions with finite Dirichlet sum.

We consider a connected, locally finite graph $G = (V, E)$ with the set of vertices V and the set of oriented edges E, and assume that a positive weight r on E is endowed. Given exponents $p > 1$, $q > 1$ with $1/p + 1/q = 1$, we introduce a norm on the space $C_1(G)$ of finite 1-chains on G by $\|I\|_q = (\frac{1}{2}\sum_{e\in E} r(e)|I(e)|^q)^{1/q}$, $I \in C_1(G)$, and denote by $\ell^q_r C_1(G)$ the completion of $C_1(G)$ relative to the norm. Similarly we define a norm on the space $C^1(G)$ of 1-cochains on G by $\|\omega\|_p = (\frac{1}{2}\sum_{e\in E} |\omega(e)|^p/r(e)^{p-1})^{1/p}$, $\omega \in C^1(G)$, and denote by $\ell^p_r C^1(G)$ the space of 1-cochains whose norms are finite. Then we have a bijection between $\ell^q_r C_1(G)$ and $\ell^p_q C^1(G)$, called the resistance operator $\mathcal{R}$, which send a 1-chain I in $\ell^q_r C_1(G)$ to a 1-cochain $\mathcal{R}(I)$ defined by $\mathcal{R}(I)(e) = r(e)I(e)|I(e)|^{q-2}$, $e \in E$. The resistance operator $\mathcal{R}$ keeps the norms in such a way that $\|\mathcal{R}(I)\|_p^p = \|I\|_q^q$.

We are concerned with Kirchhoff's equations. Given a 0-chain j on G, Kirchhoff's laws are expressed by the following equations in the unknown 1-chain I in $\ell^q_r C_1(G)$: [I] (Kirchhoff's nodes law) $\partial I(x)(= \sum_{y\sim x} I([x,y])) = j(x)$, $x \in V$; [II] (Kirchhoff's loop law) $< \mathcal{R}(I), z > (= \sum_{e\in E} \mathcal{R}(I)(e)z(e)) = 0$, $\forall z \in Z_1(G)$, where $Z_1(G)$ stands for the set of finite cycles. A 1-chain I in $\ell^q_r C_1(G)$ satisfies [II] if and only if $I = \mathcal{R}^{-1}(df)$ for some 0-cochain (function) f in $L^{1,p}(G,r) = \{f \in C^0(G)|df \in \ell^p_r C^1(G)\}$. For $j \in \partial\ell^q_r C_1(G)$, we have a unique solution I^M_j of equations [I] and [II] satisfying $\|I^M_j\|_q = \inf\{\|I\|_q | I \in \ell^q_r C_1(G), \partial I = j\}$, which is called the minimal current generated by j. Then it is proved that $I^M_j = \mathcal{R}^{-1}(dg_j)$, where g_j belongs to the closure of the space of finitely supported functions in $L^{1,p}(G,r)$, denoted by $L^{1,p}_0(G,r)$. Any other solution I of [I] and [II] is expressed uniquely as $I = I^M_j + \mathcal{R}^{-1}(dh) + z$, where h belongs to $HL^{1,p}(G,r) = \{h \in L^{1,p}(G,r)|\Delta_p h := \partial\mathcal{R}^{-1}(dh) = 0\}$, and z_j is an element of the closure of $Z_1(G)$. We can write $I = R^{-1}(df)$, where f is a solution of Poisson equation $\Delta_p f = j$ in $L^{1,p}(G,r)$. If $\sum_{x\in V}|j(x)|$ is finite, then we are interested in a solution of [I] and [II] satisfying $< du, I >=< u, j >$ for any bounded function u in $L^{1,p}(G,r)$. Such a solution is unique if it exists, and obviously it is necessary for the existence to assume that $\sum_{x\in V} j(x) = 0$. A network (G,r) is called p-nonparabolic if δ_a belongs to $\partial\ell^q_r C_1(G)$ for some (any) $a \in V$.

When we have a graph morphism from an infinite network (G,r) to another one (G',r') satisfying certain conditions, we are able to describe Rayleigh's monotonicity law.

Now we turn to a problem on the existence of non-constant p-harmonic functions with finite Dirichlet sum.

Let $G = (V,E)$ be a connected, infinite graph of bounded degrees (with weight $= 1$). The graph G is endowed with the graph distance d_G. We say that a map ϕ from G to a metric space (X, d_X) is a quasi-monomorphism if there exist positive constants $\alpha > 0$ and $\beta \geq 0$ such that $d_X(\phi(a), \phi(b)) \leq \alpha d_G(a,b) + \beta$ for all $a, b \in V$, and there is a constant $\gamma > 0$ such that for any $x \in X$, the cardinality of the set of points $a \in V$ with $d_X(x, \phi(a)) \leq 1$ is bounded by γ. Then the following result is proved in [1]:

Suppose that G admits a quasi-monomorphism $\phi : G \to \boldsymbol{H}^n$ to the hyperbolic space $\boldsymbol{H}^n$ of dimension n. Then for $p > n-1$, if G is p-nonparabolic, then it possesses a lot of p-harmonic functions with finite Dirichlet sum; in fact, there exists a perfect subspace Σ of the limit set $\overline{\phi(G)} \cap \partial_\infty \boldsymbol{H}^n$ and, for any Lipschitz function η on Σ, there is uniquely a function h in $HL^{1,p}(G)$ such that $\lim_{\phi(a)\to\xi} h(a) = \eta(\xi)$ for all $\xi \in \Sigma$.

We remark that a quasi-monomorphism ϕ from G to $\boldsymbol{H}^n$ induces a graph morphism from G to a graph that is quasi-isometric to $\boldsymbol{H}^n$.

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Asymptotic geometry, harmonic functions, and finite generation of isometry groups

Bruce Kleiner (Courant Institute of Mathematics Sciences)

The lecture will discuss spaces (e.g. graphs or Riemannian manifolds) with polynomial-type growth conditions. The emphasis will be on polynomial growth harmonic functions and related topics, such as the finite generation of discrete groups of isometries.

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Applications of Hopf-Lax formulae to analysis of heat distributions

Kazumasa Kuwada (Ochanomizu University)

Let (X, d) be a metric space. Let $p \in (1, \infty)$. For $f \in C_b(X)$, we define $Q_t f \in C_b(X)$ by

$$Q_t f(x) := \inf_{y \in X} \left[f(y) + \frac{t}{p} \left(\frac{d(x,y)}{t} \right)^p \right].$$

We call it Hopf-Lax semigroup (also called Hamilton-Jacobi semigroup). When (X, d) is an Euclidean space, $Q_t f$ is nothing but the Hopf-Lax formula, which gives a solution to the Hamilton-Jacobi equation

$$\partial_t Q_t f(x) = -\frac{1}{q} |\nabla Q_t f|(x)^q$$

in an appropriate sense, where q is the Hölder conjugate of p. This property is still valid even on more abstract metric spaces. It has been revealed that the notion of Hopf-Lax semigroup is strongly related with many functional inequalities including logarithmic Sobolev inequalities and transport-entropy inequalities. The purpose of this talk is to explain recent developments in this direction in connection with the heat semigroup.

For probability measures $\mu_0, \mu_1 \in \mathcal{P}(X)$, we denote the L^p-Wasserstein distance between μ_0 and μ_1 by $W_p(\mu_0, \mu_1)$. That is,

$$W_p(\mu_0, \mu_1) := \inf \left\{ \|d\|_{L^p(\pi)} \;\middle|\; \pi \in \mathcal{P}(X \times X) \text{: coupling of } \mu_0 \text{ and } \mu_1 \right\},$$

where we call π a coupling of μ_0 and μ_1 when the marginal distribution of π is μ_0 and μ_1 respectively. The dual representation of W_p is called the Kantorovich duality. By using $Q_t f$, it can be stated as follows:

$$W_p(\mu_0, \mu_1) = \sup_{f \in C_b(X)} \left[\int_X Q_1 f \, d\mu_1 - \int_X f \, d\mu_0 \right].$$

The Hopf-Lax semigroup appears here and this fact connects the study of Hopf-Lax formula with the theory of optimal transportation.

The first application of Hopf-Lax formula in this talk is a relation between a Lipschitz estimate of Wasserstein distance and a Bakry-Émery type gradient estimate for Markov kernels which in particular we can apply to the (Feller) heat semigroup. For $f : X \to \mathbb{R}$, we define the local Lipschitz constant $|\nabla_d f|(x)$ with respect to d by

$$|\nabla_d f|(x) = \limsup_{y \to x} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Theorem 1 (cf. [4]). *Let (X, d) be a Polish length space and $\tilde{d}$ be another length metric on X. We denote the L^p-Wasserstein distance defined by using $\tilde{d}$ instead of d by $\tilde{W}_p$. Let $P(x, \cdot) \in \mathcal{P}(X)$ be a Markov kernel on X which depends continuously in $x \in X$. Then, for $p, q \in [1, \infty]$ with $p^{-1} + q^{-1} = 1$, the following are equivalent:*

(i) *For $\mu_0, \mu_1 \in \mathcal{P}(X)$, $W_p(P^*\mu_0, P^*\mu_1) \le \tilde{W}_p(\mu_0, \mu_1)$.*

(ii) *For $f \in C_b^{\mathrm{Lip}}(X)$, $|\nabla_{\tilde{d}} Pf|(x) \le P(|\nabla_d f|^q)(x)^{1/q}$ (When $q = \infty$, $\||\nabla_{\tilde{d}} Pf|\|_\infty \le \||\nabla_d f|\|_\infty$).*

The second application is on the estimate of the speed of heat distributions with respect to W_2. For simplicity, we state it when X is a Riemannian manifold.

Theorem 2. *Let X is a complete and stochastically complete Riemannian manifold and P_t the heat semigroup on X. Take $f : X \to [0, \infty)$ with $\|f\|_{L^1} = 1$ and set $\mu_t := P_t f \mathrm{vol}$. Then*

$$|\dot{\mu}_t|_{W_2}^2 := \limsup_{s \downarrow 0} \frac{W_2(\mu_{t+s}, \mu_t)^2}{s^2} = \int_X \frac{|\nabla P_t f|^2}{P_t f} \, d\mathrm{vol}.$$

This estimate is first studied in [3] on Alexandrov spaces in the context of identification problem of heat flows. On Riemannian manifolds, there are two different ways to formulate a “heat flow”. The one is a gradient flow of the Dirichlet energy in L^2-space of functions and the other is a gradient flow of the relative entropy on $\mathcal{P}(X)$ endowed with a metric structure by W_2. Thus Theorem 2 is an estimate related with the second formulation in the sense that it is a bound of the speed of curves in $\mathcal{P}(X)$ with respect to W_2 while the object μ_t is given by the first formulation. It plays a fundamental role for identifying those two formulation on non-smooth metric measure spaces as Alexandrov spaces (see [1, 3]). As a result of the identification, we can obtain the Bakry-Émery gradient estimate for the heat semigroup under a generalized notion of lower Ricci curvature bound (see [2, 3]).

The third application is a sort of extension of Theorem 1. Inequalities of the form (i) or (ii) are first introduced in connection with the notion of lower Ricci curvature bound. Recently, F.-Y. Wang introduced an extension of the Bakry-Émery gradient estimate involving an upper bound of $\dim X$ (property (v) below; see [5]). We obtain the condition corresponding to (i):

Theorem 3. *Let X be a complete and stochastically complete Riemannian manifold with $\dim X \ge 2$. Then, for $N \in [2, \infty]$ and $K \in \mathbb{R}$, the following are equivalent:*

(iii) $\dim X \le N$ *and* $\mathrm{Ric} \ge K$.

(iv) $W_2(P_{t_0}\mu_0, P_{t_1}\mu_1)^2 \le \dfrac{\mathrm{e}^{-2Kt_1} - \mathrm{e}^{-2Kt_0}}{2K(t_0 - t_1)} W_2(\mu_0, \mu_1)^2 + (t_1 - t_0) \displaystyle\int_{t_0}^{t_1} \frac{NK}{\mathrm{e}^{2Ku} - 1} du$ *for* $t_1 > t_0 > 0$ *and* $\mu_0, \mu_1 \in \mathcal{P}(X)$.

(v) $|\nabla P_t f|(x)^2 \le \mathrm{e}^{-2Kt} P_t(|\nabla f|^2) - \dfrac{1 - \mathrm{e}^{-2Kt}}{NK} (\Delta P_t f)^2$ *for* $t > 0$ *and* $f \in C_b^{\mathrm{Lip}}(X)$.

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p harmonic measure in simply connected domains revisited

John L. Lewis (University of Kentucky)

Let Ω be a bounded simply connected domain in the complex plane, $\mathbb{C}$. Let N be a neighborhood of $\partial\Omega$, let p be fixed, $1 < p < \infty$, and let u be a positive weak solution to the p Laplace equation in $\Omega \cap N$. Assume that u has zero boundary values on $\partial\Omega$ in the Sobolev sense and extend u to $N \setminus \Omega$ by putting $u \equiv 0$ on $N \setminus \Omega$. Then there exists a positive finite Borel measure μ on $\mathbb{C}$ with support contained in $\partial\Omega$ and such that

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, dA = - \int \phi \, d\mu$$

whenever $\phi \in C_0^\infty(N)$. Define the Hausdorff dimension of μ by

$$\text{H-dim } \mu = \inf \{ \alpha : \text{ there exists } E \text{ Borel} \subset \partial\Omega \text{ with } H^\alpha(E) = 0 \text{ and } \mu(E) = \mu(\partial\Omega) \} ,$$

where $H^\alpha(E)$, for $\alpha \in \mathbf{R}_+$, is the α-dimensional Hausdorff measure of E. In this talk we first discuss results concerning H-dim μ when μ is harmonic measure (the case $p = 2$). After that we outline work of coauthors and myself concerning the dimension of p harmonic measure when $1 < p < \infty$. Time permitting we will discuss a recent paper with the same title as our talk, dealing with results for p harmonic measure, similar to the well known result of Makarov for harmonic measure in simply connected domains.

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The quasisymmetric geometry of boundaries of relatively hyperbolic groups

John Mackay (University of Oxford)

The boundary of a Gromov hyperbolic group is a metric space canonically defined up to quasisymmetry, and analysis on such spaces has been of much interest in the past twenty years. In this talk I will describe the analogous boundaries for relatively hyperbolic groups, and some of their analytic properties. I will also describe a result constructing quasi-arcs in metric spaces avoiding certain obstacles. (Based on joint work with Alessandro Sisto.)

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Mean continuity for potentials of functions in Musielak-Orlicz spaces

Fumi-Yuki Maeda

The classical results on mean continuity of Riesz potentials of funcrions f in L^p have been extended to the case when f belongs to the variable exponent $L^{p(\cdot)}$ (e.g., [1]) and further to $L^{p(\cdot)}(\log L)^{q(\cdot)}$ ([3]). Here we further extends those results to potentials of functions in Musielak-Orlicz spaces.

Let $\Phi(x,t) = t\phi(x,t) : \mathbf{R}^N \times [0,\infty) \to [0,\infty)$ satisfy the following conditions:

($\Phi1$) Carathéodory condition;

($\Phi2$) $\Phi(x,1)$ and $1/\Phi(x,1)$ are bounded;

($\Phi3$) $t \mapsto t^{-\varepsilon_0}\phi(x,t)$ is uniformly almost increasing on $(0,\infty)$ for some $\varepsilon_0 > 0$;

($\Phi4$) $\Phi(x,\cdot)$ saitisfies uniform doubling condition;

($\Phi5$) for every $\gamma > 0$, there exists a constant $B_\gamma \geq 1$ such that $\Phi(x,t) \leq B_\gamma \Phi(y,t)$ whenever $|x-y| \leq \gamma t^{-1/N}$ and $t \geq 1$.

Let $\overline{\Phi}(x,t) = \int_0^t \sup_{0\leq s\leq r} \phi(x,s)\,dr$. For an open set G in $\mathbf{R}^N$, the Musielak-Orlicz space $L^\Phi(G)$ is defined by (cf. [4])

$$L^\Phi(G) = \left\{ f \in L^1_{loc}(G)\,;\, \int_G \Phi\big(y,|f(y)|\big)\,dy < \infty \right\},$$

which is a Banach space with respect to the norm

$$\|f\|_{L^\Phi(G)} = \inf\left\{ \lambda > 0\,;\, \int_G \overline{\Phi}\big(y,|f(y)|/\lambda\big)\,dy \leq 1 \right\}.$$

As a kernel function on $\mathbf{R}^N$, we consider $k(x) = k(|x|)$ (with the abuse of notation) with a function $k(r) : (0,\infty) \to (0,\infty)$ satisfying the following conditions:

(k1) $k(r)$ is non-increasing lower semicontinuous on $(0,\infty)$;

(k2) $\int_0^1 k(r) r^{N-1}\,dr < \infty$;

(k3) there exists a constant $K_1 \geq 1$ such that $k(r) \leq K_1 k(r+1)$ for all $r \geq 1$.

For $f \in L^1_{loc}(\mathbf{R}^N)$ satisfying

$$\int_{\mathbf{R}^N} k(1+|y|)|f(y)|\,dy < \infty, \tag{$*$}$$

we consider its k-potential $k * f$.

Let $\bar{k}(r) = \frac{N}{r^N}\int_0^r k(\rho)\rho^{N-1}\,d\rho$ for $r > 0$ and set

$$\Gamma(x,s) = s^{-1}\bar{k}(s^{-1/N})\Phi^{-1}(x,s) \quad (x \in \mathbf{R}^N,\ s > 0),$$

where $\Phi^{-1}(x,s) = \sup\{t > 0; \Phi(x,t) < s\}$.

Consider a function $\Psi(x,t) : \mathbf{R}^N \times [0,\infty) \to [0,\infty)$ satisfying the following conditions:

($\Psi1$) Carathéodory condition;

($\Psi2$) there is a constant $A_1 \geq 1$ such that $\Psi(x,at) \leq A_1 a \Psi(x,t)$ for all $x \in \mathbf{R}^N$, $t > 0$ and $0 \leq a \leq 1$;

($\Psi\Phi k$) there exists a constant $A_2 \geq 1$ such that

$$\Psi\big(x,\Gamma(x,s)\big) \leq A_2 s \quad \text{for all } x \in \mathbf{R}^N \text{ and } s > 0.$$

Theorem 1. *Let $f \in L^1_{loc}(\mathbf{R}^N)$ satisfy $(*)$ and set*

$$E_1 = \{x \in \mathbf{R}^N : k * |f|(x) = \infty\},$$
$$E_2 = \left\{x \in \mathbf{R}^N : \limsup_{r\to 0+} \fint_{B(x,r)} \Phi\left(z, r^N\bar{k}(r)|f(z)|\right)dz > 0\right\}.$$

Assume

(Γ) $s \mapsto s^{-\varepsilon_1}\Gamma(x,s)$ *is uniformly almost increasing for some* $\varepsilon_1 > 0$*;*

($k5$) $k(rs) \leq K_3\bar{k}(r)k(s)$ *for all* $0 < r \leq 1,\ 0 < s \leq 1$.

Then

$$\lim_{r\to 0+} \fint_{B(x_0,r)} \Psi(x, |k * f(x) - k * f(x_0)|)\, dx = 0$$

for all $x_0 \in \mathbf{R}^N \setminus (E_1 \cup E_2)$.

For a set $E \subset \mathbf{R}^N$ and an open set $G \subset \mathbf{R}^N$, we define (cf. [2])

$$C_{k,\Phi}(E; G) = \inf_{f \in S_k(E;G)} \int_G \overline{\Phi}\,(y, f(y))\, dy,$$

where $S_k(E; G)$ is the family of all nonnegative measurable functions f on $\mathbf{R}^N$ such that f vanishes outside G and $k * f(x) \geq 1$ for every $x \in E$. We say that E is of (k, Φ)-capacity zero, if

$$C_{k,\Phi}(E \cap G; G) = 0 \qquad \text{for every bounded open set } G.$$

Theorem 2. *Let* $f \in L^{\Phi}(\mathbf{R}^N)$ *satisfy* $(*)$. *Then,* E_1 *in Theorem 1 has* (k, Φ)*-capacity zero. If* Φ *satisfies a further condition*

$(\Phi 6)$ $\Phi(x, s)\, \Phi(x, t) \leq A_3\, \Phi(x, st)$ *for all* $x \in \mathbf{R}^N$, $s \geq 1$ *and* $t > 0$,

then E_2 *in Theorem 1 has* (k, Φ)*-capacity zero.*

Joint work with Yoshihiro Mizuta, Takao Ohno and Tetsu Shimomura.

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On harmonic Hardy-Orlicz spaces

Hiroaki Masaoka (Kyoto Sangyo University)

Let $(\Omega, \mathcal{H})$ be a $\mathcal{P}$-Brelot harmonic space. Suppose that there exists a countable base for the open sets of Ω and that constant functions are harmonic on Ω. Set

$$\mathcal{N} = \{\Phi \mid \Phi \text{ is non-negative, convex and strictly increasing functions on } [0, +\infty),$$
$$\Phi(0) = 0 \text{ and } \lim_{t\to+\infty} \frac{\Phi(\alpha t)}{t} = +\infty \text{ (for } \alpha > 0)\}.$$

Let Φ and Ψ be elements of $\mathcal{N}$. We showed that under the assumption that $\limsup_{t\to+\infty} \dfrac{\Phi(\alpha t)}{\Psi(t)} = +\infty$ for all positive α the following three conditions are equivalent.

(i) the Hardy Orlicz spaces $H_\Phi(\Omega)$ and $H_\Psi(\Omega)$ coincide;

(ii) $\dim H_\Psi(\Omega) < +\infty$;

(iii) $\dim H_\Phi(\Omega) < +\infty$.

In our talk we give an example for $\mathcal{P}$-harmonic space with the above condition (i). This is a joint work with Tero Kilpeläinen and Pekka Koskela.

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Sobolev's inequality for Riesz potentials in Lorentz spaces of variable exponent

Yoshihiro Mizuta (Hiroshima Institute of Technology)

In the present talk we discuss the boundedness of the maximal operator in the Lorentz space of variable exponent defined by the symmetric decreasing rearrangement in the sense of Almut [1]. As an application of the boundedness of the maximal operator, we establish the Sobolev inequality by using Hedberg's trick in his paper [9].

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Equilibrium measures for uniformly quasiregular dynamics

Yûsuke Okuyama (Kyoto Institute of Technology)

We establish the existence and fundamental properties of the equilibrium measure in uniformly quasiregular dynamics. We show that a uniformly quasiregular endomorphism f of degree at least 2 on a closed Riemannian manifold of dimension n admits an equilibrium probability measure μ_f, which is balanced and invariant under f and non-atomic, and whose support agrees with the Julia set of f. Furthermore we show that f is strongly mixing with respect to the measure μ_f. We also characterize the measure μ_f using an approximation property by iterated pullbacks of points under f up to a set of exceptional initial points of Hausdorff dimension at most $n - 1$. These dynamical mixing and approximation results are reminiscent of the Mattila-Rickman equidistribution theorem for quasiregular mappings. This is a joint work with Pekka Pankka (Helsinki).

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Computation of capacities

Thomas Ransford (Laval University)

I shall discuss the problem of computing the value of the capacity of a set for the logarithmic, Riesz, hyperbolic and analytic capacities.

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A characterization of entire functions and approximation

Ryozi Sakai (Meijo University)

We define the degree of approximation for a continuous function f on $I = [-1, 1]$ by

$$E_n(f) := \inf_{P \in \mathcal{P}_n} \|f - P\|_{L^\infty(I)},$$

where $\mathcal{P}_n$ denotes the class of all polynomials with degree $\leq n$. In [1], S.Bernstein proved that f has an analytic extension of an entire function if and only if $\lim_{n\to\infty} E_n^{1/n}(f) = 0$. R.S.Varga ([5]) considered the rate at which $E_n^{1/n}(f)$ tends to zero, and he showed that $f \in C(I)$ satisfies

$$\limsup_{n\to\infty} \left\{ \frac{n \log n}{\log(1/E_n(f))} \right\} = \lambda$$

if and only if f has an analytic extension of an entire function of order λ. Recall that an entire function f is of order λ if

$$\limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r} = \lambda,$$

where $M(r, f) := \max_{|z|=r} |f(z)|$.

In this talk, we discuss the about result for approximations on $\mathbf{R}$. Let f be a real valued L^p-function ($1 \leq p \leq \infty$) on $\mathbf{R}$, and let

$$E_{p,n}(f, w) := \inf_{P \in \mathcal{P}_n} \|w(f - P)\|_{L^p(\mathbf{R})},$$

where $w = \exp(-Q)$ is an exponential weight which belongs to a relevant class $\mathcal{F}(C^2+)$ (see, e.g., [3]). For example, $Q(x) = \exp(|x|^\alpha) - 1$ or $Q(x) = (1 + |x|)^{|x|^\alpha} - 1$ for $\alpha > 1$.

Then we can prove the following.

Theorem. *Let*

$$\limsup_{n\to\infty} \frac{n \log n}{\log 1/E_{p,n}(f; w, \mathbf{R})} =: \rho_p(f).$$

Then the function f with $wf \in L^p(\mathbf{R})$ is the restriction to $\mathbf{R}$ of an entire function with finite order λ if and only if $\rho_p(f)$ is finite. Furthermore we see

$$\frac{1}{\lambda} - \frac{1}{A} \leq \frac{1}{\rho_p(f)} \leq \frac{1}{\lambda} - \frac{1}{B},$$

where $T(x) = xQ'(x)/Q(x)$ and

$$A := \lim \inf_{|x|\to\infty} T(x), \qquad B := \lim \sup_{|x|\to\infty} T(x).$$

Especially, if $T(x)$ is unbounded then $\lambda = \rho_p(f)$ holds true.

We point out that basic and essential results of the weighted polynomial approximation are obtained by using logarithmic potential theory (cf. [4], see also [2] and [3]).

Joint work with Noriaki Suzuki.

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Rotation of planar quasiconformal maps

Eero Saksman (University of Helsinki)

We introduce interpolation on L^p-spaces with complex exponents, and apply it to obtain optimal estimates for rotation of quasiconformal maps. The talk is based on joint work with K. Astala, T. Iwaniec and I. Prause.

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Heat kernel estimates on inner uniform domains

Laurent Saloff-Coste (Cornell University)

In this talk, I will discuss two-sided heat kernel estimates for the Neumann and Dirichlet heat kernels in inner uniform domains. In the case of the Dirichlet heat kernel, one of the key ingredient is a scale-invariant Harnack boundary principle developed in ealier work of H. Aikawa, A. Ancona and others. In the case of bounded domains, these estimates sharp intrinsic ultracontractivity bounds.

Joint work with P. Gyrya and J. Lierl (Bonn).

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Morrey spaces and fractional integral operators

Yoshihiro Sawano (Tokyo Metropolitan University)

The well-known Hardy-Littlewood-Sobolev theorem is as follows:

Theorem 1.1. *Let* $0 < \alpha < n$ *and* $1 < p < q < \infty$. *If* $\frac{1}{q} = \frac{1}{p} - \alpha$, *then* $\|I_\alpha f\|_{L^q} \le C\|f\|_{L^p}$, *where*

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

This theorem appears in disguise in many fields of mathematics. Morrey spaces seem appropriate to view subtly how this smoothing effect occurs.

Let $1 < q < p < \infty$. Then define

$$\|f\|_{\mathcal{M}^p_q} = \sup_B |B|^{\frac{1}{p}-\frac{1}{q}} \left(\int_B |f(y)|^q \, dy \right)^{1/q}.$$

The well-known Adams theorem reads;

Theorem 1.2. *Let $0 < \alpha < n$. Assume that the parameters p, q, s, t satisfy*

$$1 < q \le p < \infty, \quad 1 < t \le s < \infty, \quad \frac{q}{p} = \frac{t}{s}, \quad \frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}.$$

Then I_α is bounded from $\mathcal{M}^p_q$ to $\mathcal{M}^s_t$.

In this talk, we consider why this happens ?

1. Do we need any smooth structure of the Euclidean spaces or the nice property of Lebesgue measures ? [1, 2, 5, 8, 9, 13] We work on a very generic setting proposed in [13].
2. Is I_α surjective ? If no, characterize the image. [6, 10, 14] The function space defined in [14] can be used to view Morrey spaces and fractional integral operators by taking full advantage of the structure of $\mathbb{R}^n$.
3. What happens in the bilinear case ? [3, 4] Can we consider

   $$I_\alpha[f_1, f_2](x) = \int_{\mathbb{R}^n} \frac{f_1(y_1) f_2(y_2)}{(|x - y_1| + |x - y_2|)^{2n-\alpha}} \, dy\,?$$

   Is a natural extension of the Adams theorem all when we use Morrey spaces ? What can we say about the operator of the form $(f, g) \mapsto g \cdot I_\alpha f$?
4. How about the endpoint cases ? [7, 9, 12] For example, what can we say about the case $s = \infty$?

In the talk, the speaker will present a typical result for each problem.

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Constructing a prime end boundary for non-simply connected domains in Euclidean spaces and metric measure spaces

Nageswari Shanmugalingam (University of Cincinnati)

Caratheodory's definition of prime ends is fruitful in the case that the domain under study is a simply/finitely connected planar domain. His definition has been extended in various ways to domains in higher dimensions, but again for a limited number of domains (such as quasiconformally collared domains). I will talk about a possible alternate construction of prime ends that is useful for more general domains in all dimensions (and of course, in metric space setting as well). This talk is based on joint work with Anders Bjorn, Jana Bjorn, and Tomasz Adamowicz.

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Hardy averaging operator on generalized Banach function spaces

Tetsu Shimomura (Hiroshima University)

Let $Af(x) := \frac{1}{|B(0,|x|)|}\int_{B(0,|x|)} f(t)\,dt$ be the n-dimensional Hardy averaging operator. It is well known that A is bounded on $L^p(\Omega)$ with an open set $\Omega \subset \mathbf{R}^n$ whenever $1 < p \leq \infty$. In this talk, we improve this result within the framework of generalized Banach function spaces. We in fact find the 'source' space S_X, which is strictly larger than X, and the 'target' space T_X, which is strictly smaller than X, under the assumption that the Hardy-Littlewood maximal operator M is bounded from X into X, and prove that A is bounded from S_X into T_X. We prove optimality results for the action of A on such spaces and present applications of our results to variable Lebesgue spaces $L^{p(\cdot)}(\Omega)$, as an extension of [2] in the case when $n = 1$ and Ω is a bounded interval.

Joint work with Yoshihiro Mizuta and Aleš Nekvinda.

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Two-phase quadrature domains and harmonic balls

Tomas Sjödin (Linköping University)

This talk is mainly going to be a survey of the recent theory of two-phase quadrature domains and the related topic of harmonic balls. In particular I will focus on my work together with Stephen Gardiner (UCD Dublin) regarding two-phase quadrature domains for harmonic and subharmonic functions and my work with Henrik Shahgholian (KTH) about two-phase quadrature domains for analytic functions and harmonic balls.

Roughly speaking, two-phase quadrature domains consists of a pair of disjoint open sets D_+, D_- together with two measures μ_+, μ_- such that μ_+ and μ_- has compact support in D_+ and D_- respectively, and such that we for some suitable class of functions h have an integral equality or inequality between the integrals (where λ denotes Lebesgue measure)

$$\int_{D_+} h d\lambda - \int_{D_-} h d\lambda,$$

and

$$\int h d\mu_+ - \int h d\mu_-.$$

Natural choices for h can be analytic, harmonic or such that h is subharmonic in D_+ and superharmonic in D_-. Unlike the classical (one-phase) case we assume more about the behaviour of the functions at the boundaries of D_+ and D_-, and not just that h is integrable over D_+ and D_- (otherwise we would just have two disjoint one-phase quadrature domains). We will discuss what natural choices are, and also relate this concept to two-phase modified Schwarz potentials and Schwarz functions which also has natural definitions.

After this we shall also discuss the concept of harmonic balls, which is closely related to the above. It is well known that if $\alpha\delta_x$ is a point mass at $x \in \mathbb{R}^n$ and B is an open set such that the Newtonian potential of $\alpha\delta_x$ and $\lambda|_B$ are equal in the complement of B, then B is the ball with center x and total mass α. Harmonic balls are defined relative to a domain K, and we say that $B \subset K$ is a harmonic ball with respect to $\alpha\delta_x$ ($x \in K$) if the Green potentials in K for $\alpha\delta_x$ and $\lambda|_B$ agree in $K \setminus B$. We will discuss some known results and also some interesting open questions regarding these.

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A concept of harmonicity for families of planar curves

Eleutherius Symeonidis (Katholische Universität Eichstätt-Ingolstadt)

Let $\Omega \subseteq \mathbb{R}^2$ be a simply connected domain, $t \mapsto (x_0(t), y_0(t)) \in \Omega$ a smooth curve, parametrized over an interval I. Moreover, let J be an interval containing 0, $J \times I \ni (s,t) \mapsto (x(s,t), y(s,t)) \in \Omega$ a conformal mapping such that $x(0,t) = x_0(t)$, $y(0,t) = y_0(t)$ for all $t \in I$.

Let h be a harmonic function on Ω, $\tilde{h}$ a harmonic conjugate to h. If $I \ni t \mapsto h(x_0(t), y_0(t))$ is integrable, and if

$$\lim_{t \to \inf I} \tilde{h}(x(s,t), y(s,t)) = \lim_{t \to \sup I} \tilde{h}(x(s,t), y(s,t))$$

holds for all $s \in J$, then for all these s,

$$\int_I h(x(s,t), y(s,t))\,\mathrm{d}t = \int_I h(x_0(t), y_0(t))\,\mathrm{d}t\,,$$

which means that the integral of h over the different curves of the family $(t \mapsto (x(s,t), y(s,t))\,)_{s \in J}$ is invariant. Therefore, it is natural to speak of a *harmonic deformation* of the initial curve. We remark that the condition on $\tilde{h}$ is automatically satisfied in the case in which the curves of the family are closed and I is compact.

We show how a conformal mapping as above can be derived from a specific potential, and we present examples with bounded and unbounded curves or with such ones with multiple points.

Finally, we discuss the question of invariance of certain weighted integrals $\int_I h(x(s,t), y(s,t)) w(s,t)\,\mathrm{d}t$ in the same framework.

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A representation for harmonic Bergman function and its application

Kiyoki Tanaka (Osaka City University)

Let Ω be a bounded smooth domain in $\mathbb{R}^n$. For $1 \leq p < \infty$, we denote by $b^p(\Omega)$ the harmonic Bergman space in Ω, i.e., the set of all real-valued harmonic functions f on Ω such that $\|f\|_p := \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}} < \infty$, where dx denotes the usual n-dimensional Lebesgue measure on Ω. It is known that b^2 is the reproducing kernel Hilbert space. The reproducing kernel for $b^2(\Omega)$ is called the harmonic Bergman kernel.

In this talk, we discuss a representation for the harmonic Bergman function and interpolation theorem. B. R. Choe and H. Yi [2] studied the representation theorem and interpolation theorem for harmonic Bergman functions in the upper half space. As a recent result, we introduce the following representation theorem for the harmonic Bergman function in a bounded smooth domain.

Theorem 1 (cf. Theorem 1 in [5]). *Let $1 < p < \infty$ and Ω be a bounded smooth domain. Then, we can choose a sequence $\{\lambda_i\}$ in Ω such that $A : \ell^p \to b^p$ is a bounded onto map, where the operator A is defined by*

$$A\{a_i\}(x) := \sum_{i=1}^{\infty} a_i R(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n},$$

where $R(x, y)$ denotes the harmonic Bergman kernel and $r(x)$ denotes the distance between x and $\partial\Omega$.

Conversely, we consider the map from b^p to ℓ^p. The following theorem is called interpolation theorem.

Theorem 2. *Let $1 < p < \infty$ and Ω be a bounded smooth domain. There exists a positive constant ρ_0 such that if $\rho(\lambda_i, \lambda_j) > \rho_0$ for any $i \neq j$, then $V : b^p \to \ell^p$ is bounded onto map, where $\rho(x, y)$ is pseudo-hyperbolic distance and $Vf := \{r(\lambda_i)^{\frac{n}{p}} f(\lambda_i)\}_i$.*

The previous theorems do not refer to b^1-functions. A representation for b^1-functions is achieved by using the another kernel. The following kernel is defined by B. R. Choe, H. Koo and H. Yi [1].

Definition 1. Let η be a defining function of Ω with condition that $|\nabla\eta|^2 = 1 + \eta\omega$ for some $\omega \in C^\infty(\bar{\Omega})$. We define the modified harmonic Bergman kernel by

$$R_1(x, y) = R(x, y) - \frac{1}{2}\Delta_y\big(\eta^2(y) R(x, y)\big)$$

for any $x, y \in \Omega$, where Δ_y is the Laplacian with respect to y.

By using the modified harmonic Bergman kernel, we can give the representation for b^1-functions.

Theorem 3 (cf. [6]). *Let $1 \leq p < \infty$ and Ω be a bounded smooth domain. Then, we can choose a sequence $\{\lambda_i\}$ in Ω such that $A_1 : \ell^p \to b^p$ is a bounded onto map, where the operator A_1 is defined by*

$$A_1\{a_i\}(x) := \sum_{i=1}^{\infty} a_i R_1(x, \lambda_i) r(\lambda_i)^{(1-\frac{1}{p})n}.$$

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Distortion of dimension by projections and Sobolev mappings

Jeremy Tyson (University of Illinois at Urbana-Champaign)

We will discuss a series of recent results on the metric and measure-theoretic properties of projection mappings. These include estimates for the distortion of Hausdorff dimension for images of fixed subsets under generic projections, and also for images of generic fibers of such projections under Sobolev and quasiconformal mappings. We will discuss first the case of Euclidean spaces, where the projection mappings are linear, and then the case of the sub-Riemannian Heisenberg group. In the latter case the projections onto vertical homogeneous subgroups are neither linear nor Lipschitz.

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The first boundary value problem of the biharmonic equation for the half-space

Minoru Yanagishita (Chiba University)

Let $\mathbf{T}_{n+1}$ $(n \geq 2)$ be the half-space $\{M = (X, y) \in \mathbf{R}^{n+1} : y > 0\}$, and let $\partial\mathbf{T}_{n+1}$ be its boundary.

Let f_0 and f_1 be two functions defined on $\partial\mathbf{T}_{n+1}$. A solution of the first boundary value problem of the biharmonic equation for $\mathbf{T}_{n+1}$ with respect to f_0 and f_1 is a biharmonic function w in $\mathbf{T}_{n+1}$ such that

$$\lim_{M \to N,\ M \in \mathbf{T}_{n+1}} w(M) = f_0(N), \quad \lim_{M \to N,\ M \in \mathbf{T}_{n+1}} \frac{\partial w}{\partial y}(M) = f_1(N)$$

for every point $N \in \partial\mathbf{T}_{n+1}$.

Schot [1] gave a particular solution of the first boundary value problem of the biharmonic equation for $\mathbf{T}_{n+1}$. With respect to the Dirichlet problem for the half-space $\mathbf{T}_{n+1}$, Yoshida [2] constructed the generalized Poisson integral $H_{l,n+1}f(M)$ $(l \geq 1)$ for slowly growing boundary function f. From this, for slowly growing regular boundary functions f_0 and f_1, we shall give a particular solution $W_{l,n+1}(f_0, f_1)(M)$ $(l \geq 1)$ by using generalized Poisson integrals $H_{l,n+1}f_i(M)$ $(i = 1, 2)$ and generalize the result of Schot. A solution of this boundary value problem for any regular boundary functions is also given.

The next result concerns a type of uniqueness of solutions of this boundary value problem. We denote by $\mathcal{M}(\cdot; r)$ the mean with respect to the surface element on the upper half sphere of radius r centered at the origin of $\mathbf{R}^{n+1}$.

Let l $(l \geq 3)$ be an integer. Let w be a solution of the first boundary value problem of the biharmonic equation for $\mathbf{T}_{n+1}$ with respect to slowly growing regular boundary functions f_0 and f_1.

If w satisfies

$$\mathcal{M}(yw^+;r) = O(r^{l+2}) \quad (r \to \infty),$$
$$\mathcal{M}(y^2 \left(\frac{\partial w}{\partial y}\right)^- ;r) = O(r^{l+2}) \quad (r \to \infty),$$
$$\mathcal{M}(y^3(\Delta w)^+;r) = O(r^{l+2}) \quad (r \to \infty),$$

then

$$w(X,y) = W_{l,n+1}(f_0,f_1)(X,y) + y^2 \sum_{j=0}^{[\frac{l}{2}]+1} \alpha_j y^{2j} \Delta^j P_{l-1}(X) + y^3 \sum_{j=0}^{[\frac{l}{2}]} \beta_j y^{2j} \Delta^j P_{l-2}(X)$$

for every $(X,y) \in \mathbf{T}_{n+1}$, where $P_k(X)$ is a polynomial of X of degree less than $k+1$ $(k = \{l-1,\ l-2\})$ and

$$\alpha_j = \begin{cases} (-1)^j \frac{2!(j+1)}{(2j+2)!} & (j = 0,1,2,\cdots,[\frac{l}{2}]+1), \\ 0 & (j = [\frac{l}{2}]+1,\ l \text{ is even}), \end{cases} \qquad \beta_j = (-1)^j \frac{3!(j+1)}{(2j+3)!} \ (j = 0,1,2,\cdots,[\frac{l}{2}]).$$

Joint work with Naohiro Yaginuma.

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A potential theoretic approach to the curvature equation

Tanran Zhang (Tohoku University)

This presentation is on the estimate of the class of conformal metrics with negative curvature. This kind of estimate can be taken as the asymptotic behavior near the isolated singularity and this research is done on the basis of potential theory. The asymptotic behavior of conformal metrics with negative curvatures was well studied in 2008 in [4]. But only the first and the second order derivatives were given. That offered us a way to consider the higher order derivatives by means of some potential theoretic approach. In fact, our results are sharp. We can verify it using the generalized hyperbolic metric. The explicit formula for the generalized hyperbolic metric $\lambda_{\alpha,\beta,\gamma}$ on the thrice-punctured sphere was given in 2011 in [5]. Since the Gaussian curvature of the generalized hyperbolic metric is some constant, here we take it to be -1, it makes a persuasive case in our study. For the generalized hyperbolic metric, we obtain a stronger version of the estimate near its isolated singularity and give some limits as the asymptotic behavior of $\lambda_{\alpha,\beta,\gamma}$ in higher order case near the singularity. Our study is on the basis of potential theory (see [3]), hypergeometric functions (see [2], [1]) and the uniformization theorem. The following one is our main estimate.

Theorem. *Let $\kappa : \mathbb{D} \to \mathbb{R}$ be a locally Hölder continuous function with $\kappa(0) < 0$. If $u : \mathbb{D}^* \to \mathbb{R}$ is a C^2-solution to $\Delta u = -\kappa(z)e^{2u}$ in $\mathbb{D}^*$, then u has the order $\alpha \in (-\infty, 1]$ and*

$$u(z) = -\alpha \log|z| + v(z), \qquad \text{if } \alpha < 1,$$
$$u(z) = -\log|z| - \log\log(1/|z|) + w(z), \qquad \text{if } \alpha = 1,$$

where the remainder functions $v(z)$ *and* $w(z)$ *are continuous in* $\mathbb{D}$*. That is, the origin is an isolated singularity of* $u(z)$*. Moreover, if* $\kappa(z) \in C^{n-2,\eta}$ *for an integer* $n \geq 2$ *and* $0 < \eta \leq 1$*, then* $u(z) \in C^{n,\eta}$ *and for* $n_1, n_2 \geq 1$*,* $n_1 + n_2 = n$*, near the origin the remainder functions* $v(z)$*,* $w(z)$ *satisfy*

$$\begin{aligned} \partial^n v(z),\ \bar{\partial}^n v(z),\ \bar{\partial}^{n_1}\partial^{n_2} v(z) &= O(|z|^{2-2\alpha-n}), \\ \bar{\partial}^n w(z),\ \partial^n w(z) &= O(|z|^{-n}\log^{-2}(1/|z|)), \\ \bar{\partial}^{n_1}\partial^{n_2} w(z) &= O(|z|^{-n}\log^{-3}(1/|z|)), \end{aligned}$$

where

$$\partial^n = \frac{\partial^n}{\partial z^n}, \quad \bar{\partial}^n = \frac{\partial^n}{\partial \bar{z}^n}$$

for a positive natural number n*.*

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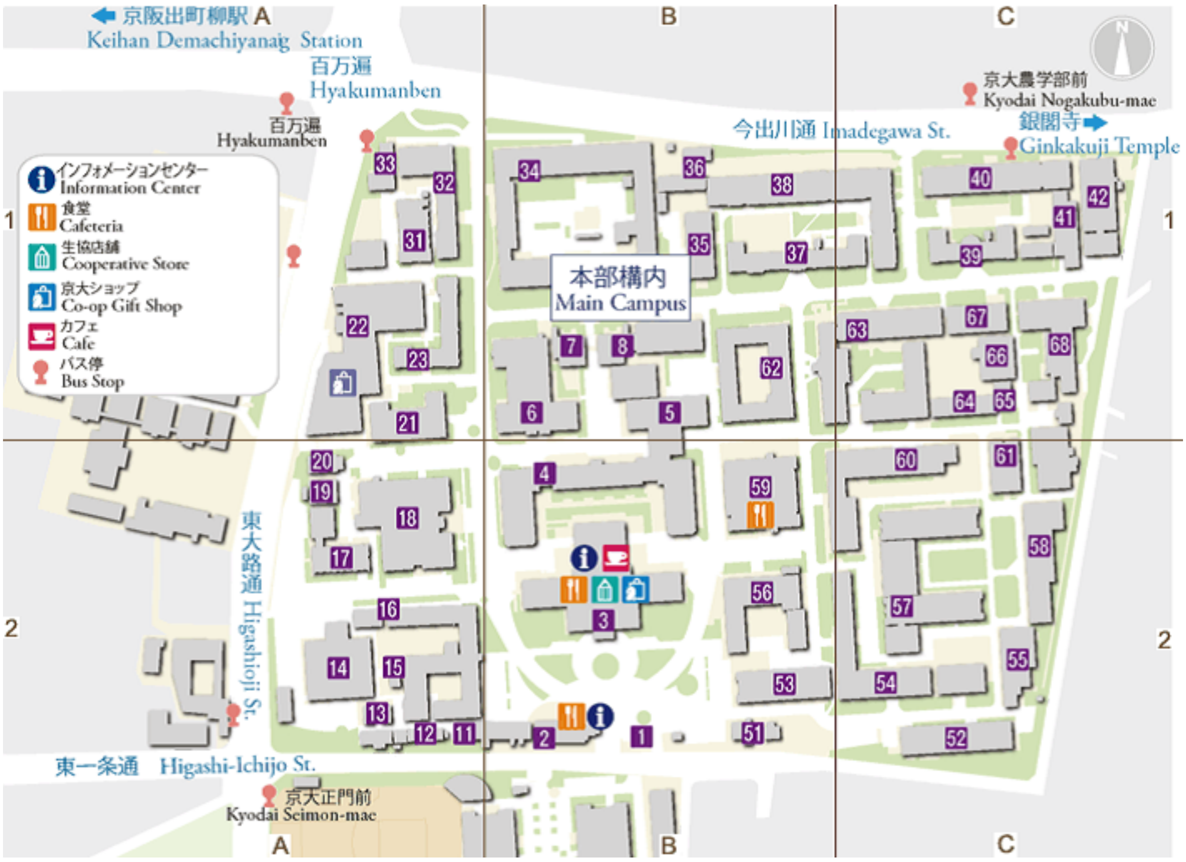
[5] D. Kraus, O. Roth and T. Sugawa, *Metrics with conical singularities on the sphere and sharp extensions of the theorems of Landau and Schottky*, Math. Z. **267** (2011), 851-868.

Main Campus Map



Our room is in the building No. 59 in this map. You can have lunch at cafeterias in the buildings No. 2, 3, 59. Also, you can find restaurants outside the campus. The place of a dinner party is in the building No. 2.