Asymptotic enumeration of large genus maps using random walks

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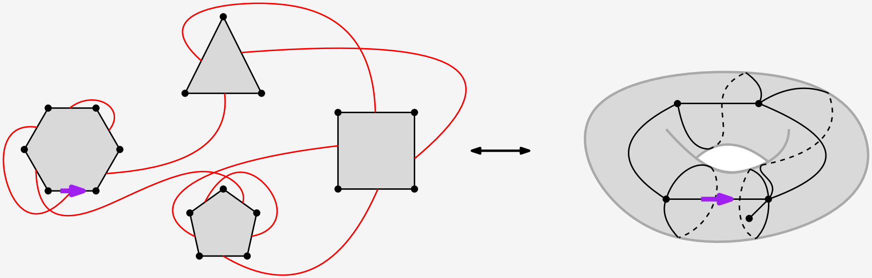
Maps

What are maps?

- A *map* is a gluing of polygons along their edges to form a (compact, connected, oriented) surface.



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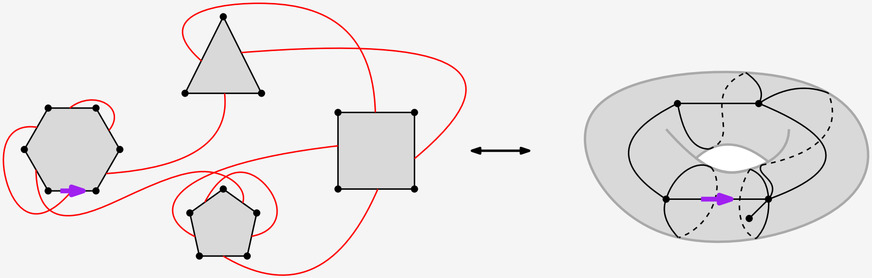

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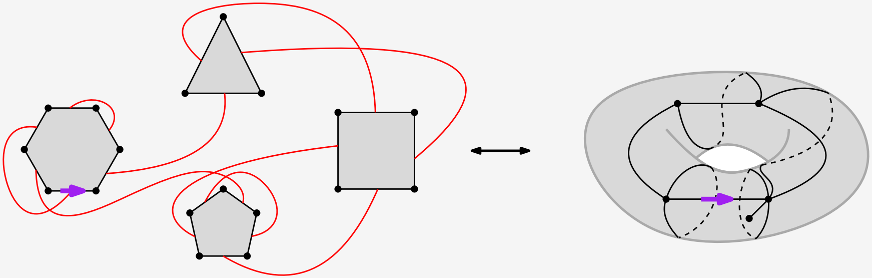
- Euler's formula relates the number of vertices (V), edges (E), faces (F), and the genus g:

$$V - E + F = 2 - 2g.$$



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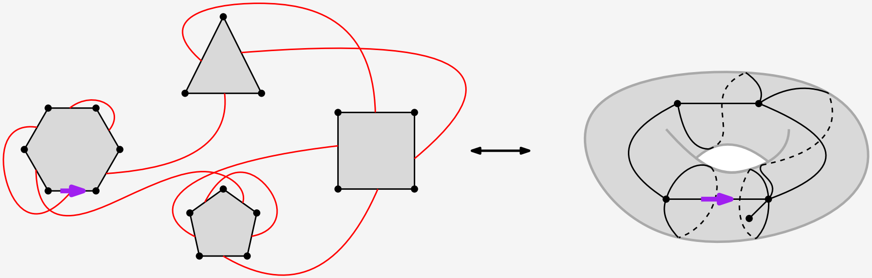

Question

How many maps are there?

Counting maps

- Difficult problem studied by many people [Tutte 63], [Bender, Canfield 86], [Gao 93], [Schaeffer 98], [Bouttier, Di Francesco, Guitter 04], [Chapuy 09], [Chapuy, Marcus, Schaeffer 13], [Albenque, Poulalhon 15], ...
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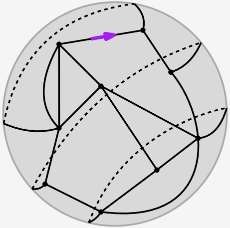
- Let $A(n,g)$ be the number of maps with n edges of genus g
- Exact answer for planar maps, i.e. in genus 0 [Tutte 63]:

$$A(n,0) = \frac{2 \cdot 3^n}{(n+1)(n+2)}\binom{2n}{n}$$



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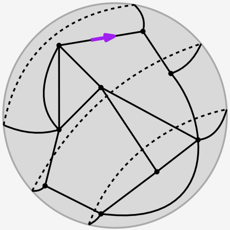

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$$A(n,g) \sim C_g 12^n n^{5/2(g-1)}$$

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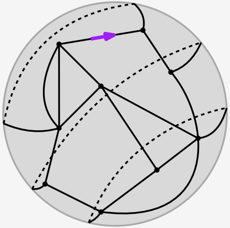

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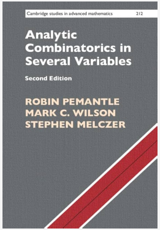
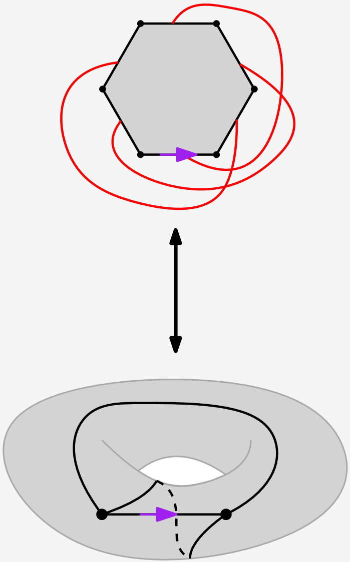


- What if n and g both go to infinity?
  - Difficult question of bivariate asymptotics
  - Recently a lot of progress in Analytic Combinatorics in Several Variables [Pemantle, Wilson, Melczer 2024]
  - Maps do not (yet?) fit in this case.

We focus on unicellular maps



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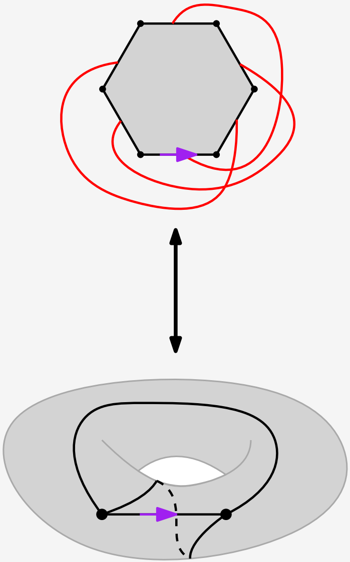

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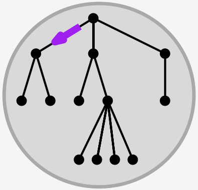

$$E(n, 0) = \frac{1}{n+1}\binom{2n}{n}$$

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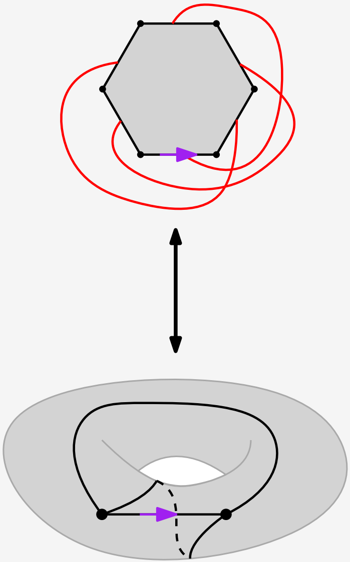

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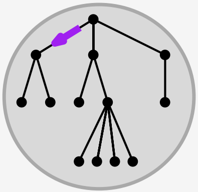

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Our goal

Study the number $E(n, g)$ of unicellular maps with n edges and genus g as $n, g \to \infty$.

Unicellular maps: what is known?

Theorem [Harer, Zagier 86]

Let $E(n,g)$ be the number $E(n,g)$ of unicellular maps with n edges and genus g. Then, $E(0,0) = 1$ and for $n \geq 1$ and $n \geq 2g$ we have

$$(n+1)E(n,g) = 2(2n-1)E(n-1,g) + (n-1)(2n-1)(2n-3)E(n-2,g-1). \qquad \textbf{(HZ)}$$

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Asymptotic enumeration known for:

- g fixed [Walsh, Lehman 72], [Goupil, Schaeffer 98]
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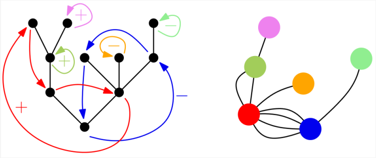
Combinatorial methods:

- Bijection between unicellular maps and decorated trees [Chapuy, Féray, Fusy 12] (second case)
- Core/kernel decomposition (third case)



(Image: G. Chapuy)

Main result: we close the gap and unify the results

Theorem [Elvey Price, Fang, Louf, W 25]

Let $g \equiv g_n$ such that $\frac{n-2g}{\log n} \to \infty$ as $n \to \infty$ (i.e. $n - 2g \gg \log n$). Then

$$E(n,g) \sim \frac{1}{2\sqrt{\pi}} \frac{\sqrt{g}(g/e)^g}{g!} n^{2g-2} e^{nf(\frac{g}{n})} J\left(\frac{g}{n}\right),$$

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$$f(\theta) = -\theta \log(1 - 4\lambda(\theta)) - (1 - 2\theta)\log(\lambda(\theta)) + 2(\log(2) - 1)\theta,$$

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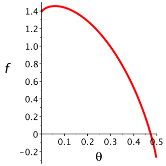

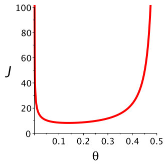

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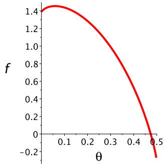
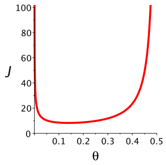
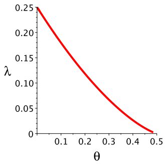
and for every $\theta \in [0,1/2]$ the value $\lambda \equiv \lambda(\theta) \in [0,1/4]$ is the unique value satisfying

$$\theta(\lambda) = \frac{1}{2} - \frac{\lambda \log\left(\frac{1+\sqrt{1-4\lambda}}{1-\sqrt{1-4\lambda}}\right)}{\sqrt{1-4\lambda}} = \frac{1}{2} - 2\lambda \sum_{n\geq 0} \frac{(1-4\lambda)^n}{2n+1}.$$







Proof ideas – the random walk method

Asymptotic guess-and-check (or making $\approx$ to $=$)

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Idea 1: The recursive nature

If we guess an explicit formula $\Omega(n,g)$ that satisfies

1. recurrence (**HZ**) and
2. initial condition $\Omega(0,0) = E(0,0)$,

then $E(n,g) = \Omega(n,g)$ for all n, g.

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Goal: Find explicit asymptotic formula $\Omega(n,g)$ that satisfies

1. **asymptotic recurrence**:

$$(n+1)\Omega(n,g) \approx 2(2n-1)\Omega(n-1,g) + (n-1)(2n-1)(2n-3)\Omega(n-2,g-1)$$

2. **asymptotic initial condition**: $\Omega(n,0) \sim E(n,0)$

Then (hopefully)

$$\Omega(n,g) \sim E(n,g).$$

How to guess $\Omega(n, g)$? Quotients!

1. Use the (**HZ**) to compute $E(n, g)$ exactly for $n, g \leq 1000$

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   1. Let $K \in \mathbb{N}$. Then

$$\frac{E(Kg,g)}{E(K(g-1),g-1)} \qquad \text{grows like } g^2.$$

For example, $E(3g,g) \sim c_1 g^{2g-2} \mu_1^g$ with $c_1 \approx 0.042$ and $\mu_1 \approx 117.92$.

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      $$\frac{E(n,\theta n)}{E(n-1,\theta(n-1))} \qquad \text{converges but depends on } \theta = \frac{g}{n} \in (0,1/2).$$

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3. Plug (**G**) into (**HZ**): This gives for $g = \theta n$ and $n \to \infty$ the (solvable!) differential equation for f:

$$1 = 4e^{-2\theta - f(\theta) + \theta f'(\theta)} + 4e^{-4\theta - 2f(\theta) + 2\theta f'(\theta) - f'(\theta)}.$$

How to prove our guess? Divide!

1. The recurrence (**HZ**) has the generic form

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Goal: Prove that $Q(n,g) \sim 1$ in the following new recurrence

$$Q(n,g) = \alpha(n,g)Q(n-1,g) + \beta(n,g)Q(n-2,g-1),$$

where

$$\alpha(n,g) := a(n,g)\frac{\Omega(n-1,g)}{\Omega(n,g)} \qquad \text{and} \qquad \beta(n,g) := b(n,g)\frac{\Omega(n-2,g-1)}{\Omega(n,g)}.$$

An interpretation in terms of (random) walks

Iterating the recurrence we get

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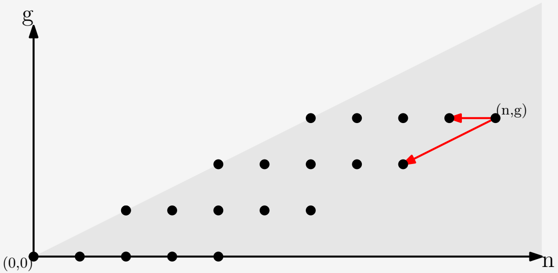

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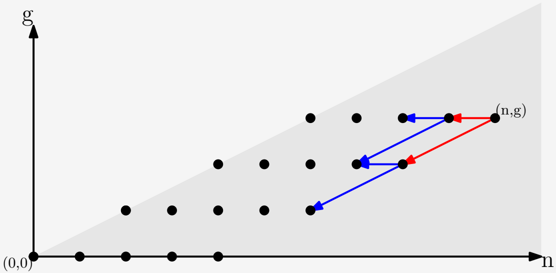

An interpretation in terms of (random) walks

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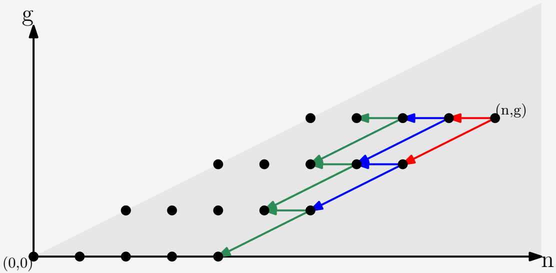

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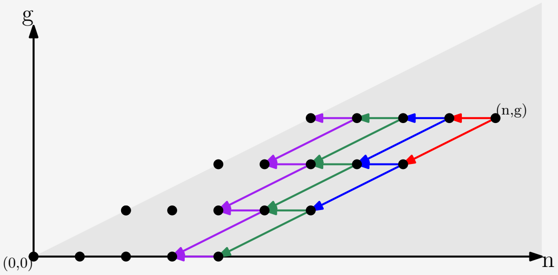

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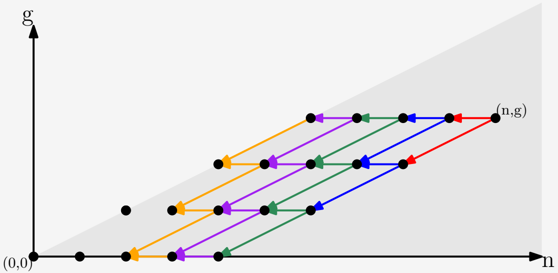

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Q(n,g) &= \alpha(n,g){\color{red}Q(n-1,g)} + \beta(n,g){\color{red}Q(n-2,g-1)}\\
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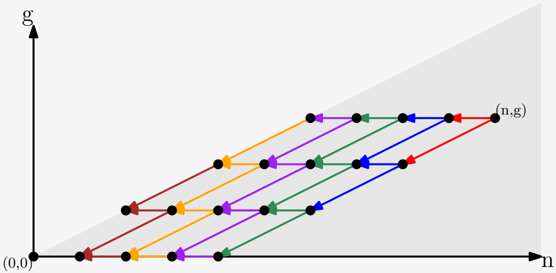

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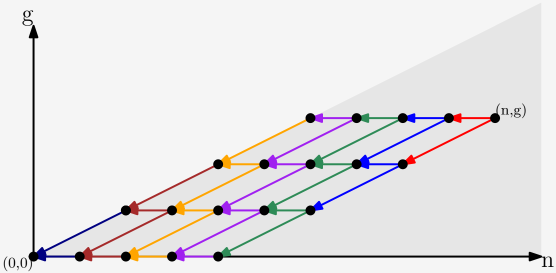

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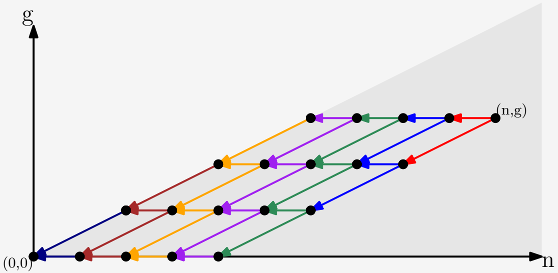

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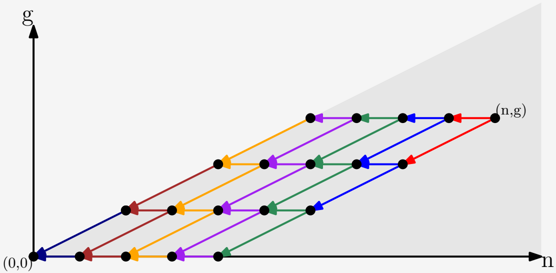

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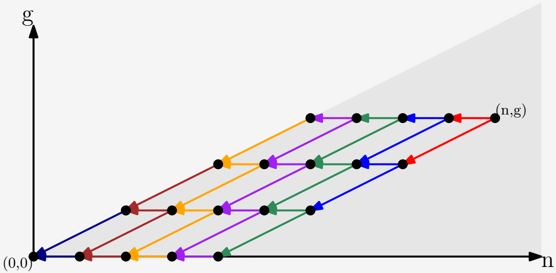

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$\Rightarrow$ **Approximation goal:** Adapt our explicit $\Omega(n,g)$ such that

$$\alpha(n,g) + \beta(n,g) \approx 1.$$

Key property: $\alpha + \beta \approx 1$

Setup:

$$\Omega(n,g) := \frac{1}{2\sqrt{\pi}} \frac{\sqrt{g}(g/e)^g}{g!} n^{2g-2} e^{nf(\frac{g}{n})} J\left(\frac{g}{n}\right) \frac{\sqrt{2\pi}\,(n-2g)^{n-2g+1}}{e^{n-2g}\,\Gamma(n-2g+3/2)},$$

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Proposition (Quasi-probabilities with summable error term)

For $n > 2g$ and $g > 0$:

$$\alpha(n,g) + \beta(n,g) = 1 + \mathcal{O}\left(\frac{1}{n \log(n)^2}\right).$$

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Proof: Homework in Advanced Calculus: Everything is explicit (but very technical). □

$\Rightarrow$ This means the approximation by random walks will be valid!

Defining a "real" random walk $(N_k, G_k)_{k\geq 0}$ that approximates ours

Setup: Start from $(N_0, G_0) = (n, g)$ and stop when $G_k = 0$ or $N_k = 2G_k$.

$$(N_{k+1}, G_{k+1}) = (N_k - 1, G_k) \quad \text{with probability} \quad \frac{\alpha(N_k, G_k)}{\alpha(N_k, G_k) + \beta(N_k, G_k)},$$

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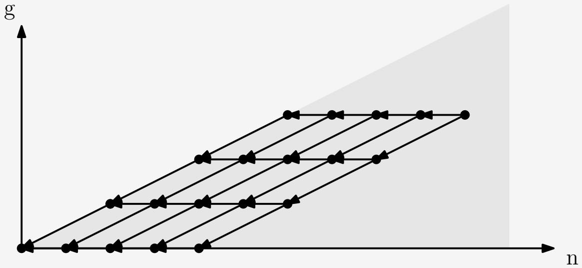

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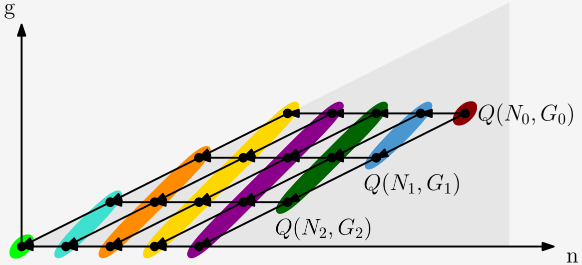
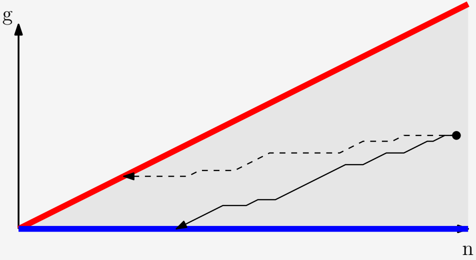


Conserved quantity

$$\mathbb{E}\left(Q(N_{k+1}, G_{k+1})\right) \approx \mathbb{E}\left(Q(N_k, G_k)\right)$$

Typical behavior of the random walk

Let $\tau = \tau(n, g)$ be the stopping time, i.e., when the walk hits an axis: $G_k = 0$ or $N_k = 2G_k$.



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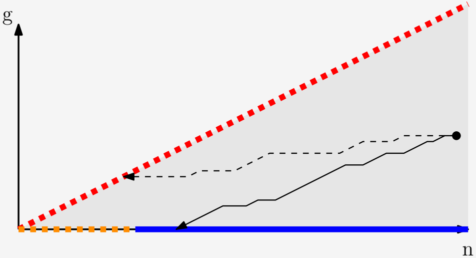

Proposition (Hitting $g = 0$ with high prob.)

For $\frac{n-2g}{\log n} \to \infty$ as $n \to \infty$ and any fixed $L > 0$:

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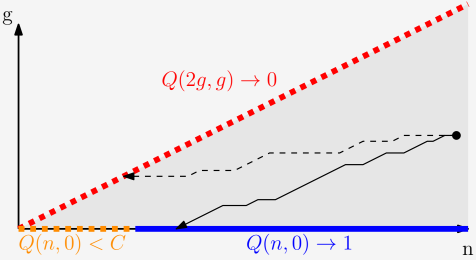

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Proof of our main result as a corollary:

Recall $Q(n, g) = \frac{E(n,g)}{\Omega(n,g)}$.

1. The asymptotic initial condition holds for $g = 0$: $Q(n, 0) \to 1$ for $n \to \infty$

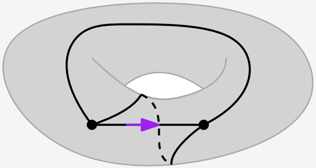
$$\text{Trees for genus 0:} \quad E(n, 0) = \frac{1}{n+1}\binom{2n}{n} \qquad \text{and} \qquad \Omega(n, 0) \sim \frac{4^n}{\sqrt{\pi n^3}}$$

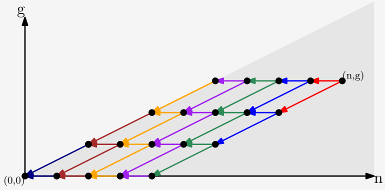
2. The random walk hits the right axis:

$$Q(n, g) \overset{\text{(Def.)}}{=} Q(N_0, G_0) \overset{\binom{\text{Conserved}}{\text{quantity}}}{=} \mathbb{E}(Q(N_\tau, G_\tau)) + o(1) \overset{\binom{\text{RW hits}}{g=0 \text{ w.h.p.}}}{=} 1 + o(1). \qquad \square$$

Conclusion

- Idea: Approximating a linear bivariate recurrence by a random walk
- Method works also for regime $n - 2g \ll \log(n)$, but asymptotics changes
  $\rightarrow$ Phase transition at $n - 2g = \Theta(\log(n))$
- Robust method: independent of combinatorics of the model or "simple" generating function
  Can we characterize asymptotics for classes of recurrences?
- Perspectives
  - linear $\rightarrow$ quadratic
  - other coefficients: e.g., depending on n and g
  - different steps, drift, boundary behavior





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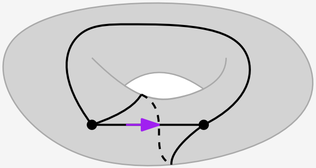

Thank you!