Polytopal Perspectives on the Alexander Polynomial of Special Alternating Links

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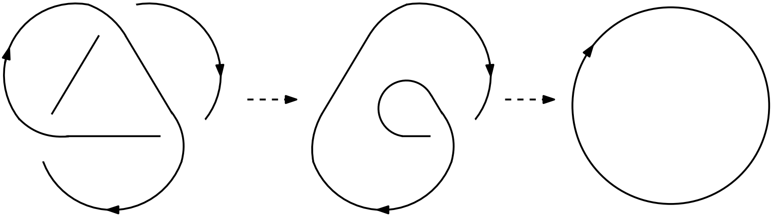
Joint work with Elena S. Hafner and Karola Mészáros

Knot Theory Basics

- We ordinarily present links in two dimensions via their *diagrams*. Link diagrams are incredibly non-unique.



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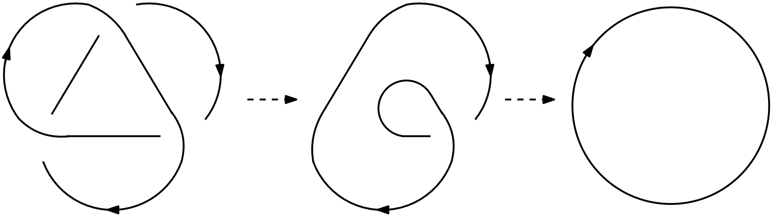
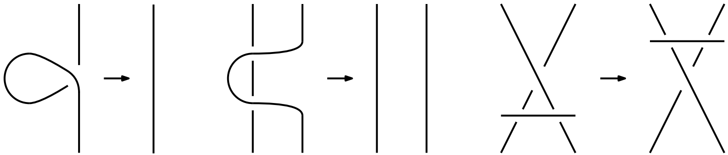


- The primary focus of study for us is an **isotopy invariant**; namely, it is a polynomial $\Delta_L(t)$ defined for a link L such that if L and L' are isotopic, then $\Delta_L(t)$ and $\Delta_{L'}(t)$ are "equivalent."

Knot Theory Basics

Theorem (Reidemeister, 1927)

Fix links L and L', as well as diagrams for each. L and L' are isotopic if and only if their diagrams can be related by sequential application of the following three Reidemeister moves.

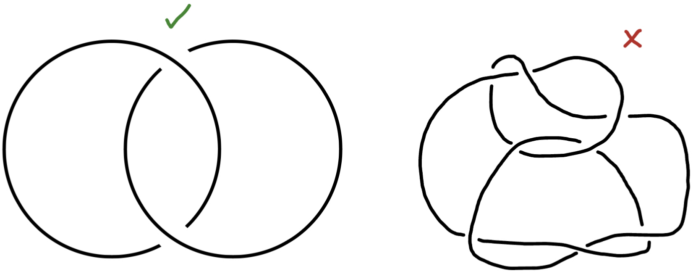


Alternating Links

Definition

A link diagram is **alternating** if, when tracing each component of the link, crossings alternate under and over.
A link is alternating if it admits an alternating diagram.

Examples:



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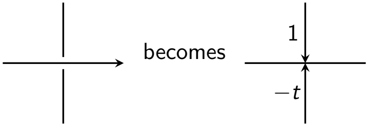
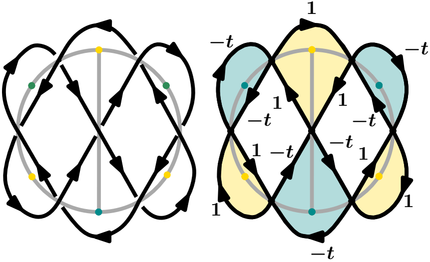
Many combinatorial methods exist for computing the Alexander polynomial. Let's work on an example.

A Combinatorial Formula (Crowell, 1959)

Let L be an alternating link. Let $\overrightarrow{G(L)}$ be the edge-weighted digraph obtained by the following convention:

Example:



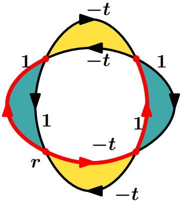


Definition

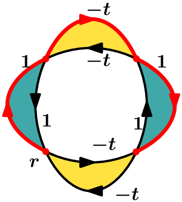
Given a directed planar graph G and a vertex r of G, an **arborescence rooted at** r is a connected subgraph A such that, for each vertex v, there is a unique directed path from r to v in A.

A Combinatorial Formula (Crowell, 1959)

Example: Hopf Link



$\prod_{e \in E(T_1)} var(e) = -t$



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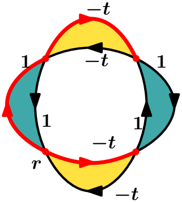

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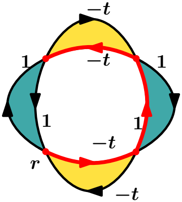

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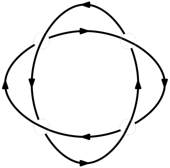
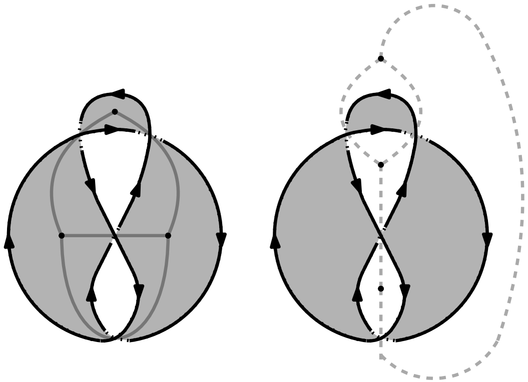


$\Delta_L(t) \sim 2t^2 - 2t$

Special Alternating Links - Checkerboard Graph

Example:



Special Alternating Links - Checkerboard Graph

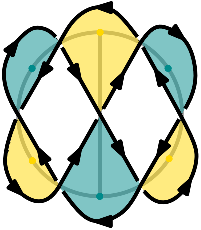
Polytopes in the literature: *Sutured Floer homology and hypergraphs* (Juhász-Kálmán-Rasmussen, 2011)

Definition

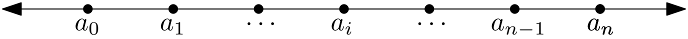
A **special alternating** link admits an alternating diagram such that one of its checkerboard graphs is bipartite.

Convention: The bipartite graph is the one without the exterior.

Example:



Trapezoidal and Log-Concave Sequences



Trapezoidal: $a_0 < a_1 < a_k = \cdots = a_m > a_{m+1} > \ldots a_n$ for some k and m.

Example: $1, 2, 3, 6, 6, 6, 4, 1$

Log-concave: $a_i^2 \geq a_{i-1} a_{i+1}$ for all i.

Positive and log-concave implies trapezoidal.

Part 1: Fox's Conjecture and Generalized Permutahedra

Conjecture (Fox, 1962)

Let L be an alternating link. Then the absolute values of the coefficients of $\Delta_L(-t)$ form a trapezoidal sequence.

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The coefficients of the Alexander polynomial $\Delta_L(-t)$ of a special alternating link L form a log-concave sequence with no internal zeros. In particular, they are trapezoidal.

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Polytopal Perspective, in Brief:

- *Crowell's* method for computing the Alexander polynomial $\rightarrow$ multivariate "M-polynomial"
- This "M-polynomial" is supported on a generalized permutahedron $\rightarrow$ Lorentzian polynomial

Part 2: Eulerian Directed Graphs and Root Polytopes

Murasugi and Stoimenow define the *Alexander polynomial* $P_H(t)$ for *any* Eulerian digraph H.

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Theorem (Hafner–Mészáros–V., 2024)

Let H be an Eulerian digraph, and let M be the oriented graphic matroid associated to H. Let A_H be a totally unimodular matrix representing M^*, the oriented dual of M, and let m be the size of a basis of M^*. Then,

$$P_H(t) = \sum_{A' \text{ has property } *} Vol(\mathcal{Q}_{A'})(t-1)^{\#col(A')-m}, \qquad (1)$$

*where a matrix A' has property * if it is obtained by deleting a set of columns from A_H without decreasing the rank of the matrix, and $\mathcal{Q}_{A'}$ denotes the root polytope of A'.*

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Conjecture (Stoimenow, 2014)

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Coefficients of $\Delta_L(t)$ alternate in sign for alternating links L (Crowell, Murasugi).

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- Certain diagrammatic Murasugi sums of special alternating links (Azarpendar–Juhász–Kálmán, 2024)
- Special alternating links, using different methods (Kálmán–Mészáros–Postnikov, 2025)

The Idea

Theorem (Hafner–Mészáros–V., 2023)

The coefficients of the Alexander polynomial $\Delta_L(-t)$ of a special alternating link L form a log-concave sequence with no internal zeros. In particular, they are trapezoidal.

- Define a multivariate Alexander polynomial that we can show is supported on the lattice points of a generalized permutahedron and has $0, 1$ coefficients $\to$ this polynomial is (denormalized) *Lorentzian*

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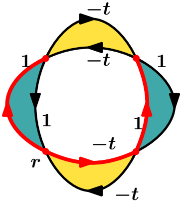
- Define it so that after specializing it back to the (homogenized) Alexander polynomial we preserve the denormalized Lorentzian property
- Thus conclude log-concavity of the Alexander polynomial



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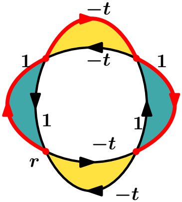

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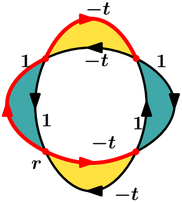

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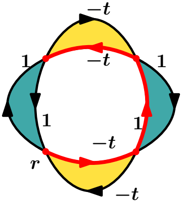

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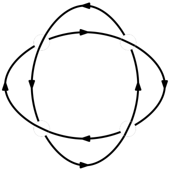

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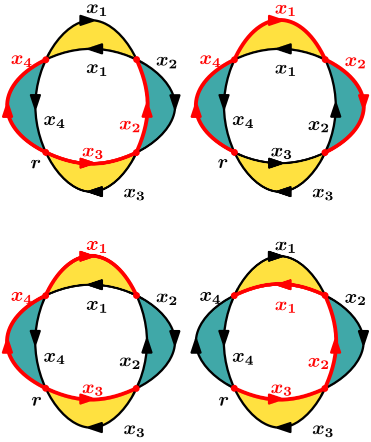
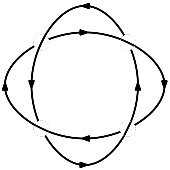
From Alexander to M-Polynomial





$$M_{\overrightarrow{G(L)},r}(x_1, x_2, x_3, x_4) = x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2$$

The M-Polynomial of a Special Alternating Link

Definition (Hafner–Mészáros–V., 2023)

Let L be a special alternating link, and let $r \in \mathrm{V}(\overrightarrow{G(L)})$. Let $C_1, \dots C_k$ denote the set of clockwise oriented cycles bounding planar regions of the alternating dimap $\overrightarrow{G(L)}$. Each edge $e \in E(\overrightarrow{G(L)})$ belongs to exactly one of the C_i. Assign a variable $var(e) = x_i$ to each edge $e \in C_i$, $i \in [k]$. Define

$$M_{\overrightarrow{G(L)},r}(x_1, \dots, x_k) = \sum_{A \in \mathcal{A}(\overrightarrow{G(L)},r)} \prod_{e \in E(A)} var(e). \quad (2)$$

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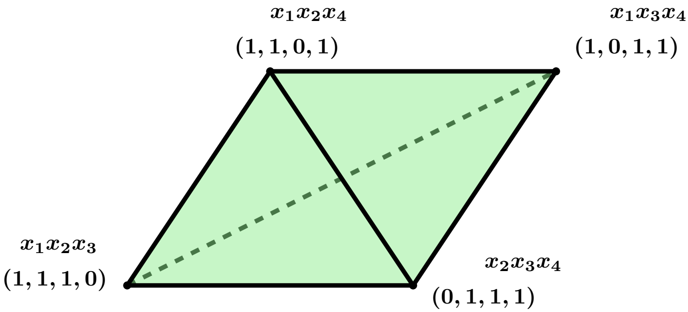
$$M_{\overrightarrow{G(L)},r}(x_1, \ldots, x_k) = \sum_{A \in \mathcal{A}(\overrightarrow{G(L)},r)} \prod_{e \in E(A)} var(e). \qquad (2)$$

This polynomial is

- independent of the choice of root
- has support the lattice points of a generalized permutahedron
- $0, 1$ coefficients

Specialization

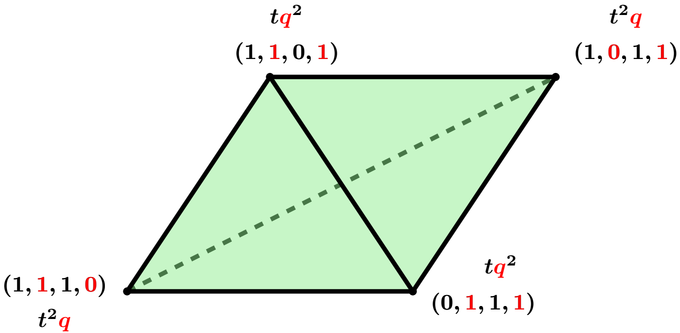
$$M_{\overrightarrow{G(L)}}(x_1, x_2, x_3, x_4) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$$



$$\{(p_1, p_2, p_3, p_4) \in \mathbb{R}^4 \mid p_1 + p_2 + p_3 + p_4 = 3\}$$

Specialization

$$M_{\overrightarrow{G(L))}}(t, q, t, q) = 2tq^2 - 2t^2q$$



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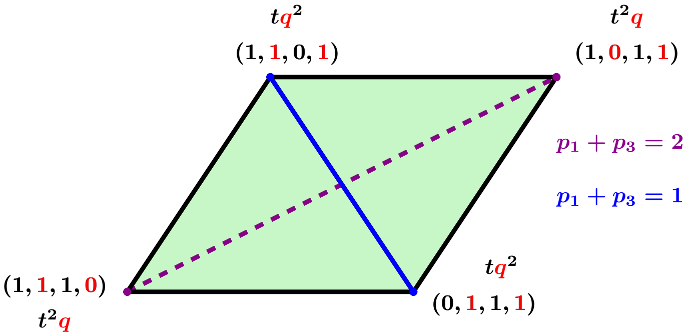
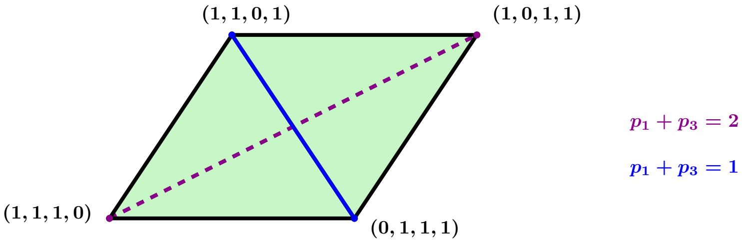


$$\{(p_1,p_2,p_3,p_4) \in \mathbb{R}^4 \mid p_1+p_2+p_3+p_4 = 3\}$$

Takeaway

The coefficients of $\Delta_L(-t)$ count the number of lattice points in the Newton polytope of $M_{\overrightarrow{G(L)}}(x_1, \ldots, x_k)$ which intersect the hyperplanes $x_{i_1} + x_{i_2} + \ldots x_{i_k} = d$, $d \in \mathbb{Z}$.

$\Delta_L(-t) \sim 2t + 2t^2$



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Part 2: Eulerian Directed Graphs and Root Polytopes

Alexander Polynomial in terms of Volumes

The following theorem follows from work by Li and Postnikov (2013) and Kálmán–Mészáros–Postnikov (2025). We give a different proof, as well as a generalization.

Theorem

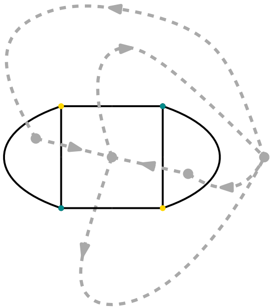
Let L be a special alternating link diagram with bipartite checkerboard graph G. The Alexander polynomial of L can be written as:

$$\Delta_L(-t) \sim \sum_{\substack{K \subset G \\ K \text{ connected}}} Vol(\mathcal{Q}_K)(t-1)^{|E(K)|-|V(G)|+1},$$

where $\mathcal{Q}_K$ denotes the root polytope of K.

Bipartite and Eulerian Graphs

The dual of any planar bipartite graph is Eulerian. It can furthermore be oriented as an **alternating dimap**.



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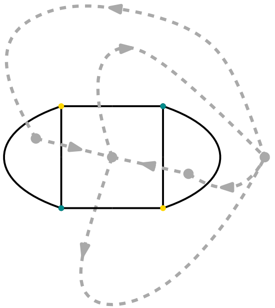

Theorem (Postnikov; Kálmán–Tóthmérész; Tóthmérész)

Let G be a planar bipartite graph with dual H, oriented as an alternating dimap. The normalized volume $Vol(\mathcal{Q}_G)$ is the number of arborescences of H.

Another Combinatorial Formula (Murasugi–Stoimenow, 2003)

Let L be a special alternating link. Orient the (Eulerian) dual of the bipartite checkerboard graph, H.

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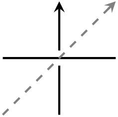

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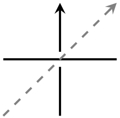

Again, fix a root $r \in \mathrm{V}(H)$. Again, sum over spanning trees. Each spanning tree incurs weight $t^{\#\{\text{edges pointing "towards" } r\}}$.

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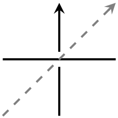

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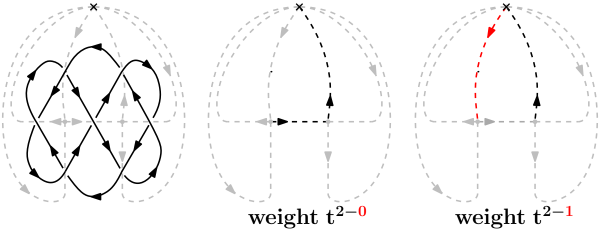

Another Combinatorial Formula (Murasugi–Stoimenow, 2003)

Note: A "k-spanning tree" refers to a spanning tree with $|\mathrm{V}(H)| - 1 - k$ edges pointing "towards" the chosen root.

Proposition (Murasugi–Stoimenow, 2003)

Let L be a special alternating link and fix a special alternating diagram with dual checkerboard graph H. For each $k \in \mathbb{Z}$, let $c_k(H, r)$ denote the number of k-spanning trees of H. Then,

$$\Delta_L(-t) \sim \sum_{k=0}^{|\mathrm{V}(H)|-1} c_k(H, r) t^{|\mathrm{V}(H)|-1-k}.$$

Proof Outline

Let G be a bipartite graph with oriented dual H and associated special alternating link L.

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1. Recall $Vol(\mathcal{Q}_G) = c_0(H, r)$.
2. By an inclusion-exclusion argument: $c_k(H, r) = \sum_{i=0}^{k}(-1)^i \sum_{\substack{\text{acyclic } E' \subset E(H) \\ |E'|=k-i}} \binom{|V(H)|-1-(k-i)}{i} c_0(H/E', r)$.

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4. By standard results in graph theory, deleting elements of a graph is equivalent to contracting the corresponding elements in its dual.

In Order to Generalize

- Alternating dimap $\to$ Eulerian digraph

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- Alternating dimap $\rightarrow$ Eulerian digraph
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- Root polytope of bipartite dual $\rightarrow$ Root polytope of the co-Eulerian dual oriented matroid of the graphic matroid of H (Tóthmérész, 2022)

Our Generalization

Theorem (Hafner–Mészáros–V., 2024)

Let H be an Eulerian digraph, and let M be the oriented graphic matroid associated to H. Let A_H be a totally unimodular matrix representing M^*, the oriented dual of M, and let m be the size of a basis of M^*. Then,

$$P_H(t) = \sum_{A' \text{ has property } *} Vol(\mathcal{Q}_{A'})(t-1)^{\#col(A')-m}, \tag{3}$$

*where a matrix A' has property * if it is obtained by deleting a set of columns from A_H without decreasing the rank of the matrix, and $\mathcal{Q}_{A'}$ denotes the root polytope of A'.*

Thank you!

Lorentzian Polynomials

Lorentzian Polynomials

The theory of Lorentzian polynomials was developed by Petter Brändén and June Huh and, in parallel, by Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant. In this talk, we follow Brändén and Huh's convention.

Definition (Brändén–Huh 2019)

A homogeneous polynomial f of degree d with nonnegative coefficients is *Lorentzian* if

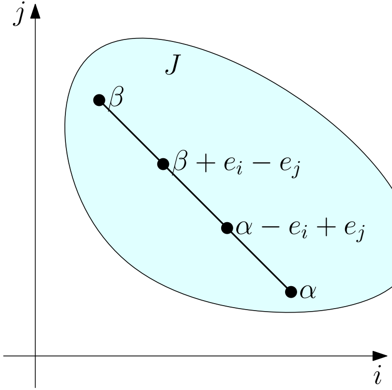
- f has M-convex support
- $\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_{d-2}}} f$ has at most one positive eigenvalue.

M-Convexity

Definition

A subset $J \subseteq \mathbb{Z}^n$ is *M-convex* if for any $\alpha, \beta \in J$ and any index i with $\alpha_i > \beta_i$, there is an index j satisfying

$$\alpha_j < \beta_j, \quad \alpha - e_i + e_j \in J, \quad \text{and} \quad \beta + e_i - e_j \in J.$$



f has SNP

+

Newton(f) is a generalized permutahedron

↕

f has M-convex support

Example and Nonexample

$f(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$

Matrix: $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$

Eigenvalues: $1/2$ and $3/2$

Nonexample!

$N(f(x_1, x_2)) = \frac{x_1^2}{2} + x_1 x_2 + \frac{x_2^2}{2}$

Matrix: $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$

Eigenvalues: 0 and 1

Example!

Lorentzian Polynomials and Log-Concavity

Definition

Let N be the *normalization operator* defined by

$$N({x_1}^{\alpha_1} \cdots {x_n}^{\alpha_n}) = \frac{{x_1}^{\alpha_1} \cdots {x_n}^{\alpha_n}}{\alpha_1! \cdots \alpha_n!}.$$

Extend linearly.

Theorem (Brändén–Huh 2019)

If $f = \sum_\alpha c_\alpha x^\alpha$ is nonzero and $N(f)$ is Lorentzian, then

- *(Discrete) For every α and $1 \leq i, j \leq n$,*

$$c_\alpha^2 \geq c_{\alpha+e_i-e_j} c_{\alpha-e_i+e_j}.$$

Namely, a single-variable polynomial whose normalized homogenization is Lorentzian has log-concave coefficients.

Proof Outline of M-polynomial Support

Proof Outline

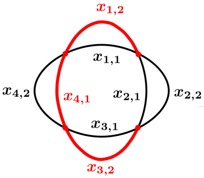
1 Denote

- $\mathcal{T}(G(L))$: (unoriented) spanning trees of $G(L)$
- $e_{i,1}, \ldots, e_{i,|C_i|}$: edges of clockwise oriented cycle C_i.

The *integer point enumerator* of the spanning tree polytope of $G(L)$ is

$$\sigma_{G(L)}(x_{1,1}, \ldots, x_{n,|C_k|}) = \sum_{T \in \mathcal{T}(G(L))} \prod_{e_{i,j} \in E(T)} x_{i,j}.$$

Its support is the set of lattice points of a generalized permutahedron.



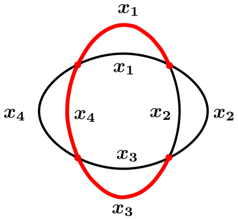
$$\begin{aligned}\sigma_{G(L)}(\mathbf{x}) = {} & x_{1,1}x_{2,1}x_{3,1} + x_{2,1}x_{3,1}x_{4,1} + x_{3,1}x_{4,1}x_{1,1} + x_{4,1}x_{1,1}x_{2,1} \\ & + x_{1,2}x_{2,1}x_{3,1} + x_{2,2}x_{3,1}x_{4,1} + x_{3,2}x_{4,1}x_{1,1} + x_{4,2}x_{1,1}x_{2,1} \\ & + x_{1,1}x_{2,2}x_{3,1} + x_{2,1}x_{3,2}x_{4,1} + x_{3,1}x_{4,2}x_{1,1} + x_{4,1}x_{1,2}x_{2,1} \\ & + x_{1,1}x_{2,1}x_{3,2} + x_{2,1}x_{3,1}x_{4,2} + x_{3,1}x_{4,1}x_{1,2} + x_{4,1}x_{1,1}x_{2,2} \\ & + x_{1,2}x_{2,2}x_{3,1} + x_{2,2}x_{3,2}x_{4,1} + x_{3,2}x_{4,2}x_{1,1} + x_{4,2}x_{1,2}x_{2,1} \\ & + x_{1,1}x_{2,2}x_{3,2} + x_{2,1}x_{3,2}x_{4,2} + x_{3,1}x_{4,2}x_{1,2} + x_{4,1}x_{1,2}x_{2,2} \\ & + x_{2,2}x_{2,1}x_{3,2} + x_{2,2}x_{3,1}x_{4,2} + x_{3,2}x_{4,1}x_{1,2} + x_{4,2}x_{1,1}x_{2,2} \\ & + x_{2,2}x_{2,2}x_{3,2} + x_{2,2}x_{3,2}x_{4,2} + x_{3,2}x_{4,2}x_{1,2} + x_{4,2}x_{1,2}x_{2,2}\end{aligned}$$

Proof Outline

2 Specialize to

$$f_{G(L)}(x_1, \dots, x_k) = \sum_{T \in \mathcal{T}(G(L))} \prod_{i=1}^{k} x_i^{a_i(T)},$$

where $a_i(T)$ is the number of edges of T in the cycle C_i, $i \in [k]$. This polynomial also has support the set of lattice points of a generalized permutahedron.

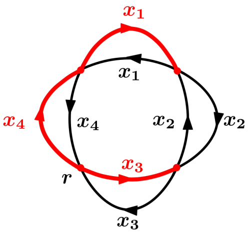


$$f_{G(L)}(\mathbf{x}) = 8x_1x_2x_3 + 8x_2x_3x_4 + x_3x_4x_1 + 7x_3x_4x_1 + 8x_4x_1x_2$$

Proof Outline

3 The polynomials $f_{G(L)}(x_1, \dots, x_k)$ and $M_{\overrightarrow{G(L)},r}$ have the same support.

4 The coefficients of $M_{\overrightarrow{G(L)},r}(x_1, \dots, x_k)$ are all either 0 or 1.



$$M_{\overrightarrow{G(L)},r}(\mathbf{x}) = x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2$$

Proof Outline

1. The support of $\sigma_{G(L)}(x_{1,1}, \ldots, x_{n,|C_k|}) = \sum_{T \in \mathcal{T}(G(L))} \prod_{e_{i,j} \in E(T)} x_{i,j}$ is the set of lattice points of a generalized permutahedron
2. The specialization

$$f_{G(L)}(x_1, \ldots, x_k) = \sum_{T \in \mathcal{T}(G(L))} \prod_{i=1}^{k} x_i^{a_i(T)},$$

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Let $\{C_1,\ldots,C_l\}$ and $\{C_{l+1},\ldots,C_k\}$ be the elements of $\{C_1,\ldots,C_k\}$ labeled with $-t$'s and 1's respectively. So, $\mathrm{Homog}_q(\Delta_L(-t)) \sim M_{\overrightarrow{\mathcal{G}(L)}}(t,\ldots,t,q,\ldots,q)$ is denormalized Lorentzian,

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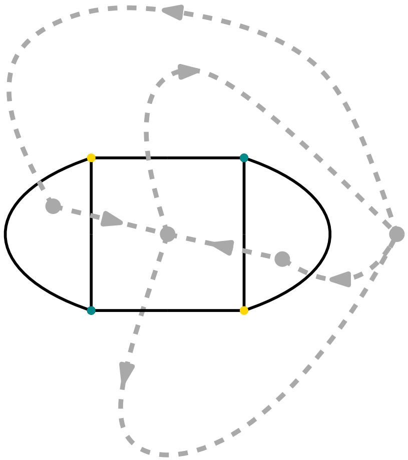
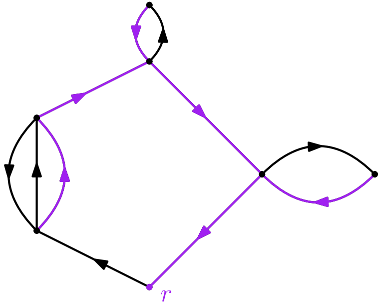
More on Murasugi–Stoimenow

Bipartite and Eulerian Graphs I

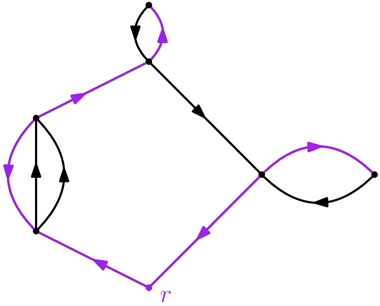
The dual of any bipartite graph is Eulerian. It can furthermore be oriented as an **alternating dimap**.



Oriented spanning tree:



4-spanning tree:



Remark

Let H be a digraph. An arborescence of H is a $|\mathrm{V}(H)| - 1$-spanning tree.

Another Combinatorial Formula (Murasugi–Stoimenow, 2003)

Using a special alternating diagram of a special alternating link L, we can compute $\Delta_L(-t)$ *exclusively* using the checkerboard dual.

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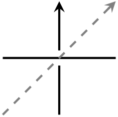
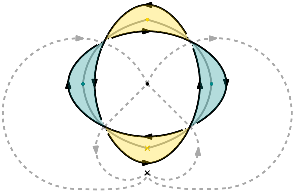
Orient the dual checkerboard graph as follows.



This yields an alternating dimap, dual to the bipartite checkerboard graph



Another Combinatorial Formula (Murasugi–Stoimenow, 2003)

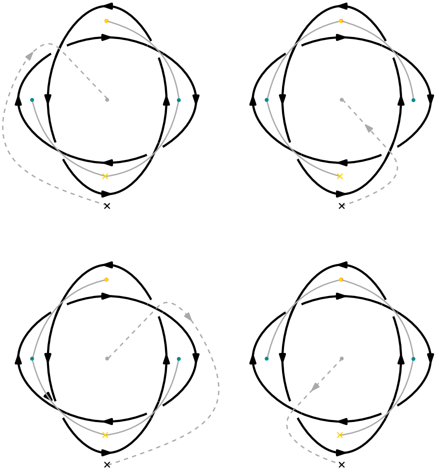
The Alexander polynomial of special alternating links can be written in terms of its dual checkerboard graph.

Proposition

Let L be a special alternating link and fix a special alternating diagram with dual checkerboard graph H. For each $k \in \mathbb{Z}$, let $c_k(H, r)$ denote the number of k-spanning trees of H. Then,

$$\Delta_L(-t) \sim \sum_{k=0}^{|V(H)|-1} c_k(H, r) t^{|V(H)|-1-k}.$$

Example



$c_1(H, r) = 2$

$c_0(H, r) = 2$

$$\Delta_L(-t) \sim 2t + 2$$

More on Oriented Matroids

Generalization to Eulerian Digraphs

Remark

Every alternating dimap is the dual checkerboard graph of a special alternating link, oriented in Murasugi and Stoimenow's convention.

One can generalize to all Eulerian digraphs as follows, for which alternating dimaps are a special case.

Definition (Murasugi–Stoimenow, 2003)

Let H be an Eulerian digraph. The **Alexander polynomial** of H is

$$P_H(t) = \sum_{k=0}^{|\mathrm{V}(H)|-1} c_k(H,r)t^{|\mathrm{V}(H)|-1-k}$$

Oriented Matroid Basics

Let M be a regular matroid represented by the totally unimodular matrix A, with columns $\{a_1, \ldots, a_m\}$. For each circuit $C = \{i_1, \ldots, i_j\}$ with a corresponding linear dependence relation $\sum_{k=1}^{j} \lambda_k a_{i_k} = 0$, we may partition the elements into two sets: $C^+ = \{i_k \mid \lambda_k > 0\}$ and $C^- = \{i_k \mid \lambda_k < 0\}$. We orient M with these bipartitions, making it an *oriented matroid*.

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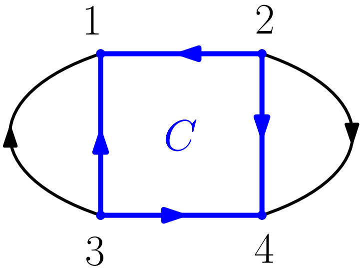
For this result, we deal only with regular oriented matroids.

Definition

Each oriented matroid M on groundset E admits a unique **dual oriented matroid** on groundset E such that, for each pair of signed circuits $C_1 = {C_1}^+ \sqcup {C_1}^-$ and $C_2 = {C_2}^+ \sqcup {C_2}^-$ of M and M^*, respectively, either $C_1 \cap C_2 = \emptyset$, or $({C_1}^+ \cap {C_2}^+) \cup ({C_1}^- \cap {C_2}^-)$ and $({C_1}^+ \cap {C_2}^-) \cup ({C_1}^- \cap {C_2}^+)$ are both nonempty.

Example

The following is an example of a **graphic** oriented matroid. (A is the incidence matrix of a directed graph.)



$$\begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix}$$

$\mathrm{a}_1 \quad \mathrm{a}_2 \quad \quad \mathrm{a}_3 \quad \mathrm{a}_4$

$$\mathrm{a}_1 - a_2 + a_3 - a_4 = 0$$
$$\mathrm{C}^+ = \{a_1,\ a_3\},\ C^- = \{a_2,\ a_4\}$$

Oriented Co-Eulerian Matroids

Definition (Tóthmérész, 2022)

A regular oriented matroid is **co-Eulerian** if for each circuit C, $|C^+| = |C^-|$.

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For graphic matroids of bipartite graphs, oriented so that all edges point out of one color class and into the other, their dual oriented matroids are also graphic, and are the matroids of alternating dimaps.

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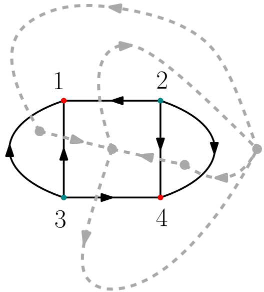

Root Polytopes: Bipartite Graphs

Definition (Postnikov, 2005)

Given a bipartite graph G on vertex set $[n] \sqcup [\overline{m}]$, the **root polytope** $\mathcal{Q}_G$ is the convex hull of vectors $(e_i - e_{\bar{j}})$, $\{i, \bar{j}\} \in \mathrm{E}(G)$ in R^{n+m}.

Theorem (Postnikov; Kálmán–Tóthmérész; Tóthmérész)

Let G be a bipartite graph and let $H = G^*$, oriented as an alternating dimap. The (normalized) volume $Vol(\mathcal{Q}_G)$ is the number of arborescences of H.

Root Polytopes: More Generally

Definition (Tóthmérész, 2022)

Let A be a totally unimodular matrix with columns $a_1, \ldots, a_m$. The **root polytope** of A is the convex hull $\mathcal{Q}_A := conv(a_1, \ldots, a_m)$.

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Proposition (Tóthmérész, 2022)

For a co-Eulerian regular oriented matroid M represented by a totally unimodular matrix A and a basis $B = \{i_1, \ldots, i_j\}$ of M, the simplex $\Delta_B := conv(a_{i_1}, \ldots, a_{i_j})$ is unimodular. That is, its normalized volume is 1.

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Theorem (Tóthmérész, 2022)

Let H be an Eulerian digraph, and let A be any totally unimodular matrix representing the oriented dual of the oriented graphic matroid of H. Let $r \in V(H)$ and

$\mathcal{H} = \{B \subset E(H) \mid E(H) - B$ *is an arborescence of H rooted at r*$\}$.

Then $\{\Delta_B \mid B \in \mathcal{H}\}$ is a triangulation of $\mathcal{Q}_A$.

Check-In

Recall we are aiming to prove the following result.

Theorem (Hafner–Mészáros–V., 2024)

Let H be an Eulerian digraph, and let M be the oriented graphic matroid associated to H. Let A_H be a totally unimodular matrix representing M^*, the oriented dual of M, and let m be the size of a basis of M^*. Then,

$$P_H(t) = \sum_{A' \text{ has property } *} Vol(\mathcal{Q}_{A'})(t-1)^{\#col(A')-m}, \tag{4}$$

*where a matrix A' has property * if it is obtained by deleting a set of columns from A_H without decreasing the rank of the matrix, and $\mathcal{Q}_{A'}$ denotes the root polytope of A'.*

Proof Outline

1 Tóthmérész proves, by the aforementioned triangulation, that $Vol(\mathcal{Q}_{A_H}) = c_0(H, r)$.

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2 By an inclusion-exclusion argument: $c_k(H, r) = \sum_{i=0}^{k} (-1)^i \sum_{\substack{\text{acyclic } E' \subset E(H) \\ |E'|=k-i}} \binom{|V(H)|-1-(k-i)}{i} c_0(H/E', r)$.

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3. This and the result by Tóthmérész above prove that $c_k(H, r) = \sum_{i=0}^{k}(-1)^i \sum_{\substack{\text{acyclic } E' \subset E(H) \\ |E'|=k-i}} \binom{|V(H)|-1-(k-i)}{i} Vol(\mathcal{Q}_{A_{H/E'}})$.

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4. By standard results in matroid theory, deleting elements of M^* (or columns of A) is equivalent to contracting the corresponding elements in M.