



# Chow functions for partially ordered sets

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Build a new class of polynomials associated to various objects in combinatorics.

- Do these polynomials have interesting features?
- When you specialize these polynomials to specific cases do you recover any known result?

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iii) Coxeter groups

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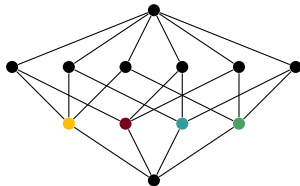
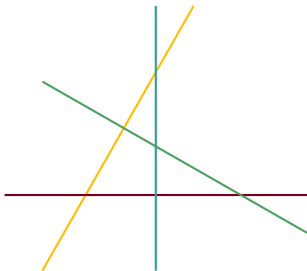
iii) Coxeter groups

Joint work with Luis Ferroni (IAS – University of Pisa) and Jacob Matherne (NCSU).

# Posets

# Hyperplane arrangements

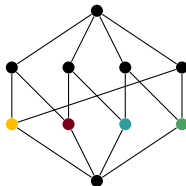
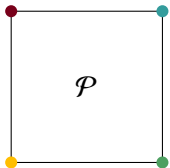
Consider a hyperplane arrangement over some field  $\mathbb{K}$ .



This is a geometric lattice (atomistic and semimodular).

# Polytopes

Let  $\mathcal{P}$  be a polytope. Consider its face lattice



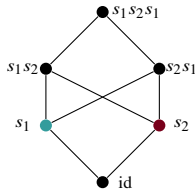
This is an Eulerian poset.

## Bruhat order

Let  $(W, S)$  be a Coxeter group and  $T = \{wsw^{-1} \mid w \in W, s \in S\}$ .  
The *Bruhat order* is defined as

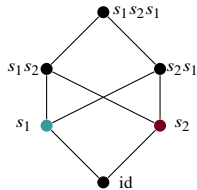
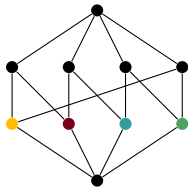
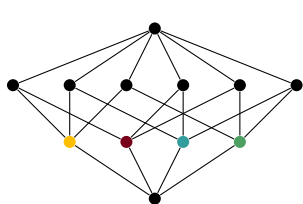
$$u \leq v \Leftrightarrow \exists w_0, \dots, w_n \text{ such that } w_0 = u, w_n = v, w_i^{-1}w_{i+1} \in T.$$

$$\mathfrak{S}_3 = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2 s_1)^2 = 1 \rangle$$



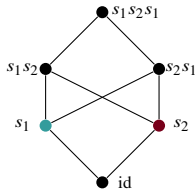
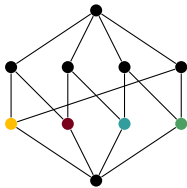
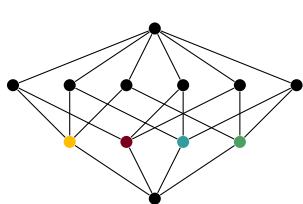


# Polynomials for posets



These posets are bounded and graded.

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Usually, one associates polynomials to posets using the language of the incidence algebra.

# Incidence algebra

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$$I(P) = \{f : P \times P \rightarrow \mathbb{Z}[t], f([x, y]) = f_{xy}(t)\}$$

- $(f + g)_{xy}(t) = f_{xy}(t) + g_{xy}(t)$
- $(fg)_{xy}(t) = \sum_{x \leq z \leq y} f_{xz}(t)g_{zy}(t)$

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- Identity

$$\delta_{xy}(t) = \begin{cases} 1 & x = y \\ 0 & \text{otherwise} \end{cases}$$

## Incidence algebra (ctd.)

Restrict to the subalgebra

$$\mathcal{I}(P) = \{f \in I(P) \mid \deg f_{xy}(t) \leq \rho(y) - \rho(x)\}.$$

$$\text{rev} : \mathcal{I}(P) \rightarrow \mathcal{I}(P), \quad f_{xy}^{\text{rev}}(t) = t^{\rho(y) - \rho(x)} f_{xy}(t^{-1}).$$

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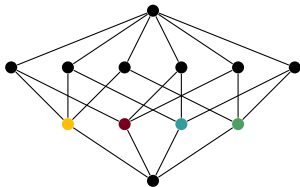
**Definition (Kazhdan-Lusztig, Stanley, Brenti)**

An element  $\kappa \in \mathcal{I}(P)$  is a  $P$ -kernel if and only if

$$\kappa^{-1} = \kappa^{\text{rev}}.$$

# Characteristic polynomial

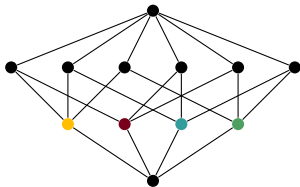
$$\chi_{xy}(t) = \sum_{x \leq z \leq y} \mu_{xz} t^{\rho(y) - \rho(z)}.$$





# Characteristic polynomial

$$\chi_{xy}(t) = \sum_{x \leq z \leq y} \mu_{xz} t^{\rho(y) - \rho(z)}.$$



$\chi$  is a  $P$ -kernel for every poset!

# Eulerian $P$ -kernel

## Theorem (Stanley)

*A poset  $P$  is Eulerian if and only if*

$$\varepsilon : [x, y] \mapsto (t - 1)^{\rho(y) - \rho(x)}$$

*is a  $P$ -kernel.*

## Coxeter $P$ -kernel

For every  $x \leq y \in W$ , if  $\rho(ys) < \rho(y)$  we define

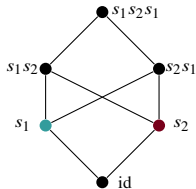
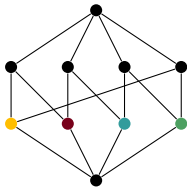
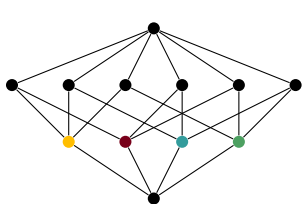
$$R_{xy}(t) := \begin{cases} 1 & x = y \\ R_{xs,ys}(t) & \rho(xs) < \rho(x) \\ tR_{xs,ys}(t) + (t-1)R_{x,ys}(t) & \rho(xs) > \rho(x) \end{cases}$$

### Theorem (Björner–Brenti)

*The function  $R : [x, y] \mapsto R_{xy}(t)$  is a  $P$ -kernel for every Coxeter group.*

# **The new stuff**

## What we have so far



$$\chi_P(t) = (t-1)(t^2-3t+3)$$

$$\varepsilon_P(t) = (t-1)^3$$

$$R_P(t) = (t-1)(t^2-t+1)$$

## Reduced $P$ -kernels

### Lemma

*If  $\kappa$  is a  $P$ -kernel, then  $(t - 1) \mid \kappa_{xy}(t)$  for every  $x < y$ .*

### Definition (Ferroni–Matherne–V.)

$$\bar{\kappa}_{xy}(t) := \begin{cases} -1 & x = y \\ \frac{\kappa_{xy}(t)}{t-1} & x < y \end{cases}$$

# Chow functions

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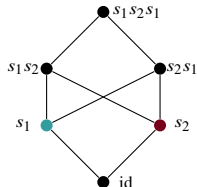
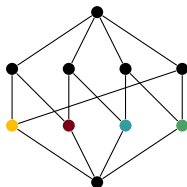
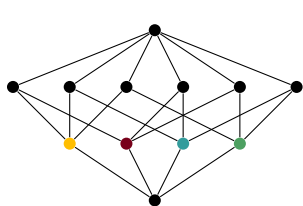
$$H := (-\bar{\kappa})^{-1}.$$

$$H_{xy}(t) = \sum_{x < z \leq y} \bar{\kappa}_{xz}(t) H_{zy}(t)$$

$$H_P(t) := H_{\widehat{01}}(t).$$



# Examples



$$\chi_P(t) = (t-1)(t^2-3t+3)$$

$$\varepsilon_P(t) = (t-1)^3$$

$$R_P(t) = (t-1)(t^2-t+1)$$

$$H_P(t) = t^2 + 7t + 1$$

$$H_P(t) = t^2 + 6t + 1$$

$$H_P(t) = t^2 + 3t + 1$$

# Palindromicity...

## Theorem (Ferroni–Matherne–V.)

*For every  $P$  and  $\kappa$ ,*

- *$H_P(t)$  is a palindromic polynomial of degree at most  $\rho(P) - 1$ .*
- *If  $\kappa$  is monic then so is  $H$ .*

## ... and more

A condition on the Kazhdan–Lusztig–Stanley functions of the poset gives us much more.

### Theorem (Ferroni–Matherne–V.)

*If  $f$  or  $g$  is non-negative, then  $H$  is non-negative and unimodal.*

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*For matroids, polytopes and Bruhat intervals,  $H$  is non-negative and unimodal.*

# Characteristic Chow functions

Feichtner–Yuzvinsky define

$$\underline{\text{CH}}(M) = \frac{\mathbb{Q}[x_F \mid F \in P \setminus \widehat{0}]}{I + J}$$

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(PD)  $\implies$  palindromicity

(HL)  $\implies$  unimodality



# Real-rootedness

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If  $H$  is non-negative and palindromic

real-rooted  $\implies \gamma$ -positive.

$$H(t) = \sum_i \gamma_i t^i (1+t)^{d-2i} \quad \gamma_H(t) = \sum_i \gamma_i t^i \quad \gamma_i \geq 0.$$

## $\gamma$ -positivity

### Theorem (Ferroni–Matherne–Stevens–V., Stump)

*If  $P$  is a geometric lattice ( $R$ -labelled), then the characteristic Chow polynomial is  $\gamma$ -positive.*

Proof: Combinatorial interpretation of  $\gamma$  by working on the extended  $ab$ -index  $\text{ex } \Psi(a, b)$ .

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### Conjecture (Ferroni–Matherne–V.)

*The characteristic Chow polynomial of a Cohen–Macaulay poset only has real roots.*

# Eulerian Chow functions

The  $f$ -polynomial of a simplicial complex is given by

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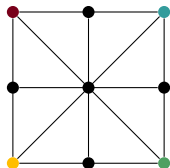
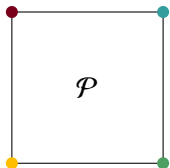
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## Theorem (Ferroni–Matherne–V.)

*If  $P$  is an Eulerian poset, the Eulerian Chow polynomial coincides with the  $h$ -polynomial of the order complex of  $P$ .*

## Example

In the simpler case of a face lattice of a polytope  $\mathcal{P}$ , this corresponds to the baricentric subdivision  $\text{sd}(\mathcal{P})$ .



# Real-rootedness

## Conjecture (Brenti–Welker)

*The Eulerian Chow polynomial of a polytope only has real roots.*

Proved when  $\mathcal{P}$  is a simplicial polytope.

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## Conjecture (Athanasiadis–Kalampoglia–Evangelinou)

*The Eulerian Chow polynomial of an Eulerian Cohen–Macaulay poset only has real roots.*

## $\gamma$ -positivity

### Theorem

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### Proof

- Gal shows that  $\gamma_P(t) = \Phi(1, 2t)$ , where  $\Phi(c, d)$  is the  $cd$ -index.
- Karu proves that  $\Phi(c, d)$  is non-negative when  $P$  is Eulerian and Cohen–Macaulay.

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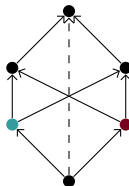
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### Question

*Is the Chow polynomial of an Eulerian poset non-negative?*

## Coxeter Chow functions

Let  $B(x, y)$  be the Bruhat graph of  $[x, y]$ , where  $z_1 \rightarrow z_2$  if  $z_1^{-1}z_2 \in T$ .



### Theorem (Ferroni–Matherne–V.)

$H$  enumerates paths in the Bruhat graph,

$$H_{xy}(t) = \sum_{\Delta \in B(x,y)} t^{\frac{\rho(y) - \rho(x) - \ell(\Delta)}{2} + \text{des}(\Delta)}.$$



# Real-rootedness

Conjecture (Ferroni–Matherne–V.)

*The Coxeter Chow polynomials only have real roots.*

Checked on all intervals of  $\mathfrak{S}_n$  for  $n \leq 7$ .

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- Billera–Brenti define a more general version of the  $cd$ -index called the *complete  $cd$ -index*  $\tilde{\Psi}(c, d)$ .

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Theorem (Ferroni–Matherne–V.)

$$\gamma(t^2) = t^{\rho(y) - \rho(x)} \tilde{\Psi}(t^{-1}, 2).$$

Conjecture (Billera–Brenti, Ferroni–Matherne–V.)

*The complete  $cd$ -index (resp.  $\gamma$ ) is non-negative.*

# Upshot

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We now have a new way of computing polynomials that are

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at least for every matroid, polytope and Coxeter group.

## Conclusion/Open questions

- What are other nice  $P$ -kernels that provide well-behaved families of Chow polynomials?
- What does real-rootedness mean for a Hilbert series?
- This new language lets us collect under the same object a number of real-rootedness conjectures.
- Borrowing tools from one area and applying them to another seems to be effective ( $cd$ -index and complete  $cd$ -index).

**Thank you! 😊**