New perspectives on quasisymmetry

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$$R_1(f) = 0^4x_1 + 0^4x_2 + x_1^4x_2 - 0x_1^3x_2 = x_1^4x_2$$
$$R_2(f) = x_1^40 + x_1^4x_2 + 0^4x_2 - x_10^3x_2 = x_1^4x_2$$
$$R_3(f) = x_1^4x_2 + 0^4x_2 + x_1^40 - x_1x_2^30 = x_1^4x_2.$$

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Nil-Hecke relations: $\partial_i^2 = 0$, $\partial_i\partial_j = \partial_j\partial_i$ for $|i-j| \geq 2$ and $\partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}$

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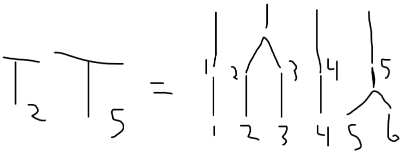

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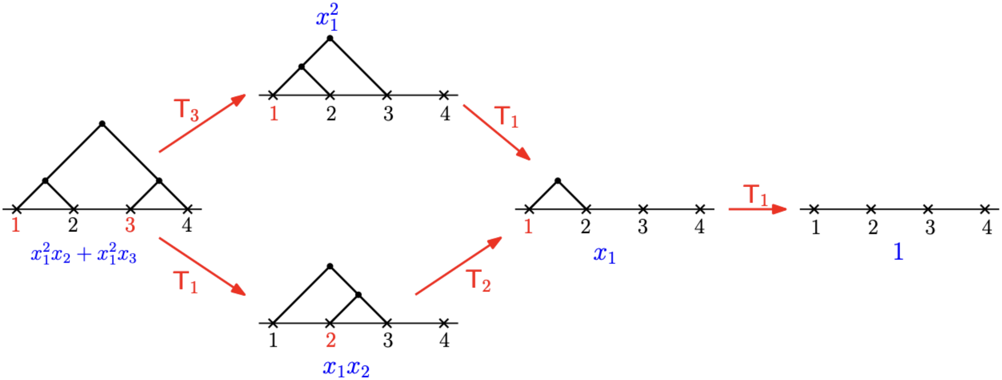
Theorem

(NST'24) The Forest polynomials p_F are the unique family of homogenous polynomials indexed by plane binary indexed forests such that $F_\emptyset = 1$ and

$$T_i p_F = \begin{cases} p_{F/i} & F \text{ has a node with children the leaves } i, i+1 \\ 0 & \text{otherwise.} \end{cases}$$

An example of trimming forest polynomials



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Success!

Most details worked out for Grassmannians, arbitrary types, connection to cluster varieties, ...

Thank You

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