(P, ϕ)-Tamari lattices

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LACIM
Laboratoire d'algèbre, de combinatoire et d'informatique mathématique

2025.7 FPSAC Hokkaido University, Sapporo

SUMMARY:

BACKGROUND

I) POSETS AND LATTICES

NEW CONSTRUCTION

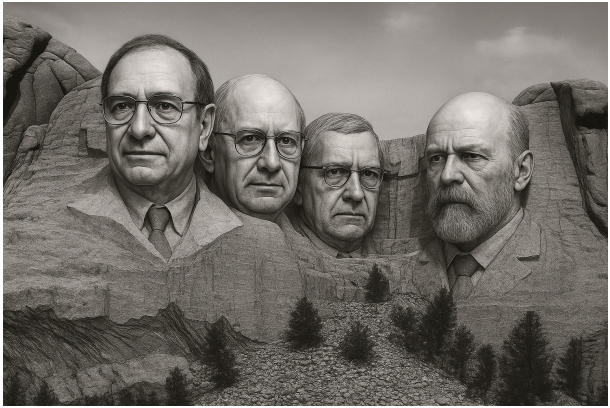
II) (P, ϕ)-TAMARI

III) HIGHER TORSION CLASSES:

TYPE A AUSLANDER AND NAKAYAMA ALGEBRAS

ALGEBRAIC MOTIVATION

I) POSETS AND LATTICES

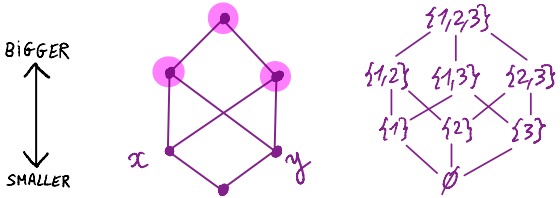


GIAN-CARLO ROTA, ROBERT DILWORTH, GARRETT BIRKHOFF, ERNST SCHRÖDER

- A **LATTICE** $(L, \leq)$ IS A POSET SUCH THAT ALL $\{x, y\} \subseteq L$ HAVE A LEAST UPPER BOUND AND A GREATEST LOWER BOUND.

  MEET $x \wedge y$ JOIN $x \vee y$

THE **CHAINS** ARE THE TOTALLY ORDERED SUBSETS. THE **LENGTH** OF L IS $\ell(L) :=$ SIZE OF A MAXIMUM-LENGTH CHAIN.



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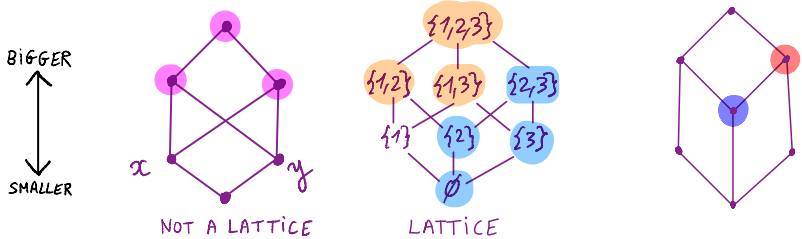
AN ORDER **IDEAL** (**FILTER**) IS THE SUBSET OF ELEMENTS BELOW (ABOVE) SOME ELEMENTS. $I_p(x) := \{y \mid y \leq x\}$.



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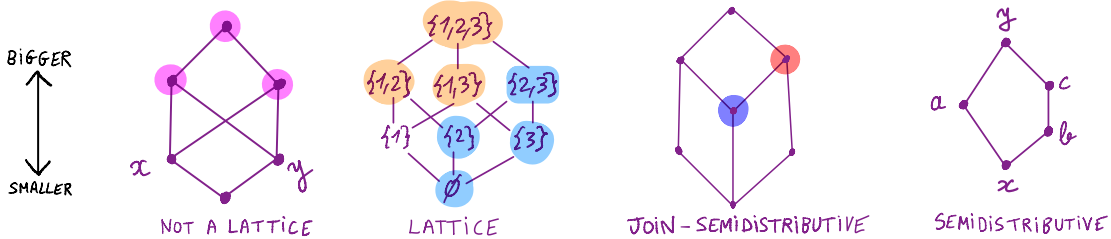

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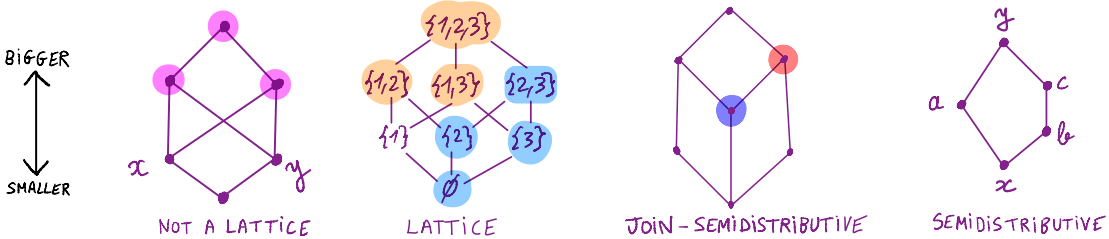

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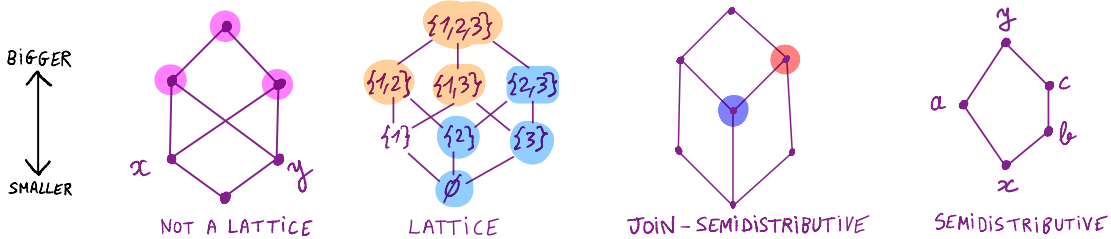
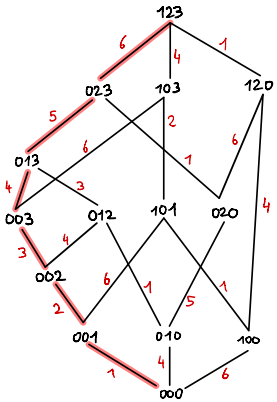
- $a \in L$ IS **LEFT MODULAR** IF $\forall b < c$, $(b \vee a) \wedge c = b \vee (a \wedge c)$. IF L HAS A LEFT MODULAR MAXIMAL CHAIN, THEN L IS LEFT MODULAR.



- THE ORDER COMPLEX $\Delta(P)$ IS THE SIMPLICIAL COMPLEX WHOSE FACES ARE THE CHAINS OF P.
- IT IS SHELLABLE IF WE CAN ORDER THE FACETS $F_1, F_2, \dots, F_m$ SUCH THAT FOR ALL k, $\left(\bigcup_{1 \le i < k} \langle F_i \rangle\right) \cap \langle F_k \rangle$ IS A UNION OF PURE COMPLEXES OF DIMENSION $\dim(\langle F_k \rangle) - 1$.



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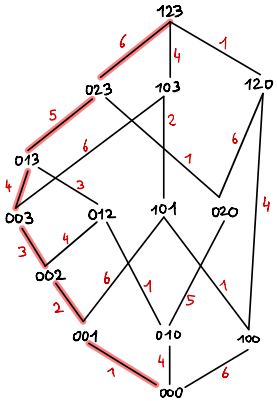

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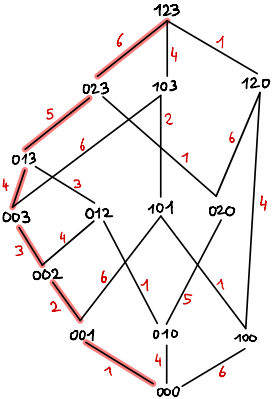

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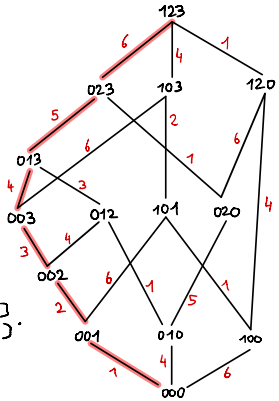

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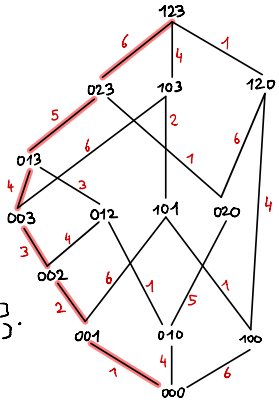
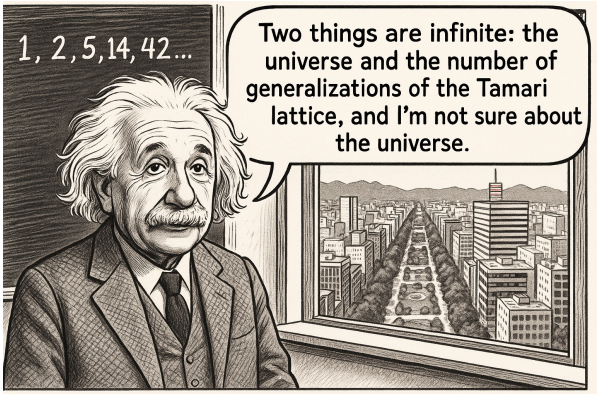


II) (P, ϕ)-TAMARI



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TAM(P,ϕ) IS THE POSET ORDERED BY INCLUSION ON THE ORDER FILTERS $T \subseteq \mathcal{C}_P^{\phi}$ SATISFYING

FOR ALL $0 \leq i < j < m$, IF $(x,i) \in T$ AND $(\phi(i+1), j) \in T$, THEN $(x,j) \in T$.

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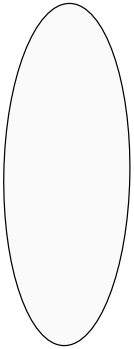

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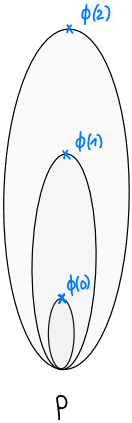

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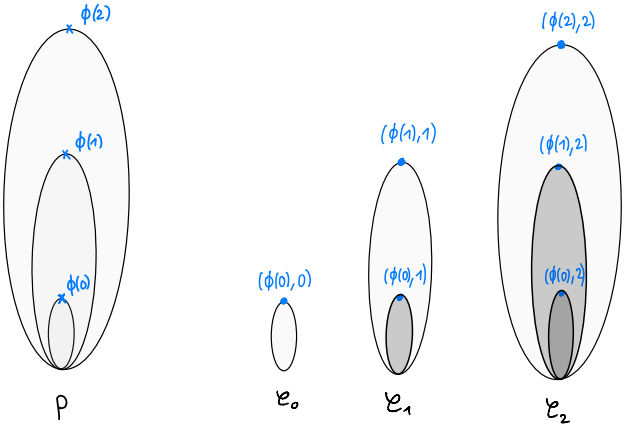

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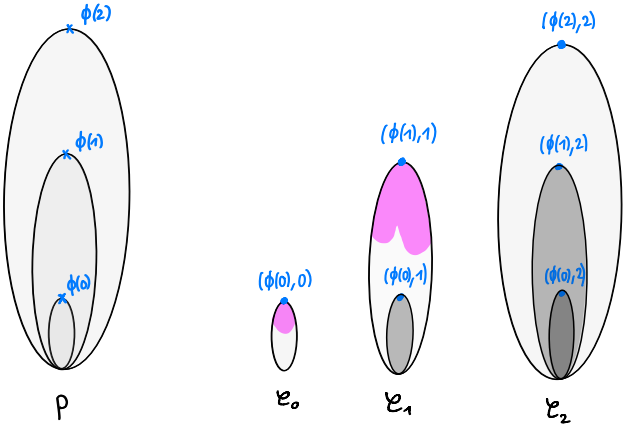

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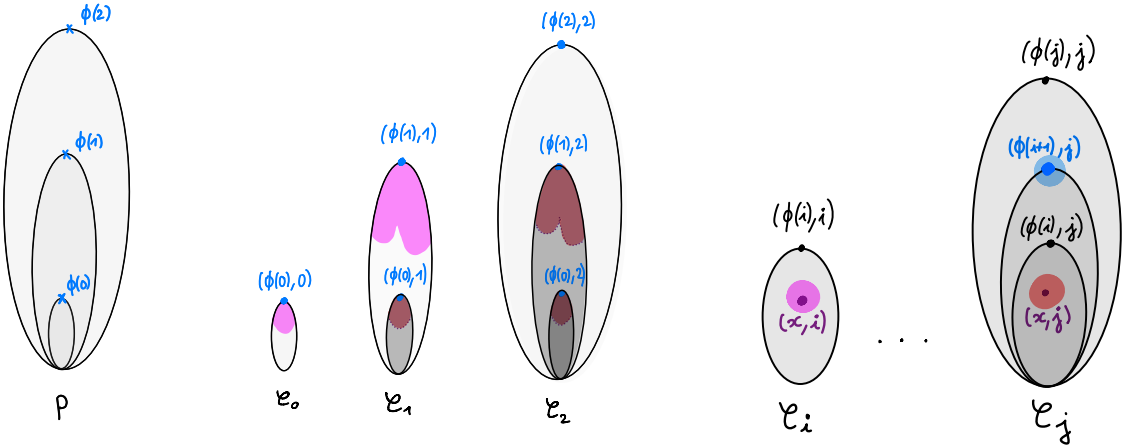

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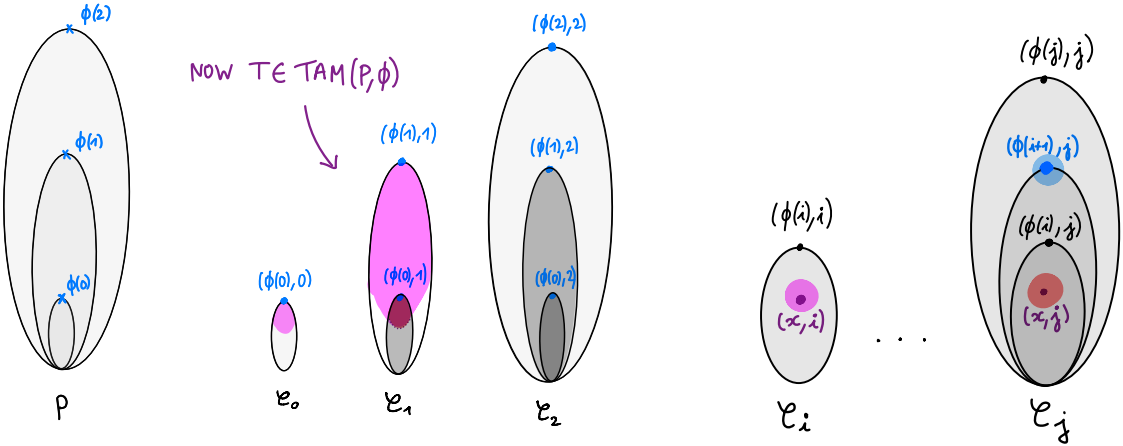

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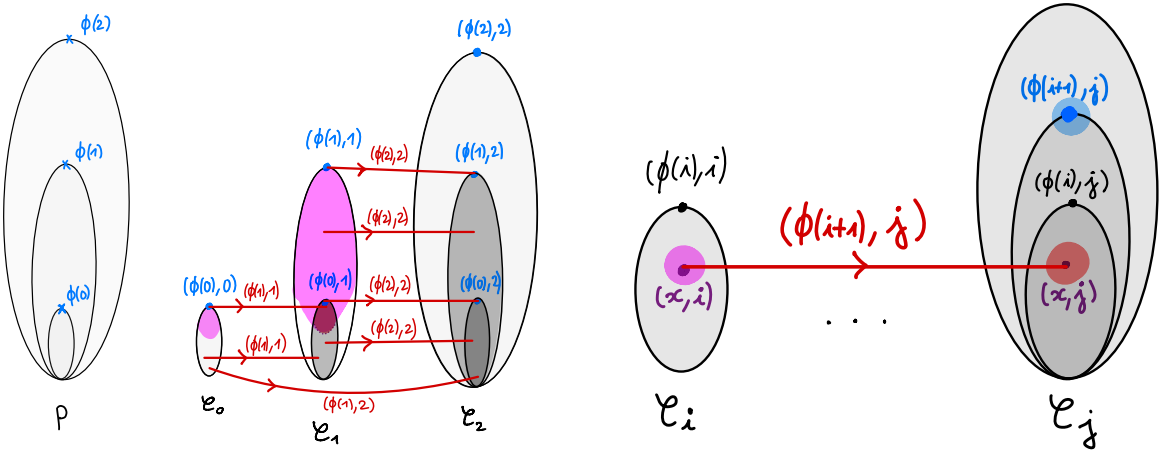

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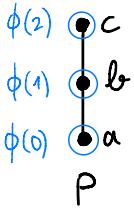


ORDER FILTERS:

$\emptyset = 000$

$\{(a,0)\} = 100$

$\{(b,1)\} = 010$

$\{(c,2)\} = 001$

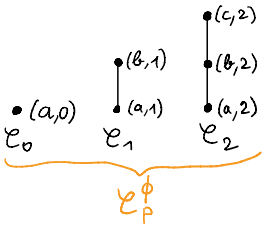
$\{(b,1),(a,1)\} = 020$

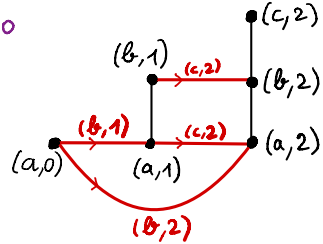
$\vdots$

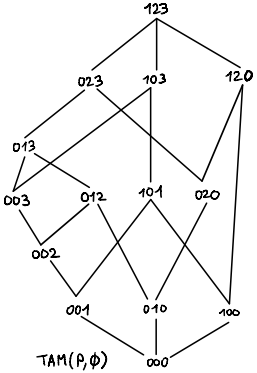
$\{(a,0),(b,1)\} = 110$

$\vdots$

$\mathcal{C}_P^{\phi} = 123$







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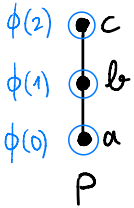

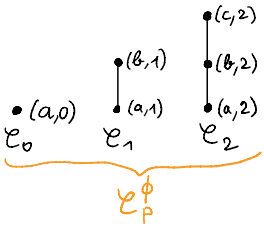

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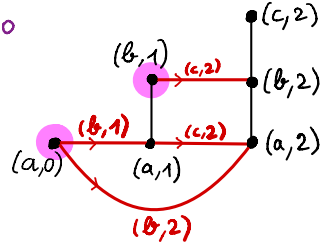
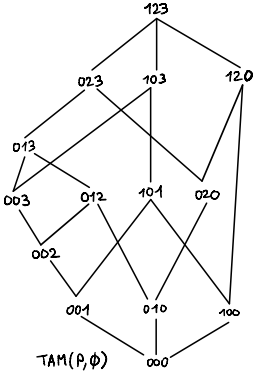
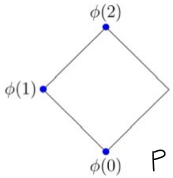




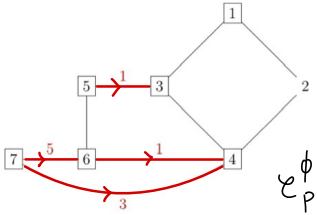
WHY NOT 110 ?

SOME RESULTS: DENOTE $L := \mathrm{TAM}(P,\phi)$ AND $C_m := 0<1<\cdots<m-1$.

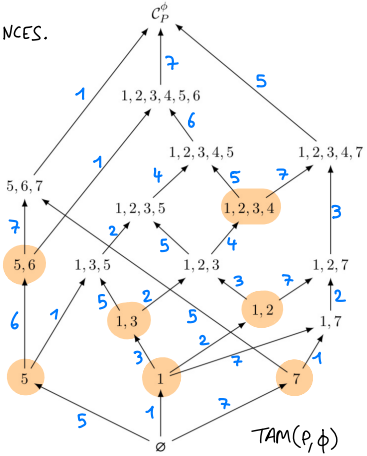
- L IS A LATTICE WITH MEET GIVEN BY INTERSECTION.
- $\ell(L) = |JI(L)|$, THUS L IS JOIN-EXTREMAL.
- L IS JOIN-SEMIDISTRIBUTIVE AND WE KNOW WHEN IT IS SEMIDISTRIBUTIVE.
- L IS JOIN-CONGRUENCE UNIFORM AND WE UNDERSTAND ITS CONGRUENCES.
- L IS LEFT MODULAR, THUS EL-SHELLABLE.
  FOR ALL $x \leq y$, $\mu(x,y) \in \{-1,0,1\}$.
- $\mathrm{TAM}(C_m, C_m)$ IS THE TAMARI LATTICE OF ORDER $m+1$.



P

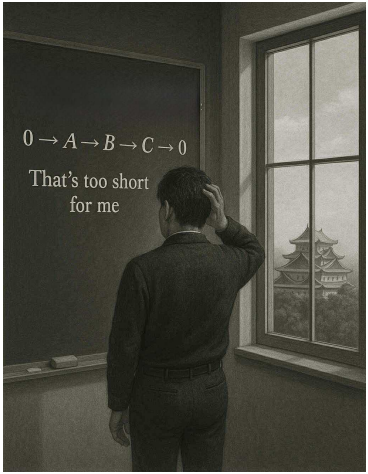


$\mathcal{C}_P^\phi$



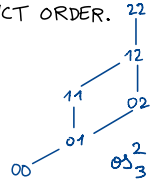
$\mathrm{TAM}(P,\phi)$

III) HIGHER TORSION CLASSES: TYPE A AUSLANDER AND NAKAYAMA ALGEBRAS

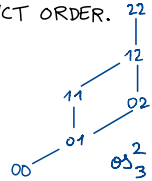


OSAMU IYAMA

DENOTE $\mathcal{OS}_m^d$ THE POSET OF NON-DECREASING WORDS OF LENGTH d ON $\{0,1,\dots,m-1\}$ WITH PRODUCT ORDER.



DENOTE $\mathfrak{S}_n^d$ THE POSET OF NON-DECREASING WORDS OF LENGTH d ON $\{0,1,\dots,n-1\}$ WITH PRODUCT ORDER.



A TORSION CLASS IN $\operatorname{mod} A$ IS A SUBCATEGORY CLOSED BY QUOTIENTS AND EXTENSIONS.

Q: $0 \to 1 \to \cdots \to n-1$ $\quad A = kQ$ $\quad$ TORS$(\operatorname{mod} A)$ IS THE TAMARI LATTICE OF ORDER $n+1$.

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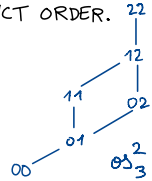

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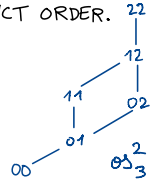

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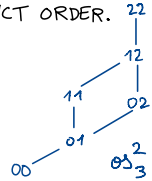

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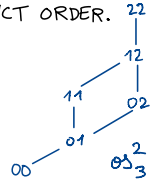

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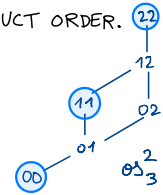

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LET $\phi : i \mapsto ii \cdots i$.

PROP [S.]: $[(1)+(2)] \iff [(1) + \text{FOR ALL } 0 \leq i < j < m, \text{ IF } (x,i) \in T \text{ AND } (\phi(i+1), j) \in T, \text{ THEN } (x,j) \in T]$.

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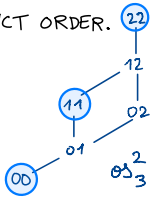

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