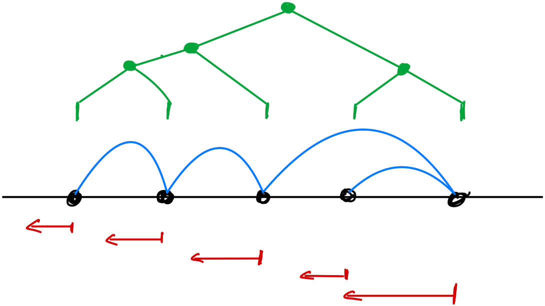
From linear programming to particle collisions



Raman Sanyal

Goethe-Universität Frankfurt

joint with

Alex Black and Niklas Lütjeharms

This talk is based on a true story.

Linear programs and the simplex algorithm

Linear Program (LP)

$$\begin{array}{lll} \max & c_1\, x_1 + \cdots + c_d\, x_d & \\ \text{s.t.} & a_{i1}x_1 + \cdots + a_{id}x_d \leq \; b_i & \text{for } i = 1, \ldots, n \end{array}$$

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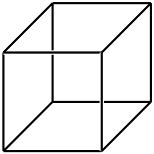

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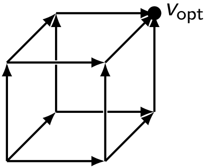

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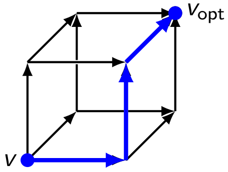
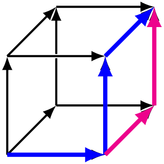


simplex algorithm produces a path from *any* starting node v to the sink v_{opt}

Pivot rules

The pivot rule chooses the path from *any* starting vertex.



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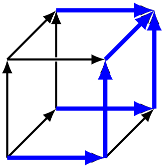

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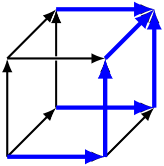

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For fixed LP (P, c), memory-less pivot rules are given by arborescences

$$\mathcal{A} : V \setminus v_{\text{opt}} \to V \qquad \mathcal{A}(v) \in \mathrm{Nb}_{+}(v) \quad \text{for all } v \neq v_{\text{opt}} ,$$

where $\mathrm{Nb}_{+}(v)$ are the improving neighbors of v.

Some pivot rules

(P, c) fixed and a generic weight $\omega \in \mathbb{R}^d$.

- Greatest improvement

$$\langle c, u - v \rangle$$

- p-Steepest Edge

$$\frac{\langle c, u - v \rangle}{\|u - v\|_p}$$

- Max-slope

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For a weight ω and normalization $\eta : \mathbb{R}^d \to \mathbb{R}$
the associated normalized-weight pivot rule is $\mathcal{A}^\omega : V \setminus v_{\text{opt}} \to V$

$$\mathcal{A}^\omega(v) \;:=\; \operatorname{argmax} \left\{ \frac{\langle \omega, u - v \rangle}{\eta(u - v)} : u \in \mathrm{Nb}_+(v) \right\} .$$

A polytope of pivot rules

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For an arborescence $\mathcal{A} : V \setminus v_{\text{opt}} \to V$ define

$$\Psi(\mathcal{A}) \; := \; \sum_{v \neq v_{\text{opt}}} \frac{\mathcal{A}(v) - v}{\eta(\mathcal{A}(v) - v)}$$

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Theorem (Black, De Loera, Lütjeharms, S.'22)

For generic $\omega \in \mathbb{R}^d$ and arborescence $\mathcal{A}$ the following are equivalent:

1. $\Psi(\mathcal{A})$ *is the vertex of* $\Pi(P, c)$ *maximizing* ω.
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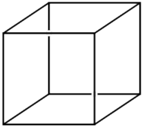
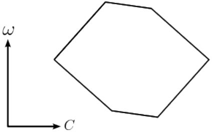
(coherent) triangulations? GKZ-vectors? secondary/fiber polytopes?

Max-slope and monotone path polytopes

c, ω define a projection $\pi : \mathbb{R}^d \to \mathbb{R}^2$





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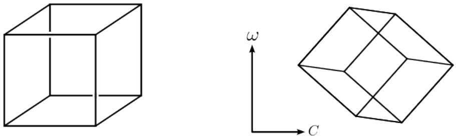

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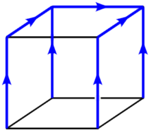

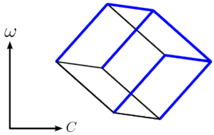

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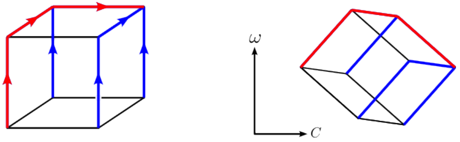

Path from minimizer $v_{-\text{opt}}$ to v_{opt} is a c-monotone path.

This is a coherent monotone path in the sense of Billera–Sturmfels.

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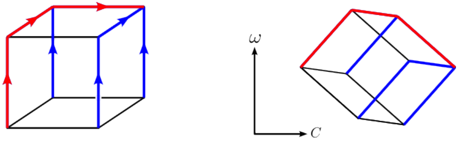

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Theorem (Billera-Sturmfels'92)

The monotone path polytope $\Sigma(P, c)$ parametrizes coherent c-monotone paths.

Theorem (Black, De Loera, Lütjeharms, S.'22)

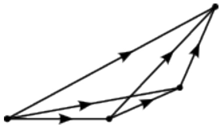
$\Sigma(P, c)$ *is a weak Minkowski summand of* $\Pi(P, c)$.

An example: max-slope pivot rules on simplices

n-dimensional simplex

$$\Delta_n \;=\; \mathrm{conv}(e_1, e_2, \ldots, e_{n+1}) \subset \mathbb{R}^{n+1}$$

Objective function $c = (c_1 < c_2 < \cdots < c_{n+1})$.



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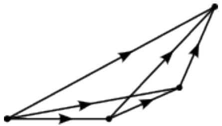

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for n in [2, 3, 4, 5, 6, 7]:
    P = polytopes.simplex(n)
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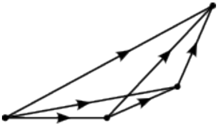

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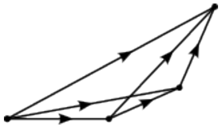

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[43]: [ PP.is_combinatorially_isomorphic( polytopes.associahedron(['A',n-1] ) ) for n, PP in PivPolytopes ]

[43]: [True, True, True, True, True, True]
```

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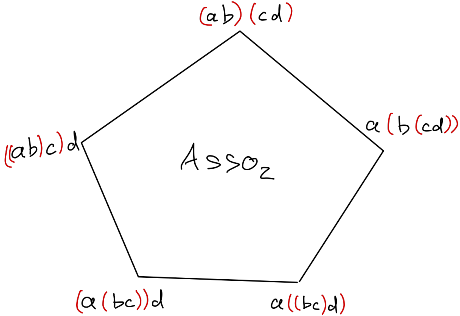

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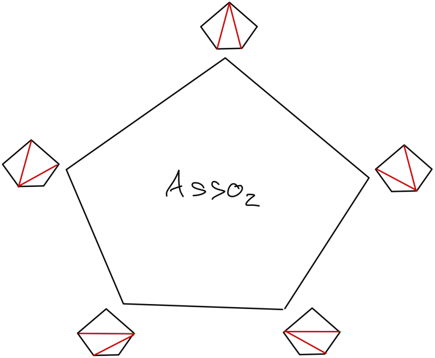

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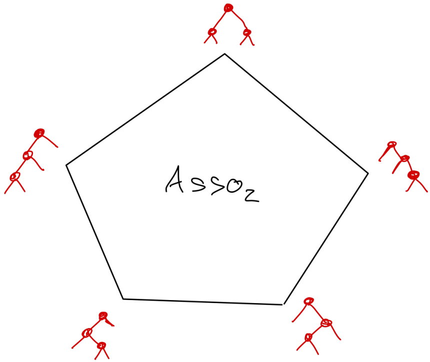

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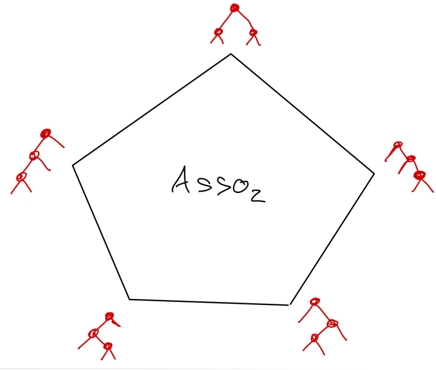

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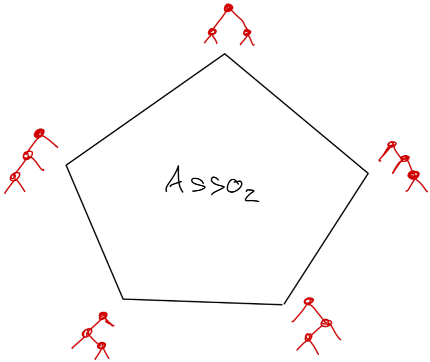

Theorem (Black-Lütjeharms-S.'24)

The max-slope pivot rule polytope of (Δ_n, c) *is isomorphic to* Asso_{n-1}.

Prisms over simplices

Prism over Δ_n is $\text{prism}(\Delta_n) = \Delta_n \times [0,1] \subset \mathbb{R}^{n+2}$.

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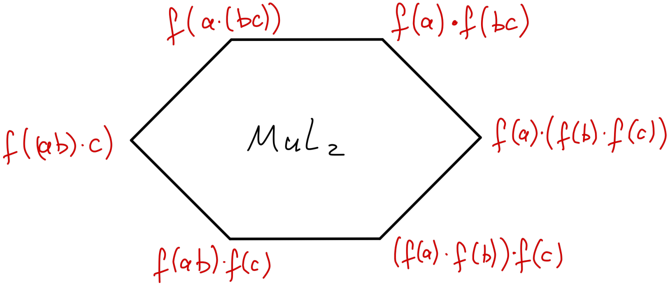

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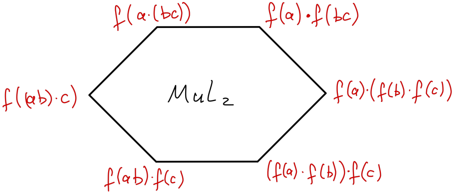

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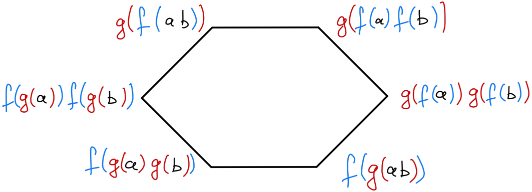

Theorem (Black-Lütjeharms-S.'24)

Max-slope pivot polytope of $(\text{prism}^k(\Delta_n), c')$ *is isom. to* k*-multiplihedron* Mul_n^k.

Products of simplices

$$\text{prism}^k(\Delta_n) = \Delta_n \times \underbrace{\Delta_1 \times \cdots \times \Delta_1}_{k}$$

Max-slope pivot rule polytopes of $\Delta_n \times \Delta_m$

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Theorem (Pilaud–Poullot'25)

$\Pi(\Delta_{n_1} \times \cdots \times \Delta_{n_l}, c)$ *is comb. isomorphic to the* $(n_1, \ldots, n_l)$*-constrainahedron.*

Piecewise-linear homeomorphism between normal fans of $\Pi(\Delta_n, c)$ and Loday's associahedron Asso_{n-1}. Extends to shuffle products of deformed permutahedra.

Max-slope pivot rules on simplices

simplex $\Delta_n = \text{conv}(e_1, e_2, \ldots, e_{n+1})$ obj. function $c = (c_1 < c_2 < \cdots < c_{n+1})$

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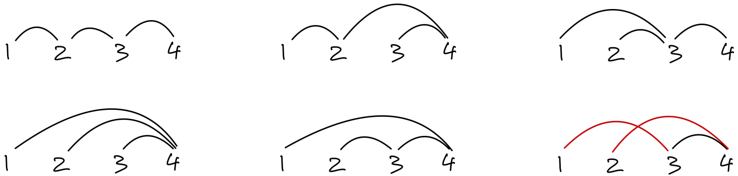

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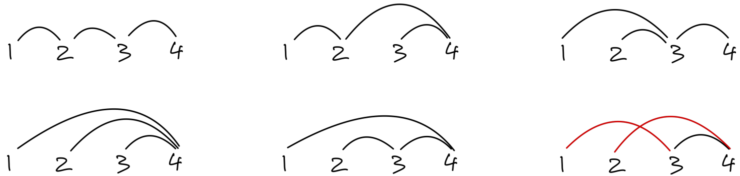
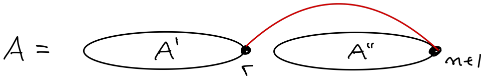


Non-crossing arborescences – Catalan recurrence!



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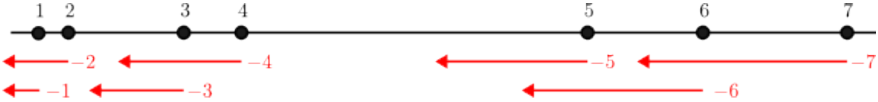

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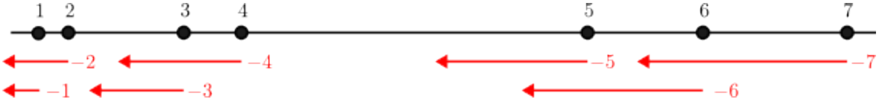

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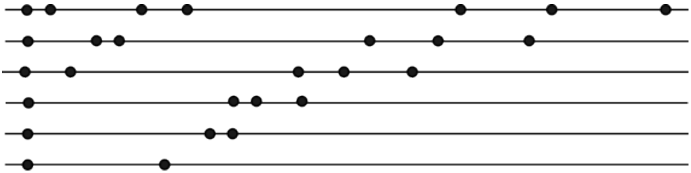

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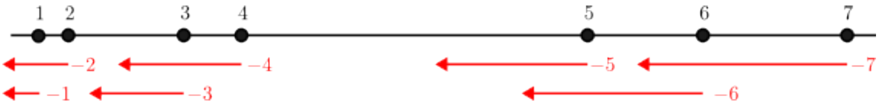

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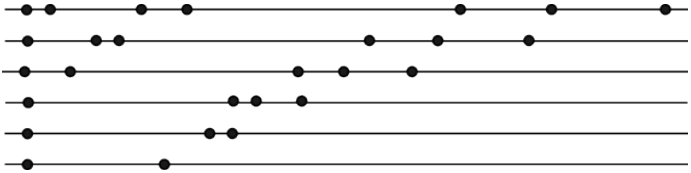

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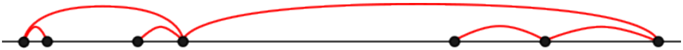

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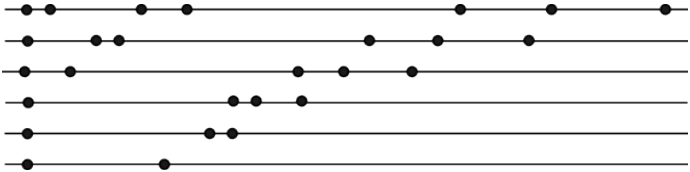

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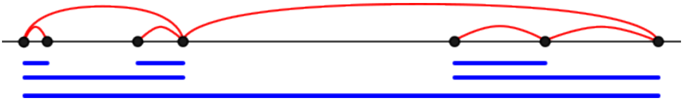

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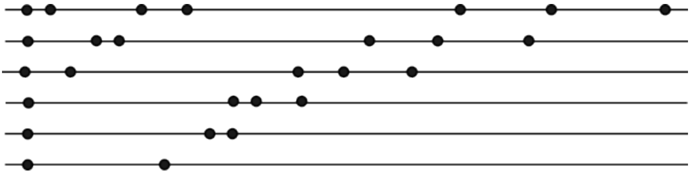

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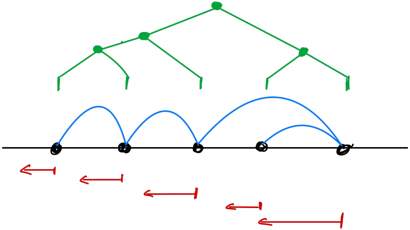
- the associated bracketing yields the isomorphism of face lattices

□

From linear programming to particle collisions



The Polyhedral Geometry of Pivot Rules and Monotone Paths
(Black, De Loera, Lütjeharms, S.), SIAGA 2023, arXiv:2201.05134

From linear programming to colliding particles
(A. Black, N. Lütjeharms, S.) arXiv:2405.08506

Applications of max-slope polytopes to (flag) matroids and flag varieties

Flag Polymatroids
(Black, S.), Adv. Math. 2024, arXiv:2207.12221