Lattices of transfer systems

Yongle Luo and Baptiste Rognerud

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- Classical combinatorical objects : binary trees, posets and lattice theory.
- Inspired by the τ-tilting theory and the lattices of torsion pairs (Tamari lattices, cambrian lattices, weak Bruhat orderings etc.).
- A rather efficient algorithm to count the elements of $\mathrm{Trs}(L)$.

Examples

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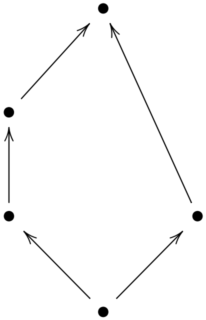

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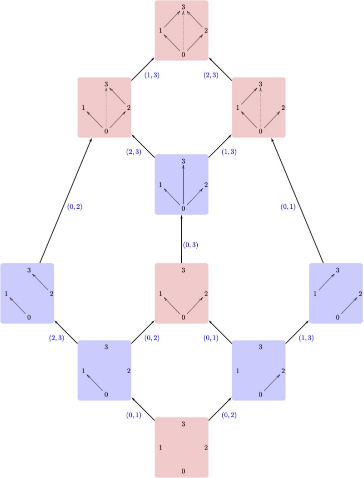
Let L be the boolean lattice of subsets of $\{1, 2\}$. Then the lattice $\mathrm{Trs}(L)$ is :



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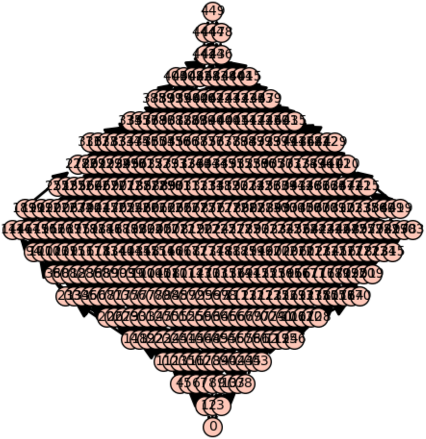

Motivation

Let G be a finite group. An N_∞-operad is an equivariant version of an E_∞-operad such that algebras over these operads are equipped with

- An operation associative and commutative up to coherent homotopies.
- Homotopy coherent multiplicative norm maps which are encoded by the fixed points of the spaces in the operad : $A^H \to A^K$ for some $H \leqslant K$.

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- These norm maps seem to be very useful in the applications (Solution of Kervaire invariant one problem by Hill, Hopkins, and Ravenel 2009).
- Classifying N_∞-operads helps to understand what norms might appear in applications. Main objective of the "homotopical combinatorics" of [Blumberg, Hill, Ormsby, Osorno and Roitzheim Nottices AMS 2024]

Main theorem [Blumberg, Hill 2015 $\cdots$ Balchin Barnes Roitzheim 2021]

Let G be a finite group. There is an equivalence of categories between $\mathrm{H}_0(N_\infty^G)$ and the poset of G-transfer systems (viewed as a category).

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3. If $H, L \subseteq K$, we have

$$\begin{array}{ccc} H \cap L & \longrightarrow & L \\ \downarrow & & \downarrow \\ H & \longrightarrow & K. \end{array}$$

If $H \lhd K$, then $H \cap L \lhd K$. Stability by pullback.

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Let $(L, \leqslant)$ be a finite lattice. A transfer system $\lhd$ on L is a poset on L such that :

1. If $x \lhd y$, then $x \leqslant y$.
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We denote by $\mathrm{Trs}(L)$ the set of all transfer systems on the lattice L. This is a subposet of the “poset of finite posets".

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Let $G = C_{p^{n-1}}$. Then $Sub(G) \cong 0 < 1 < \cdots < n$ is a total order.

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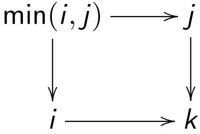

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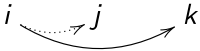

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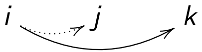
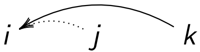


- The condition for $\lhd^{op}$ is :



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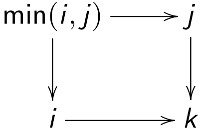

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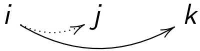

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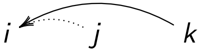

- This is an interval-poset of Châtel and Pons.

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The lattice of transfer systems on a total order with n elements is isomorphic to the Tamari lattice on the binary trees with n inner vertices.

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Proof.

If T is a binary tree, view it as a binary search tree, keep the decreasing relations of its poset. The opposite is a transfer system. □

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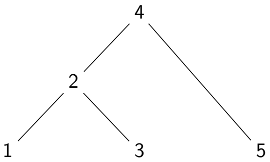

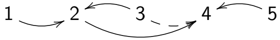

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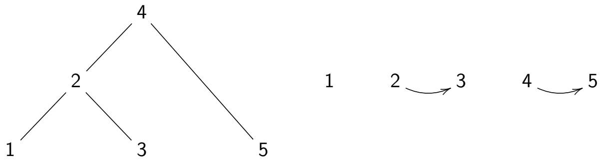

Theorem

Let Tam_n *be the Tamari lattice on the binary trees with* n *inner vertices.*

1. Tam_n *is a semidistributive lattice (Urquhart 1978).*
2. Tam_n *is a trim lattice (Thomas 2005).*
3. Tam_n *is a congruence uniform lattice (Urquhart 1978).*
4. *The congruence lattice of* Tam_n *is isomorphic to the lattice of Dyck paths (Geyer 1994).*

Theorem (Yongle Luo, R- 2024)

Let $(L, \leqslant)$ be a finite lattice. Then $\mathrm{Trs}(L)$ *is*

1. *a semidistributive lattice ;*
2. *a trim lattice ;*
3. *a congruence uniform lattice ;*
4. *Explicit description of the congruence lattice of* $\mathrm{Trs}(L)$.

Definition

A lattice $(L, \leqslant)$ is distributive if for all $x, y, z \in L$, we have

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Theorem (Luo R- 2024)

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Proof.

Let R_1 and R_2 be two transfer systems. Then $R_1 \wedge R_2 = R_1 \cap R_2$ and $R_1 \vee R_2 = (R_1 \cup R_2)^{tc}$. □

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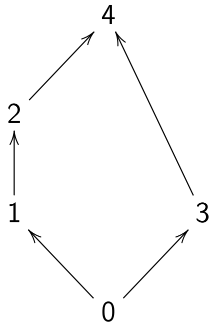

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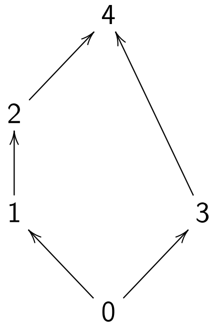

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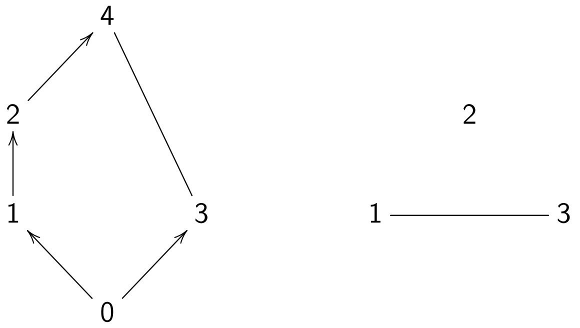
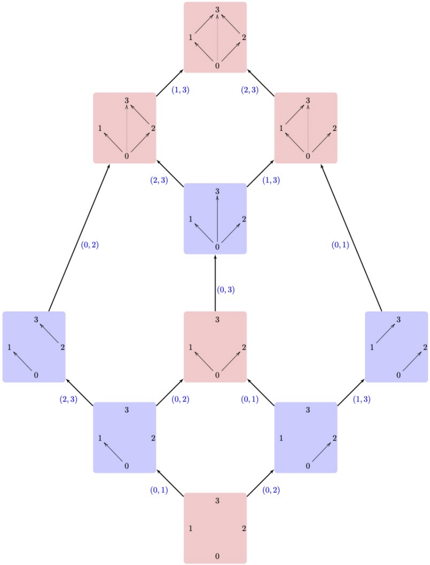


For transfer systems

Proposition (Yongle Luo, R- 2024)

Let $(L, \leqslant)$ *be a finite lattice. Then there is a bijection between the* *join irreducibles* *of* $\mathrm{Trs}(L)$ *and* $\mathrm{Rel}^*(L) = \{(a, b) \in L^2 \mid a < b\}$.

For transfer systems



For transfer systems

The elevating graph of $(L, \leqslant)$ is :

- Vertices $= \mathrm{Rel}^*(L)$.
- Edges $(a, b) - (c, d)$ if and only if (a, b) lifts on the left (c, d) : if $a \leqslant c$ and $b \leqslant d$, then $b \leqslant c$.

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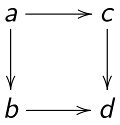

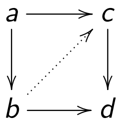

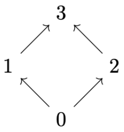

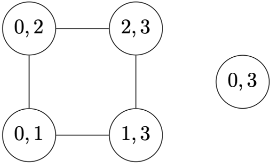

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Theorem (Luo, R- 2024)

There is an explicit bijection between cliques of the elevating graph of $(L, \leqslant)$ *and* $\mathrm{Trs}(L)$*.*

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- The algorithm is not good enough for $n = 5$.
- The boolean lattice $\mathcal{P}(\{1,\cdots,n\})$ is the lattice of subgroups of a squarefree elementary abelian group.

A lower bound for the number of transfer systems

Obvious remark

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Lemma

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- $\sum_{j=0}^{\frac{n-1}{2}} \binom{n}{j}\binom{n-j}{n-2j}$ *if n is odd,*
- $\sum_{j=1}^{\frac{n}{2}} \binom{n}{j}\binom{n-j}{n+1-2j}$ *If n is even.*

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Proof.

For $1 \leqslant k \leqslant 2n-1$. $\mathcal{R}_k : X \lhd_k Y$ if and only if $X = Y$ or $X \subset Y$ and and $|X| + |Y| \leqslant k$. □

$$
\begin{array}{c}
1 \\
2\ 1\ 2 \\
3\ 3\ 7\ 3\ 3 \\
4\ 6\ 16\ 13\ 16\ 6\ 4 \\
5\ 10\ 30\ 35\ 51\ 35\ 30\ 10\ 5 \\
6\ 15\ 50\ 75\ 126\ 121\ 126\ 75\ 50\ 15\ 6 \\
7\ 21\ 77\ 140\ 266\ 322\ 393\ 322\ 266\ 140\ 77\ 21\ 7
\end{array}
$$

Arigatô gozaimasu