Lattices of transfer systems

Yongle Luo and Baptiste Rognerud

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Main result

Application of semidistributivity

Transfer systems on the boolean attices

Starting from a finite lattice (L, \leq) construct a Tamari-like lattice Trs(L).

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Original motivation in equivariant algebraic topology

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Original motivation in equivariant algebraic topology and category theory. Lattices of transfer systems

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Application of semidistributivity

Transfer systems on the boolean lattices

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- Original motivation in equivariant algebraic topology and category theory.
- Classical combinatorical objects : binary trees, posets and lattice theory.

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- Original motivation in equivariant algebraic topology and category theory.
- Classical combinatorical objects : binary trees, posets and lattice theory.
- Inspired by the τ -tilting theory and the lattices of torsion pairs (Tamari lattices, cambrian lattices, weak Bruhat orderings etc.).

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 Original motivation in equivariant algebraic topology and category theory.

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 Classical combinatorical objects: binary trees, posets and lattice theory. Homotopica

Inspired by the τ -tilting theory and the lattices of torsion pairs (Tamari lattices, cambrian lattices, weak Bruhat orderings etc.).

A rather efficient algorithm to count the elements of Trs(L). iviain results

Transfer systems

Examples



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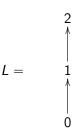
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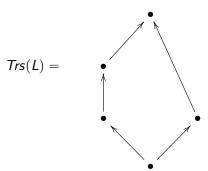
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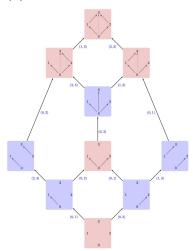
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Application of semidistributivity

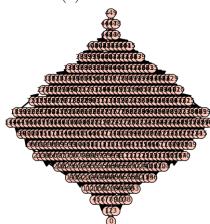
Transfer systems on the boolean

Let L be the boolean lattice of subsets of $\{1,2\}$. Then the lattice $\mathsf{Trs}(L)$ is :



Examples

Let L be the boolean lattice of subsets of $\{1,2,3\}$. Then the lattice $\mathsf{Trs}(L)$ is :



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Transfer systems on the boolean

Let G be a finite group. An N_{∞} -operad is an equivariant version of an E_{∞} -operad such that algebras over these operads are equipped with

- An operation associative and commutative up to coherent homotopies.
- ▶ Homotopy coherent multiplicative norm maps which are encoded by the fixed points of the spaces in the operad : $A^H \rightarrow A^K$ for some $H \leq K$.

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- ► These norm maps seem to be very useful in the applications (Solution of Kervaire invariant one problem by Hill, Hopkins, and Ravenel 2009).

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- ► These norm maps seem to be very useful in the applications (Solution of Kervaire invariant one problem by Hill, Hopkins, and Ravenel 2009).
- ▶ Classifying N_{∞} -operads helps to understand what norms might appear in applications. Main objective of the "homotopical combinatorics" of [Blumberg, Hill, Ormsby, Osorno and Roitzheim Nottices AMS 2024]

Main theorem [Blumberg, Hill 2015 · · · Balchin Barnes Roitzheim 2021]

Let G be a finite group. There is an equivalence of categories between $H_0(N_\infty^G)$ and the poset of G-transfer systems (viewed as a category).

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A G-transfer system \lhd is a poset on Sub(G) such that :

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- 1. If $H \triangleleft K$, then $H \subseteq K$.
- 2. If $H \triangleleft K$ and $g \in G$, then $gHg^{-1} \triangleleft gKg^{-1}$. Stability by conjugacy.

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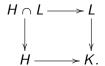
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- 3. If $H, L \subseteq K$, we have



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If $H \triangleleft K$, then $H \cap L \triangleleft K$. Stability by pullback.

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Transfer systems on the boolean Let G be a finite group. There is an equivalence of categories between $H_0(N_\infty^G)$ and the poset of G-transfer systems (viewed as a category).

Let (L, \leq) be a finite lattice. A transfer system \lhd on L is a poset on L such that :

- 1. If $x \triangleleft y$, then $x \leqslant y$.
- 2. If $x, y \leq z$, we have



If $x \triangleleft z$, then $x \land y \triangleleft y$. Stability by pullback.

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- 1. If $x \triangleleft y$, then $x \leqslant y$.
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If $x \triangleleft z$, then $x \land y \triangleleft y$. Stability by pullback.

We denote by Trs(L) the set of all transfer systems on the lattice L. This is a subposet of the "poset of finite posets".

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have:

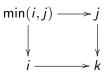
 $\min(i, j) \longrightarrow i$

order. A transfer system is a subposet such that $\forall i, j \leq k$ we

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▶ If $j \le i$, then min(i,j) = j: no condition.

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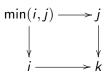
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- ▶ If $i \le i$, then min(i, j) = j: no condition.
- ▶ If i < j we have $i \lhd k$ implies $i \lhd j$.



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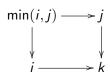


▶ The condition for \triangleleft^{op} is :



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▶ The condition for \triangleleft^{op} is :



This is an interval-poset of Châtel and Pons.





Theorem (Roitzheim Barnes Balchin 2022, Luo R- 2024)

The lattice of transfer systems on a total order with n elements is isomorphic to the Tamari lattice on the binary trees with n inner vertices.

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Theorem (Roitzheim Barnes Balchin 2022, Luo R- 2024)

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Proof.

If T is a binary tree, view it as a binary search tree, keep the decreasing relations of its poset. The opposite is a transfer system.

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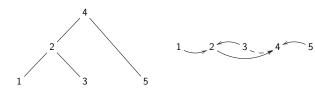
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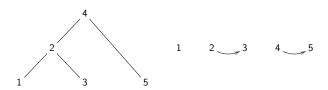
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Theorem

Let Tam_n be the Tamari lattice on the binary trees with n inner vertices.

- 1. Tam_n is a semidistributive lattice (Urquhart 1978).
- 2. Tam_n is a trim lattice (Thomas 2005).
- 3. Tam_n is a congruence uniform lattice (Urquhart 1978).
- 4. The congruence lattice of Tam_n is isomorphic to the lattice of Dyck paths (Geyer 1994).

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Theorem (Yongle Luo, R- 2024)

Let (L, \leq) be a finite lattice. Then Trs(L) is

- 1. a semidistributive lattice;
- 2. a trim lattice;
- 3. a congruence uniform lattice;
- 4. Explicit description of the congruence lattice of Trs(L).

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Definition

A lattice (L, \leq) is distributive if for all $x, y, z \in L$, we have

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ and } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

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Application of semidistributivity

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 and $(x \vee y) = (x \vee z)$.

Theorem (Luo R- 2024)

Let (L, \leq) be a finite lattice. Then Trs(L) is a semidistributive lattice.

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Application of semidistributivity

A lattice (L, \leq) is semidistributive if for all $x, y, z \in L$, we

have

 $x \land (y \lor z) = (x \land y) \lor (x \land z)$ and $x \lor (y \land z) = (x \lor y) \land (x \lor z)$

whenever $(x \wedge y) = (x \wedge z)$ and $(x \vee y) = (x \vee z)$.

Theorem (Luo R- 2024)

Let (L, \leq) be a finite lattice. Then Trs(L) is a semidistributive lattice.

Proof.

Let R_1 and R_2 be two transfer systems. Then $R_1 \wedge R_2 = R_1 \cap R_2 \text{ and } R_1 \vee R_2 = (R_1 \cup R_2)^{tc}.$

E. Barnard's work on semidistributive lattices

Let (L, \leq) be a finite semidistributive lattice.

► Each element of *L* has a canonical join representation.

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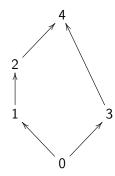
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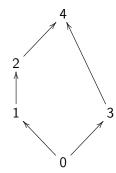
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- ► Each element of *L* has a canonical join representation.
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- ▶ A subset S of $\mathsf{Jirr}(L)$ is a canonical join representation if and only if $i \leqslant \kappa(j)$ for all $i \neq j \in S$ [Reading Speyer Thomas 2021]

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- The set $\Gamma(L)$ of all canonical join representations is a simplicial complex.

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- ▶ The complex $\Gamma(L)$ is flag : it is the clique complex of its 1-skeleton G(L).

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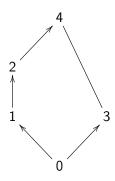
Transfer systems on the boolean

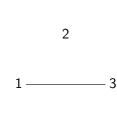
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- ▶ The set $\Gamma(L)$ of all canonical join representations is a simplicial complex.
- ▶ The complex $\Gamma(L)$ is flag : it is the clique complex of its 1-skeleton G(L).
- ▶ There is a bijection between L and the cliques of G(L).

E. Barnard's work on semidistributive lattices

Let (L, \leq) be a finite semidistributive lattice. There is a bijection between L and the cliques of G(L)





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Proposition (Yongle Luo, R- 2024)

Let (L, \leq) be a finite lattice. Then there is a bijection between the join irreducibles of Trs(L) and $Rel^*(L) = \{(a, b) \in L^2 \mid a < b\}.$

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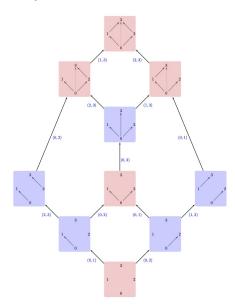
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- ▶ Vertices = $Rel^*(L)$.
- ▶ Edges (a, b) (c, d) if and only if (a, b) lifts on the left (c, d): if $a \le c$ and $b \le d$, then $b \le c$.

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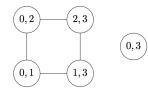
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There is an explicit bijection between cliques of the elevating graph of (L, \leq) and Trs(L).

Rather efficient algorithm to compute the number of transfer systems $\!!$

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▶ The number of transfer systems on $\mathcal{P}(\{1,2,3,4\})$ is 5389480.

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▶ The algorithm is not good enough for n = 5.

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▶ The algorithm is not good enough for n = 5.

Application of

▶ The boolean lattice $\mathcal{P}(\{1,\cdots,n\})$ is the lattice of subgroups of a squarefree elementary abelian group. semidistributivity

A lower bound for the number of transfer systems

Obvious remark

If there is a clique of size r in $\mathcal{G}(L)$, then there are at least 2^r transfer systems.

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Transfer systems on the boolean lattices

Obvious remark

If there is a clique of size r in G(L), then there are at least 2^r transfer systems.

Lemma

Let $B_n = \mathcal{P}([n])$. Then there is a clique of size $a_n =$

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- $\sum_{i=1}^{\frac{n}{2}} \binom{n}{i} \binom{n-j}{n+1-2i}$ If n is even.

Proof.

For $1 \le k \le 2n-1$. $\mathcal{R}_k : X \triangleleft_k Y$ if and only if X = Y or $X \subset Y$ and and $|X| + |Y| \leq k$.

2 1 2 33733 4 6 16 13 16 6 4 5 10 30 35 **51** 35 30 10 5 6 15 50 75 126 121 126 75 50 15 6 7 21 77 140 266 322 <mark>393</mark> 322 266 140 77 21 7

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Lattices of transfer systems

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Lattices of transfer systems

Yongle Luo and Baptiste Rognerud

, iii example

Homotopical combinatoric

rotar oraci.

Main results

Application of semidistributivity