Multigraded strong Lefschetz property for balanced simplicial complexes

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face numbers

For a $(d-1)$-dimensional simplicial complex Δ, let $f_i (= f_i(\Delta))$ be the number of i-dimensional faces of Δ.

- $(f_{-1}, f_0, \ldots, f_{d-1})$ is called the f-vector of Δ

Stanley-Reisner ring

Assumption: Δ is always pure $(d-1)$-dimensional and $V(\Delta) = [n]$.
Notation: $x_\tau := \prod_{i \in \tau} x_i$.

Let k be an infinite field.
Stanley-Reisner ring of Δ is $k[\Delta] := k[x_1, \ldots, x_n]/I_\Delta$, where $I_\Delta := (x_\tau : \tau \notin \Delta)$.

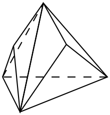
- Under natural $\mathbb{N}$-grading,
  $\mathrm{Hilb}(k[\Delta], t)(:= \sum_{i=0}^{\infty} \dim_k k[\Delta]_i t^i) = \sum_{i=0}^{d} f_{i-1} \frac{t^i}{(1-t)^i}$.

–Combinatorics part–

h-vector

The h-vector $(h_0, \dots, h_d)$ of Δ is defined by

$$\frac{\sum_{i=0}^{d} h_i t^i}{(1-t)^d} = \sum_{i=0}^{d} f_{i-1} \frac{t^i}{(1-t)^i} \quad (= \mathrm{Hilb}(k[\Delta], t)).$$



$$(f_{-1}, f_0, f_1, f_2) = (1, 6, 12, 8)$$
$$\updownarrow$$
$$(h_0, h_1, h_2, h_3) = (1, 3, 3, 1)$$

Simplicial sphere

Δ is a simplicial $(d-1)$-sphere if $\|\Delta\| \overset{\text{homeo.}}{\simeq} \mathbb{S}^{d-1}$.

Theorem (Dehn-Sommerville relation) For a simplicial $(d-1)$-sphere Δ, $h_i = h_{d-i}$ for $i = 0, \ldots, \lfloor \frac{d}{2} \rfloor$.

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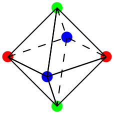
Generalized Lower Bound Inequality (Adiprasito 2018, Papadakis-Petrotou 2020)

If Δ is a simplicial $(d-1)$-sphere, $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor \frac{d}{2} \rfloor} \geq \cdots \geq h_d$.

Balancedness

A pure $(d-1)$-dim. simplicial complex Δ is balanced if there is $\kappa : V(\Delta) \to [d]$ such that for every facet σ of Δ, $|\sigma \cap \kappa^{-1}(i)| = 1$ for every $i \in [d]$.



- E.g. Barycentric subdivision, order complex

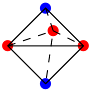
Balanced Generalized Lower Bound Inequality (Juhnke-Murai 2018 +Hard Lefschetz Theorem)

If Δ is a balanced simplicial $(d-1)$-sphere, $\frac{h_0}{\binom{d}{0}} \leq \frac{h_1}{\binom{d}{1}} \leq \cdots \leq \frac{h_{\lfloor d/2 \rfloor}}{\binom{d}{\lfloor d/2 \rfloor}}$.

$\boldsymbol{a}$-balancedness

For $\boldsymbol{a} = (a_1, \ldots, a_m) \in \mathbb{Z}^m_{>0}$, let $|\boldsymbol{a}| := a_1 + \cdots + a_m = d$.

A pure $(d-1)$-dim. simplicial complex Δ is $\boldsymbol{a}$-balanced if there is $\kappa : V(\Delta) \to [m]$ such that for every facet σ of Δ, $|\sigma \cap \kappa^{-1}(j)| = a_j$ for each $j \in [m]$.



$(2, 1)$-balanced

E.g.

- $\Delta_1 * \cdots * \Delta_m$ is $(a_1, \ldots, a_m)$-balanced if $\dim \Delta_i = a_i - 1$.
- Partial barycentric subdivision

Flag h-vector

Suppose that Δ is $\boldsymbol{a}$ ($\in \mathbb{Z}^m_{>0}$)-balanced with the coloring $\kappa : V(\Delta) \to [m]$.
$k[\Delta]$ is $\mathbb{N}^m$-graded by $\deg x_v = \boldsymbol{e}_{\kappa(v)} \in \mathbb{N}^m$.

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Define flag f, h-vector as follows:

- For $\boldsymbol{0} \le \boldsymbol{b} \le \boldsymbol{a}$, $f_{\boldsymbol{b}} := |\{\tau \in \Delta : |\tau \cap \kappa^{-1}(j)| = b_j \text{ for all } j \in [m]\}|$.
- Using variables $\boldsymbol{t} = (t_1, \dots, t_m)$, define $(h_{\boldsymbol{b}})_{\boldsymbol{0} \le \boldsymbol{b} \le \boldsymbol{a}}$ by

$$\frac{\sum_{\boldsymbol{b}} h_{\boldsymbol{b}} \boldsymbol{t}^{\boldsymbol{b}}}{\prod_{i=1}^m (1-t_i)^{a_i}} = \sum_{\boldsymbol{b}} f_{\boldsymbol{b}} \frac{\boldsymbol{t}^{\boldsymbol{b}}}{\prod_{i=1}^m (1-t_i)^{b_i}} \quad (= \mathrm{Hilb}(k[\Delta], t_1, \dots, t_m))$$

Result on flag h-vector

Theorem (O.)

Suppose that Δ is an $\boldsymbol{a}$-balanced simplicial sphere.
If $2\boldsymbol{b} + \boldsymbol{e}_i \leq \boldsymbol{a}$, then $h_{\boldsymbol{b}} \leq h_{\boldsymbol{b}+\boldsymbol{e}_i}$ holds.

$$
\begin{array}{ccccccc}
h_{(0,3)} & & h_{(1,3)} & & h_{(2,3)} & & h_{(3,3)} \\
h_{(0,2)} & & h_{(1,2)} & & h_{(2,2)} & & h_{(3,2)} \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & & & \\
h_{(0,1)} & \leq & h_{(1,1)} & \leq & h_{(2,1)} & & h_{(3,1)} \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & & & \\
h_{(0,0)} & \leq & h_{(1,0)} & \leq & h_{(2,0)} & & h_{(3,0)}
\end{array}
\qquad \boldsymbol{a} = (3,3)
$$

- Common generalization of GLBI and balanced GLBI

–Algebra part–

Artinian reduction

Let Δ be a $(d-1)$-dim. simplicial complex.

A length d sequence $\Theta = (\theta_1, \ldots, \theta_d)$ of linear forms of $k[\Delta]$ is called a linear system of parameters (l.s.o.p.) for $k[\Delta]$ if $\dim_k k[\Delta]/(\Theta) < \infty$.

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Theorem (Hochster) If Δ is a simplicial $(d-1)$-sphere, $k[\Delta]$ is Gorenstein*:

Let Θ be an l.s.o.p. for $k[\Delta]$ and let $A := k[\Delta]/(\Theta) = A_0 \oplus \cdots \oplus A_d$. Then

- $\dim_k A_i = h_i$ for each $i \in [d]$.
- Poincaré duality holds for A, namely
  - $\times : A_i \times A_{d-i} \to A_d \cong k$ is a perfect pairing for each $i \in [d]$.

Hard Lefschetz Theorem

Hard Lefschetz Theorem (Adiprasito 2018, Papadakis-Petrotou 2020)

Let $k = \mathbb{R}$ and Δ be a simplicial $(d-1)$-sphere. Let $\theta_1, \ldots, \theta_d$ be generic linear forms of $\mathbb{R}[\Delta]$. Then $A := k[\Delta]/(\Theta) = A_0 \oplus \cdots \oplus A_d$ has **Strong Lefscehtz Property**:

$\exists \ell \in A_1$ s.t. $\times \ell^{d-2i} : A_i \to A_{d-i}$ is an isomorphism for any $i \leq \lfloor d/2 \rfloor$.

- In particular GLBI holds since $\times \ell : A_i \to A_{i+1}$ is injective for $i < d/2$.

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Anisotropy (Papadakis-Petrotou 2020)

Let k be the rational function field $\mathbb{F}_2(p_{iv} : i \in [d], v \in V(\Delta))$. Then the generic Artinian reduction $A := k[\Delta]/(\Theta) = A_0 \oplus \cdots \oplus A_d$ has **anisotropy**:
For nonzero $g \in A_i$ $(i \leq \lfloor d/2 \rfloor)$, $g^2 \neq 0$.

Proof via generic Artinian reduction

Setting

- **simplicial complex** simplicial $(d-1)$-sphere Δ
- **field** $k := \mathbb{F}_2(p_{iv} : i \in [d], v \in V(\Delta))$, where p_{iv}s' are new indeterminates.
- **l.s.o.p.** $(\theta_1, \dots, \theta_d)^\top := P(x_1 \dots, x_n)^\top$, where $P[i,v] = p_{iv}$.
- **lefschetz element** $\ell := x_1 + \cdots + x_n$

Let $A := k[\Delta]/(\Theta)$.

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Let $A := k[\Delta]/(\Theta)$.

Theorem (Karu-Xiao 2023)

For $i \leq \lfloor d/2 \rfloor$, let $\mathcal{Q} : A_i \to A_d; g \mapsto g^2 \ell^{d-2i}$. Then, $\mathcal{Q}(g) \neq 0$ if $g \neq 0$.

- Differential Formula: under normalization, $\Psi : k[x_1, \ldots, x_n]_d \twoheadrightarrow A_d \overset{\simeq}{\to} k$ satisfies

$$\partial_{p_{1v_1}} \circ \cdots \circ \partial_{p_{dv_d}} \Psi(x_J) = \Psi(\sqrt{x_{v_1} \cdots x_{v_d} x_J})^2$$

Multigraded setting

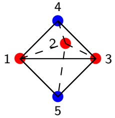
$\boldsymbol{a}$-colored s.o.p.

For $\boldsymbol{a} \in \mathbb{Z}^m_{>0}$ with $|\boldsymbol{a}| = d$, let Δ be an $\boldsymbol{a}$-balanced simplicial complex.

Proposition (Stanley 1979) There is $\mathbb{N}^m$-homogeneous l.s.o.p. $\Theta = (\theta_1, \dots, \theta_d)$ for $k[\Delta]$ called an $\boldsymbol{a}$-colored s.o.p.

- $\boldsymbol{a}$-colored s.o.p. Θ contains a_i elements of degree $\boldsymbol{e}_i$ for each $i \in [m]$

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ 0 & 0 & 0 & * & * \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$



$(2, 1)$-balanced

Artinian reduction as multigraded algebra

$$\begin{array}{cccc} h_{(0,3)} & h_{(1,3)} & h_{(2,3)} & h_{(3,3)} \\ h_{(0,2)} & h_{(1,2)} & h_{(2,2)} & h_{(3,2)} \\ h_{(0,1)} & h_{(1,1)} & h_{(2,1)} & h_{(3,1)} \\ h_{(0,0)} & h_{(1,0)} & h_{(2,0)} & h_{(3,0)} \end{array} \qquad \begin{array}{ccccccc} A_{(0,3)} & & A_{(1,3)} & & A_{(2,3)} & & A_{(3,3)} \\ A_{(0,2)} & & A_{(1,2)} & & A_{(2,2)} & & A_{(3,2)} \\ \uparrow & & \uparrow & & & & \\ A_{(0,1)} & \to & A_{(1,1)} & \to & A_{(2,1)} & & A_{(3,1)} \\ \uparrow & & \uparrow & & & & \\ A_{(0,0)} & \to & A_{(1,0)} & \to & A_{(2,0)} & & A_{(3,0)} \end{array}$$

Theorem (Juhnke-Murai 2018 + HL) If $\boldsymbol{a} = (a_1, a_2)$, in the above situation, if Θ is generic (a_1, a_2)-colored s.o.p., then $\times \ell_1^{a_1 - 2i} : A_{(i,0)} \to A_{(a_1 - i, 0)}$ is injective for generic $\ell_1 \in A_{(1,0)}$

Anisotropy/Lefschetz result

- **simplicial complex** $\boldsymbol{a}$-balanced simplicial sphere Δ
- **field** $k := \mathbb{F}_2(p_{jv})$
- **s.o.p.** $(\theta_1, \ldots, \theta_d)^\top := P(x_1, \ldots, x_n)^\top$, where
  $$P[j, v] = \begin{cases} p_{jv} & \text{if } j \in \mathcal{I}_{\kappa(v)} \\ 0 & \text{o.w.} \end{cases}$$
- **lefschetz elements** $\ell_i := \sum_{v \in \kappa^{-1}(i)} x_v$ for each $i \in [m]$

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Theorem (O.)

For $\boldsymbol{b} \leq \frac{\boldsymbol{a}}{2}$, let $\mathcal{Q} : A_{\boldsymbol{b}} \to A_{\boldsymbol{a}}; g \mapsto g^2 \ell_1^{a_1 - 2b_1} \cdots \ell_m^{a_m - 2b_m}$. Then $\mathcal{Q}(g) \neq 0$ if $g \neq 0$.

In particular, **Multigraded SLP** holds: $\times \prod_j \ell_j^{a_j - 2b_j} : A_{\boldsymbol{b}} \to A_{\boldsymbol{a}-\boldsymbol{b}}$ is an isomorphism for each $\boldsymbol{b} \leq \boldsymbol{a}/2$.

$$\begin{array}{cccc}
A_{(0,3)} & A_{(1,3)} & A_{(2,3)} & A_{(3,3)} \\
 & & \uparrow \times \ell_2 & \\
A_{(0,2)} & A_{(1,2)} & A_{(2,2)} & A_{(3,2)} \\
 & & \uparrow \times \ell_2 & \\
A_{(0,1)} & A_{(1,1)} & A_{(2,1)} & A_{(3,1)} \\
 & & \uparrow \times \ell_2 & \\
A_{(0,0)} & A_{(1,0)} \xrightarrow{\times \ell_1} & A_{(2,0)} & A_{(3,0)}
\end{array}$$

More corollaries

- (conjectured by Kalai-Nevo-Novik 2016) If Δ is $\mathbf{1}_{k+1}$-balanced and Δ is a subcomplex of a simplicial $2k$-sphere, then $f_k \leq 2f_{k-1}$.
- When $t \geq 2$, the bipartite graph of an $(t-1, t)$-balanced connected triangulated manifold is spanning in bipartite (t, t)-rigidity matroid (also known as rank t matrix completion matroid).

Thank you!