The positive orthogonal Grassmannian

joint work with Yassine El Maazouz

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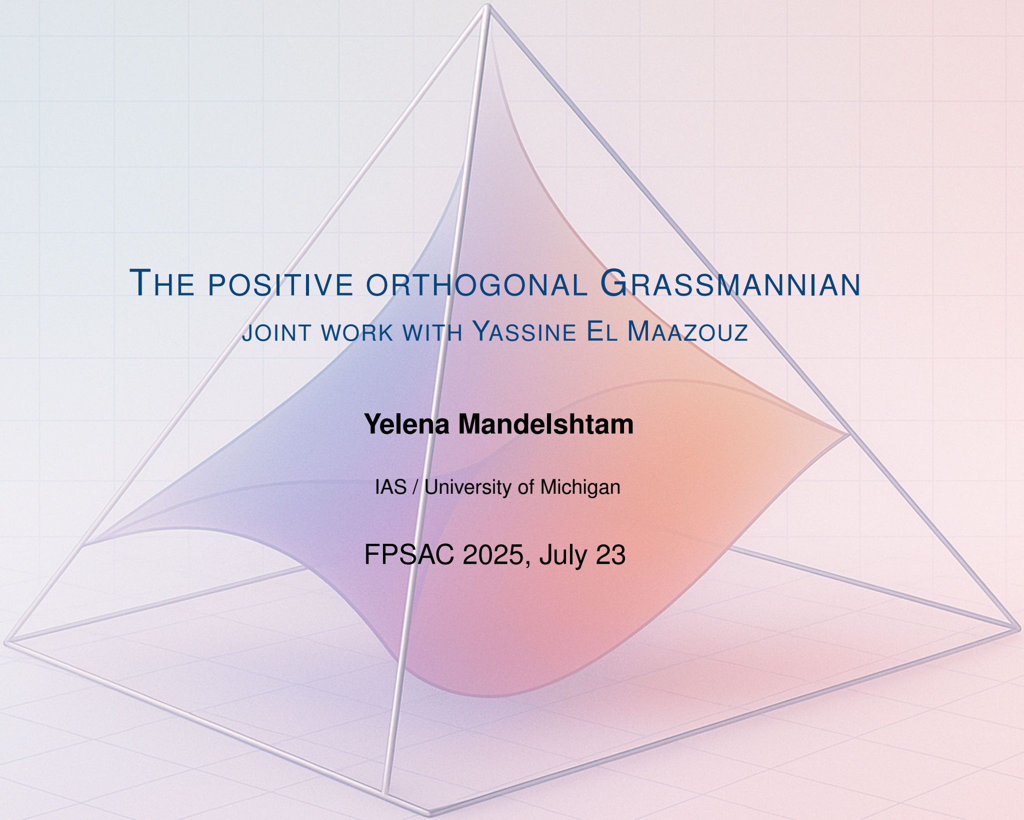

The positive Grassmannian $\mathrm{Gr}_{\geq 0}(k, n)$

- $\mathrm{Gr}_{\mathbb{R}}(k, n)$ parameterizes k-dimensional subspaces in $\mathbb{R}^n$
- $\mathrm{Gr}(k, n) = \mathrm{Mat}_{k \times n} /$ left multiplication by GL_k.
- Embed into $\mathbb{P}^{\binom{n}{k}-1}$ via $k \times k$ minors, called the Plücker coordinates and denoted p_I, for $I \in \binom{[n]}{k}$.
  - $\begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix} \rightsquigarrow [1 : c : -a : d : -b : ad - bc] \in \mathbb{P}^5$
- The Grassmannian is a projective variety cut out by the Plücker relations
  - $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}$
- The positive Grassmannian $\mathrm{Gr}_{\geq 0}(k, n)$ is the subset of $\mathrm{Gr}(k, n)$ where all p_I have the same sign.
  - Admits a stratification by positroid cells that can be indexed by combinatorial objects like Grassmann necklaces, decorated permutations, plabic graphs and Le diagrams [Postnikov].

What about type D?

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THE ORTHOGONAL GRASSMANNIAN

- Let ω be the nondegenerate bilinear form on $\mathbb{C}^n$

$$\omega(x, y) = x_1 y_1 - x_2 y_2 + \cdots + (-1)^{n-1} x_n y_n.$$

- A subspace $V \subset \mathbb{C}^n$ is isotropic if $\omega(v, w) = 0$ for all $v, w \in V$.
- The orthogonal Grassmannian

$$\mathrm{OGr}(k, n) \;=\; \{\, V \in \mathrm{Gr}(k, n) \mid \omega|_{V \times V} \equiv 0\}$$

  parametrizes all such k-dimensional isotropic subspaces.
- In coordinates, one imposes additional quadratic relations among the Plücker coordinates p_I to enforce $\omega(v, w) = 0$.
  - For example, $\mathrm{OGr}(1, n)$ is cut out by one relation, $p_1^2 - p_2^2 + p_3^2 + \ldots (-1)^{n-1} p_n^2 = 0$
- As with $\mathrm{Gr}(k, n)$, it is also interesting to consider the positive part of $\mathrm{OGr}(k, n)$, which we denote by $\mathrm{OGr}_+(k, n)$

The Case $n = 2k$: Physics and Mathematical Foundations

- The set $\mathrm{OGr}_+(k, 2k)$ was first studied by physicists. In 3D Chern–Simons–matter ABJM theory, tree-level amplitudes find a positive geometry description in $\mathrm{OGr}_+(k, 2k)$ [Huang-Wen, 2013]
- Huang-Wen and Huang-Wen-Xie stated many observations about the combinatorics of $\mathrm{OGr}_+(k, 2k)$
- Galashin-Pylyavskyy studied $\mathrm{OGr}_+(k, 2k)$ from a mathematical point of view and connected it to the Ising model in 2018
  - $\mathrm{OGr}(k, 2k)$ defined by Plücker relations and $p_I = \pm p_{[2k]\setminus I}$
  - Developed a cell decomposition of $\mathrm{OGr}_+(k, 2k)$ indexed by fixed-point-free involutions on $[2k]$
  - Detailed combinatorial description of the stratification
  - Provided parameterizations for cells (cell structure induced from the positroid cell stratification of $\mathrm{Gr}_{\geq 0}(k, n)$).

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Our goal: depart from $n = 2k$ and find interesting structure for other values of n.

Our Main Results

1. **Geometry and commutative algebra of** $\mathrm{OGr}(k, n)$: equations cutting out the ideal, Gröbner basis, dimension, degree, primality
2. **Quadric case (**$k = 1$**)**: $\mathrm{OGr}_+(1, n)$ is a positive geometry; full boundary description
3. **The case (**$n = 2k + 1$**)**:
$$\mathrm{OGr}_+(k, 2k + 1) \cong \mathrm{OGr}_+(k + 1, 2k + 2)$$
inherits the combinatorics of matchings on $[2k + 2]$
4. **General failure**: For $n > 2k + 1$, positroid cells of $\mathrm{Gr}_+(k, n)$ *do not* induce CW decomposition on $\mathrm{OGr}_+(k, n)$

Equations Cutting Out $\mathrm{OGr}^{\omega}(k, n)$

- We work in the Plücker embedding

$$\mathrm{OGr}^{\omega}(k,n) \subset \mathbb{P}^{\binom{n}{k}-1} \quad \text{with homogeneous coordinates. } (p_I)_{|I|=k}.$$

- The orthogonal relations are the $\frac{1}{2}\binom{n}{k-1}\left(\binom{n}{k-1}+1\right)$ equations of the form

$$\sum_{\ell=1}^{n}(-1)^{(\ell-1)}\varepsilon(I\ell)\,\varepsilon(J\ell)\,p_{I\ell}\,p_{J\ell} \;=\; 0, \quad I, J \in \binom{[n]}{k-1},$$

where $\varepsilon(I\ell)$ is the sign of the ordering of $I \cup \{\ell\}$.

Commutative algebra and geometry of $\mathrm{OGr}(k, n)$

- Gröbner basis: comes from "straightening-law" quadrics, where each non-standard Plücker monomial is rewritten as an alternating sum over permutations of corresponding skew Young tableau entries
- Dimension:

$$\dim \mathrm{OGr}(k,n) = \dim \mathrm{Gr}(k,n) - \tfrac{k(k+1)}{2} = k(n-k) - \tfrac{k(k+1)}{2} = \tfrac{k(2n-3k-1)}{2}.$$

- Degree:
  - Each degree ℓ piece of the homogeneous coordinate ring of $\mathrm{OGr}(k,n)$ is an irreducible representation of $\mathrm{SO}(n)$ corresponding to a specific highest weight vector [Borel-Weil-Bott]
  - Weyl dimension formula allows us to compute the dimensions of these representations and then the Hilbert polynomial of the coordinate ring.

$$D! \cdot \left(\prod_{\substack{1\le i\le k\\ k<j\le m}} \frac{1}{(2m-i-j)(j-i)} \right) \left(\prod_{1\le i<j\le k} \frac{2}{2m-i-j} \right), \qquad \text{if } n = 2m,$$

$$D! \cdot \left(\prod_{1\le i\le k} \frac{2}{2m-2i+1} \right) \left(\prod_{\substack{1\le i\le k\\ k<j\le m}} \frac{1}{(2m-i-j)(j-i)} \right) \left(\prod_{1\le i<j\le k} \frac{2}{2m-i-j+1} \right), \quad \text{if } n = 2m+1.$$

where $m := \lfloor n/2 \rfloor$ and $D := k(n-k) - \binom{k+1}{2}$

THE QUADRIC $\mathrm{OGr}_+(1, n)$

- $\mathrm{OGr}(1, n) \subset \mathbb{P}^{n-1}$: single quadric hypersurface

$$\sum_{i \in [n] \cap (2\mathbb{Z}+1)} x_i^2 = \sum_{j \in [n] \cap 2\mathbb{Z}} x_j^2 \quad \text{and} \quad x \in \mathbb{P}_+^{n-1}.$$

  Boundaries of cells are obtained by driving some of the coordinates to 0. Each boundary is a lower dimensional $\mathrm{OGr}_+(1, n')$
- Combinatorially isomorphic to the product of simplices $\Delta_{p-1} \times \Delta_{q-1}$
- We describe the cell poset structure, give parameterizations, and prove that $\mathrm{OGr}_+(1, n)$ is a positive geometry

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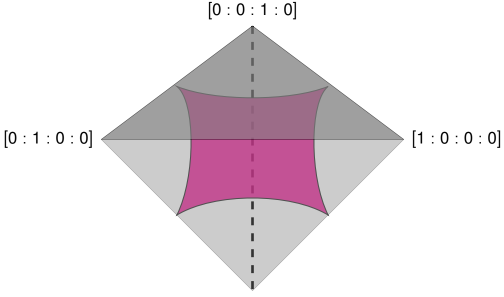
Example: $n = 4$ The points $(x_1 : x_2 : x_3 : x_4)$ in the positive orthogonal Grassmannian $\mathrm{OGr}_+(1, 4)$ in $\mathbb{P}^3$ are those that satisfy:

$$x_1^2 - x_2^2 + x_3^2 - x_4^2 = 0 \text{ and } x_1, x_2, x_3, x_4 \geq 0.$$

Then $\mathrm{OGr}_+(1, 4)$ is a curvy quadrilateral inside the 3-simplex in $\mathbb{P}_+^3$



Poset structure of $\mathrm{OGr}_+(1, n)$

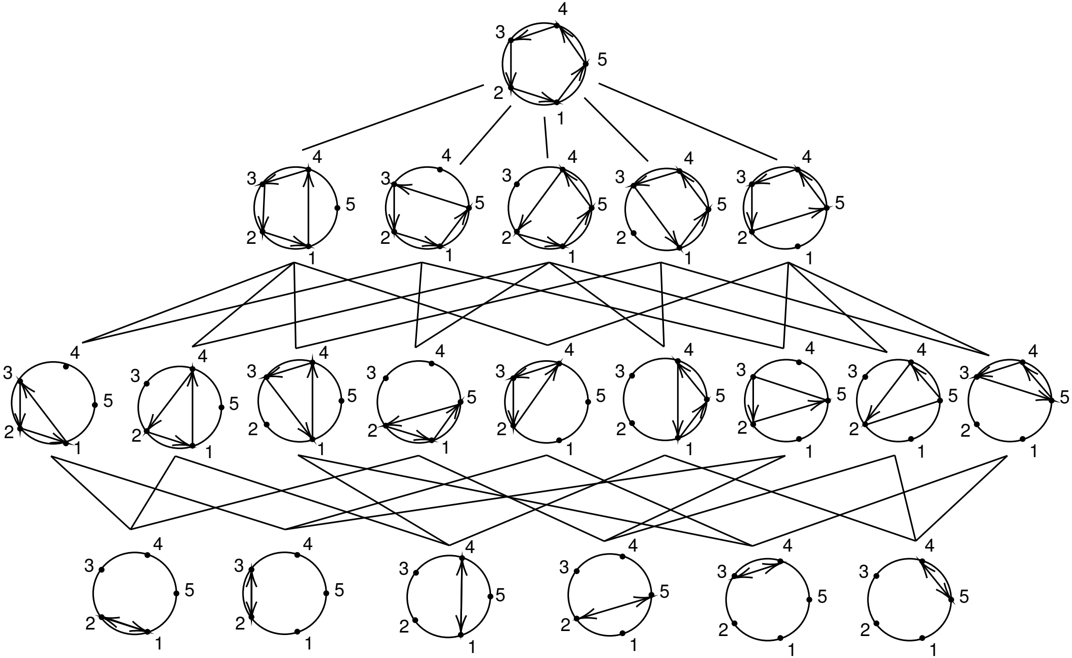


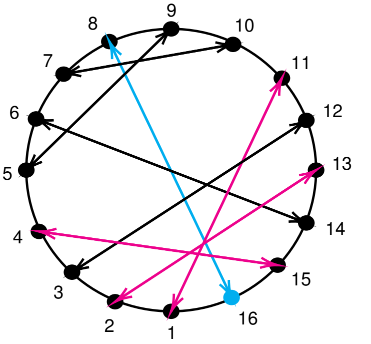
Figure. The Hasse diagram of the poset structure on $\mathfrak{S}_{1,5}$.

ISOMORPHISM FOR $n = 2k + 1$

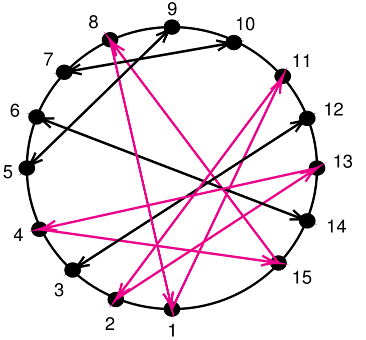
- Map sending a k-plane in $\mathrm{OGr}_+(k, 2k+1)$ to $(k+1)$-plane in $\mathrm{OGr}_+(k+1, 2k+2)$

$$\begin{array}{rccc} \Phi_k : & \mathrm{OGr}(k+1, 2k+2) & \to & \mathrm{OGr}(k, 2k+1) \\ & (q_J)_{J \in \binom{[2k+2]}{k+1}} & \mapsto & (p_I = q_{I \cup \{2k+2\}})_{I \in \binom{[2k+1]}{k}} \end{array}$$

- Cells correspond to perfect matchings on $[2k+2]$



τ

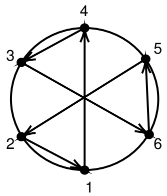


σ

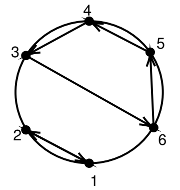
The equations that cut out $\mathrm{OGr}(k, 2k+1)$ in $\mathrm{Gr}(k, 2k+1)$ are all quadrics. So it is remarkable that we can still describe the face structure of $\mathrm{OGr}_+(k, 2k+1)$ from our understanding of the face structure of $\mathrm{OGr}_+(k+1, 2k+2)$ which is obtained by taking a linear slice of $\mathrm{Gr}_+(k+1, 2k+2)$!

WHAT GOES WRONG FOR $k > 1, n > 2k+1$?

Key example: the following orthopositroid cells σ and τ in $\mathrm{OGr}_+(2,6)$:



σ



τ

The two-dimensional cells $C_\sigma = \Pi_\sigma \cap \mathrm{OGr}_+(2,6)$ and $C_\tau = \Pi_\tau \cap \mathrm{OGr}_+(2,6)$ are described by

$$M_\sigma = \begin{bmatrix} 1 & 1 & 0 & 0 & -x & -x \\ 0 & 0 & 1 & 1 & y & y \end{bmatrix}, \qquad \text{where } x, y > 0,$$

$$M_\tau = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & a & b & c \end{bmatrix}, \qquad \text{where } \begin{cases} a, b, c > 0 \\ 1 + b^2 = a^2 + c^2 \end{cases}.$$

The closure of the cell C_σ has the combinatorial type of a triangle. Its edges are given by:

$$e_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & b & b \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 1 & 1 & b & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & b & b \end{bmatrix} \quad b \geq 0.$$

The closure of the cell C_τ is isomorphic to $\mathrm{OGr}_+(1,4)$ so it is a square.

$$M_\sigma = \begin{bmatrix} 1 & 1 & 0 & 0 & -x & -x \\ 0 & 0 & 1 & 1 & y & y \end{bmatrix}, \qquad \text{where } x, y > 0,$$

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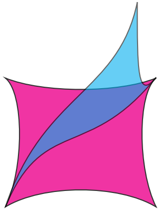
The closure of C_σ is a triangle with edges:

$$e_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & -b & -b \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 1 & 1 & b & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & b & b \end{bmatrix} \quad b \geq 0.$$

The edge e_3 is one of the diagonals of the “square" C_τ.



This problem arises as soon as $n > 2k + 1$. We can extend any 2×6 matrix in $\mathrm{OGr}_+(2, 6)$ by a $(k-2) \times (n-6)$ matrix to make an element of $\mathrm{OGr}_+(k, n)$:

$$\begin{bmatrix}
1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
0 & 0 & 1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
0 & \cdots & \cdots & & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & & & (0) & & & \\
\vdots & \cdots & \cdots & \cdots & 0 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 & & & & & & \\
 & & & & & & & & & & & & * & * & * & * & * & * \\
 & & & & & (0) & & & & & & & * & * & * & * & * & *
\end{bmatrix}.$$

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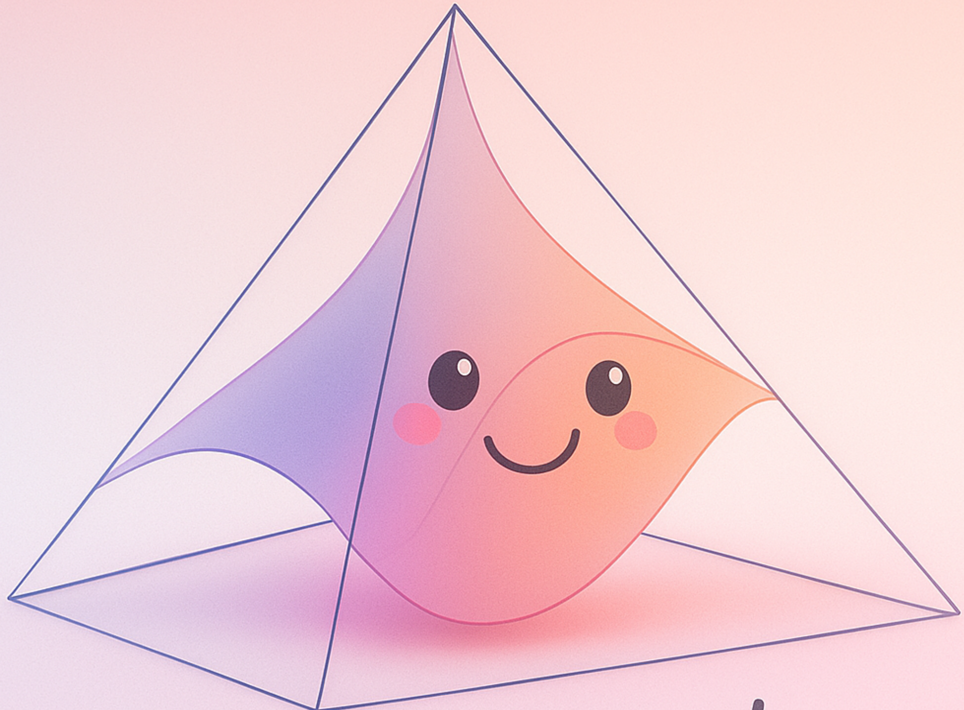
0 & 0 & 1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
0 & \cdots & \cdots & & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & & & (0) & & & \\
\vdots & \cdots & \cdots & \cdots & 0 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 & & & & & & \\
 & & & & & & & & & & & & * & * & * & * & * & * \\
 & & & & & (0) & & & & & & & * & * & * & * & * & *
\end{bmatrix}.$$

We need new combinatorics to give a CW cell decomposition of $\mathrm{OGr}_+(k,n)$ when $n > 2k+1$ and $k > 1$.

Open Questions

- General boundary classification for arbitrary (k, n)
- Alternative (more refined) cell decompositions?
- Computation of canonical forms: toward ABJM amplitude formulas
- Connections to cluster algebras in the orthogonal setting



Thank you!