Matrix Loci, Orbit Harmonics, and Shadow Play

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Schensted Correspondence

$$\begin{array}{ccc} \mathfrak{S}_n & \longrightarrow & \coprod_{\lambda \vdash n} SYT(\lambda) \times SYT(\lambda) \\ w & \longmapsto & (P(w), Q(w)) \end{array}$$

Schensted Correspondence

$$\mathfrak{S}_n \longrightarrow \coprod_{\lambda \vdash n} SYT(\lambda) \times SYT(\lambda)$$

$$w \longmapsto (P(w), Q(w))$$

$$w = [2, 6, 3, 5, 4, 8, 7, 1] \in \mathfrak{S}_8$$

$$w \longmapsto \left(\begin{array}{llll} 1 & 3 & 4 & 7 \\ 2 & 8 & & \\ 5 & & & \\ 6 & & & \end{array} \;,\; \begin{array}{llll} 1 & 2 & 4 & 6 \\ 3 & 7 & & \\ 5 & & & \\ 8 & & & \end{array} \right)$$

Schensted Correspondence

$$\mathfrak{S}_n \longrightarrow \bigsqcup_{\lambda \vdash n} SYT(\lambda) \times SYT(\lambda)$$

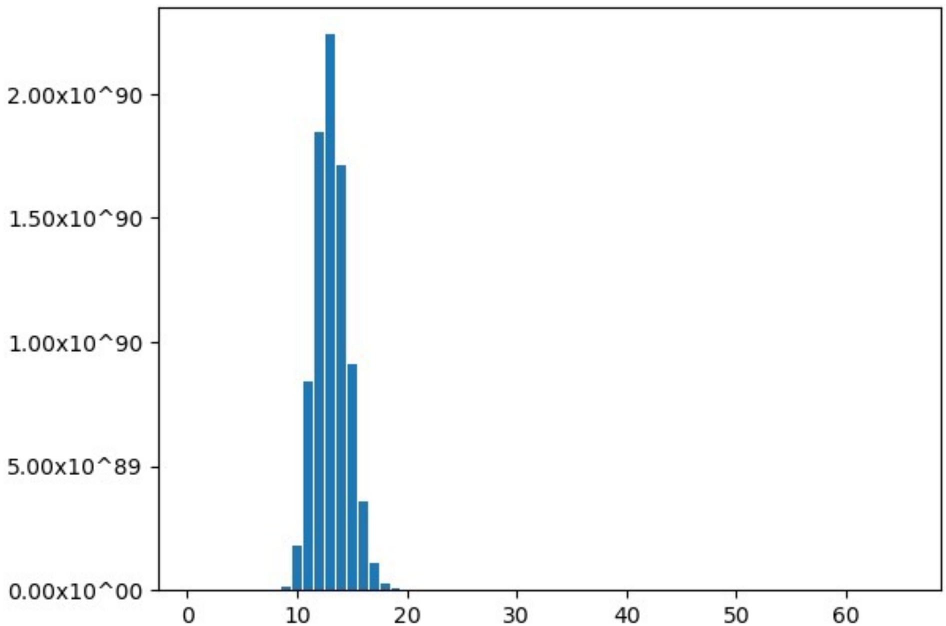
$$w \longmapsto (P(w), Q(w))$$

$$w = [\underline{2}, 6, \underline{3}, \underline{5}, 4, \underline{8}, 7, 1] \in \mathfrak{S}_8 \qquad \mathrm{lis}(w) = 4$$

$$w \longmapsto \left(\begin{array}{llll} 1 & 3 & 4 & 7 \\ 2 & 8 & & \\ 5 & & & \\ 6 & & & \end{array} \quad , \quad \begin{array}{llll} 1 & 2 & 4 & 6 \\ 3 & 7 & & \\ 5 & & & \\ 8 & & & \end{array} \right) \qquad \lambda_1 = \mathrm{lis}(w)$$

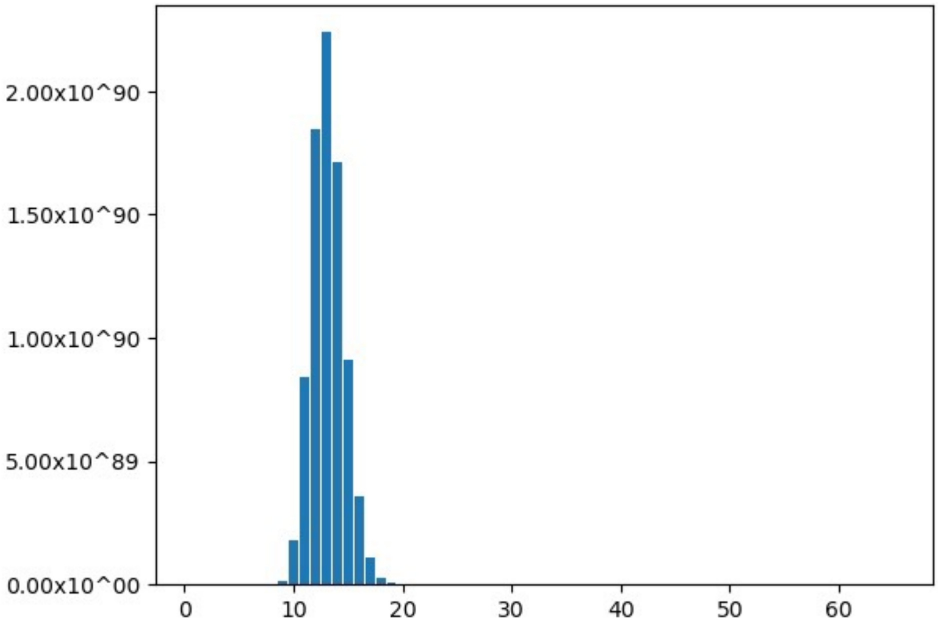
Distribution of $\mathrm{lis}(w)$

$$a_{n,k} := \#\{w \in \mathfrak{S}_n : \mathrm{lis}(w) = k\}$$



$a_{n,k}$ when $n = 65$

Distribution of lis(w)



$a_{n,k}$ when $n=65$

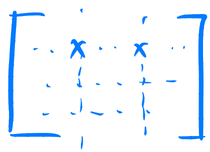
Chen's Conjecture $\{a_{n,k}\}$ is log-concave: $a_{n,k-1} \cdot a_{n,k+1} \leq a_{n,k}^2$

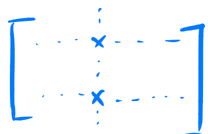
Baik-Deift-Johansson $\{a_{n,k}\}$ converges to the Tracy-Widom distribution of random GUE matrices as $n \to \infty$.

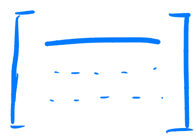
$$X_{n\times n} = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}$$ variables

$I_n \subseteq \mathbb{C}[X_{n\times n}]$ is the ideal generated by

$x_{i,j}^2$

$x_{i,j}\cdot x_{i,j'}$ 

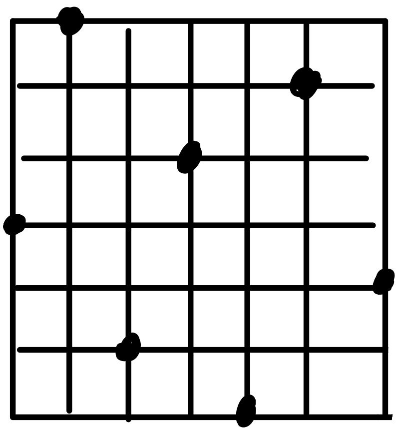
$x_{i,j}\cdot x_{i',j}$ 

$x_{i,1}+x_{i,2}+\cdots+x_{i,n}$ 

$x_{1,j}+x_{2,j}+\cdots+x_{n,j}$ 

Viennot Shadow

$w = [4, 7, 2, 5, 1, 6, 3] \in \mathfrak{S}_7$



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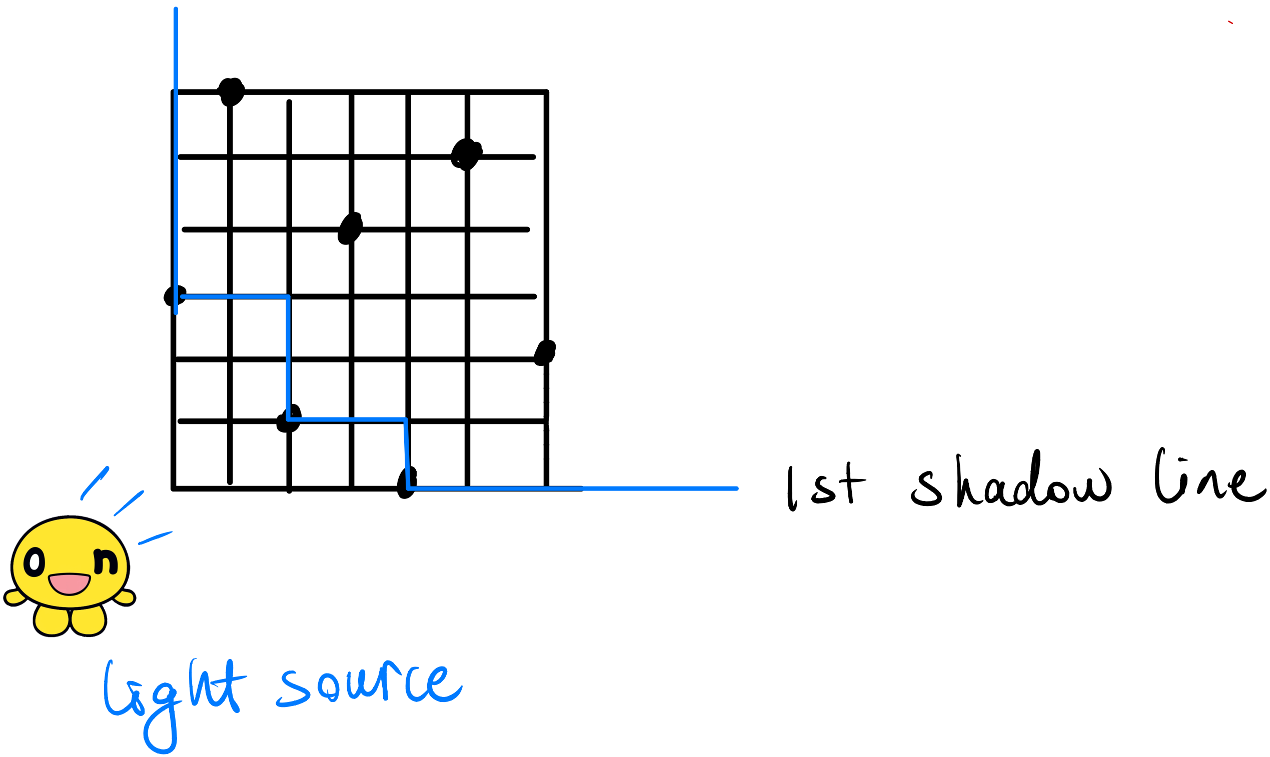

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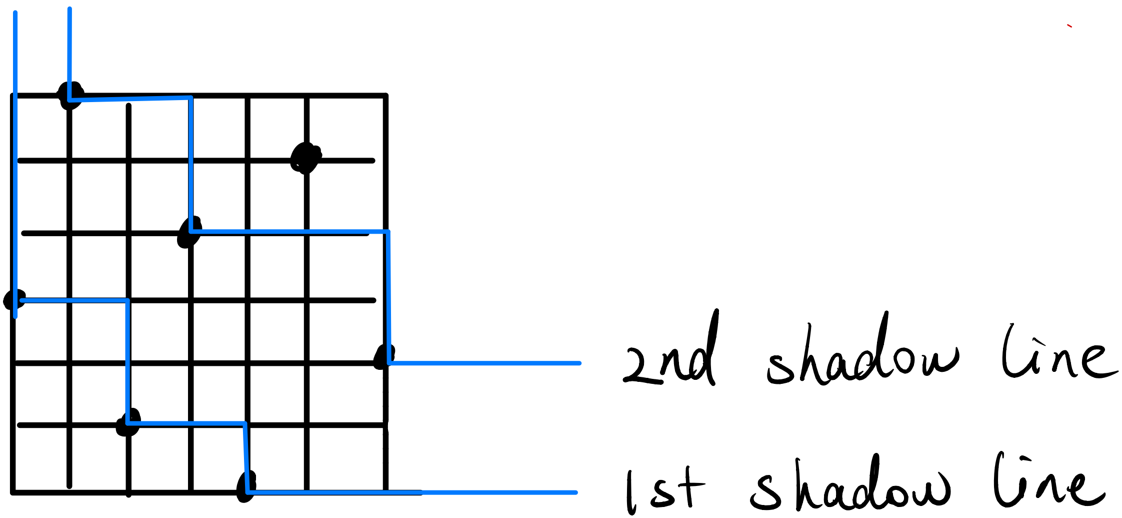

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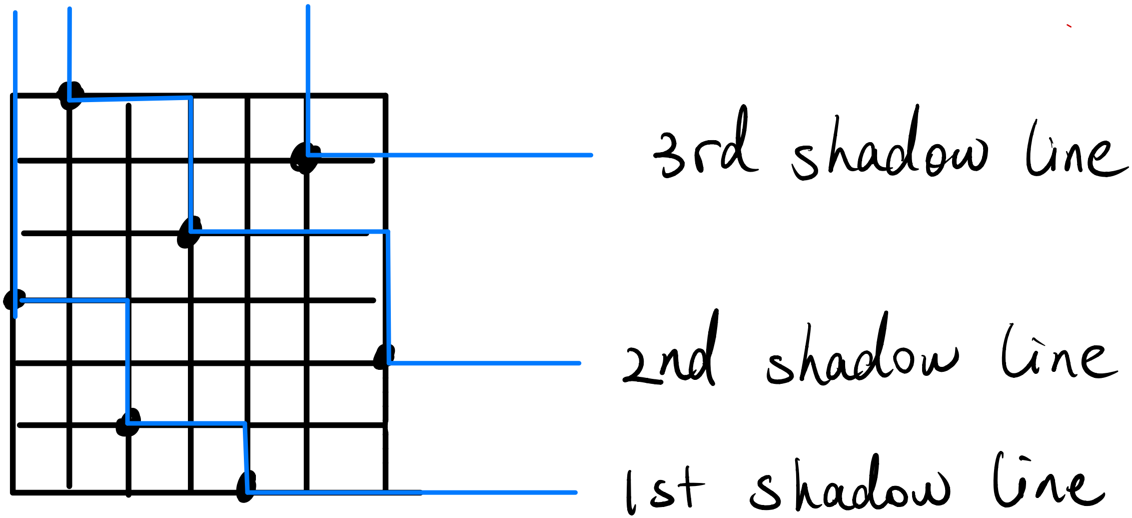

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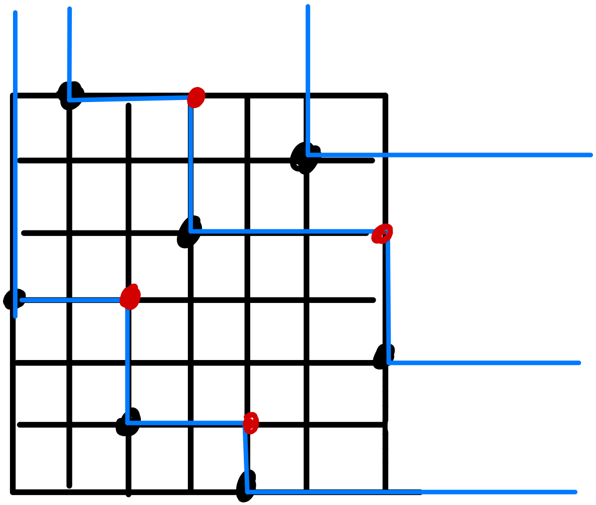

- shadow points

$S(w) = \{ \cdot \}$

$= \{(3,4), (4,7), (5,2), (7,5)\}$

Viennot Shadow

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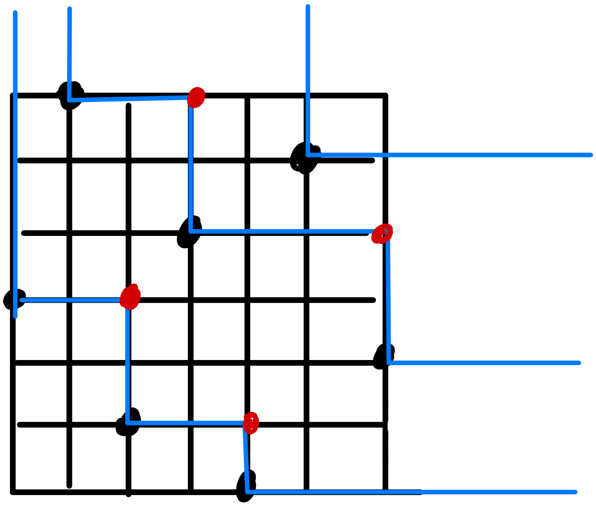

• shadow points

$$S(w) = \{\bullet\}$$
$$= \{(3,4),(4,7),(5,2),(7,5)\}$$

* $|S(w)| = n - \mathrm{lis}(w)$

Viennot Shadow

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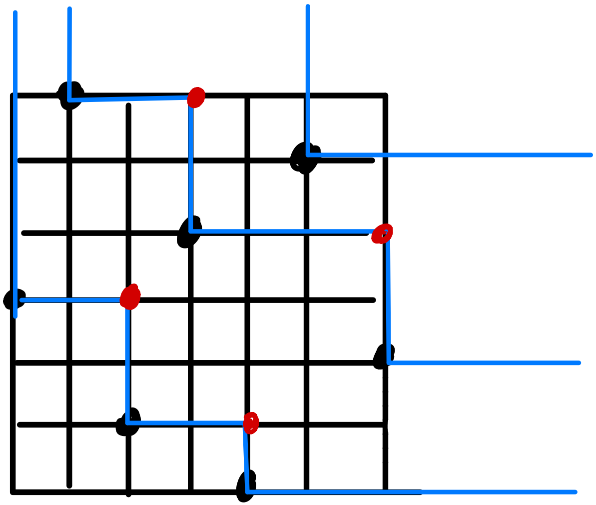

- shadow points

$s(w) = x_{34}\, x_{47}\, x_{52}\, x_{75}$

Theorem (Rhoades)

$\{s(w) : w \in \mathfrak{S}_n\}$ descends to a basis of $\mathbb{C}[X_{n\times n}]/I_n$. This is the standard monomial basis w.r.t the Toeplitz order.

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Cor $\mathrm{Hilb}(\mathbb{C}[X_{n\times n}]/I_n ; q) = a_{n,n} + a_{n,1}q + \cdots + a_{n,1}q^{n-1}$

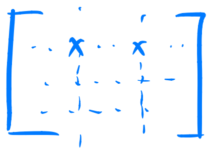
$$= \sum_{w \in \mathfrak{S}_n} q^{n - \mathrm{lis}(w)}$$

Graded Module Structure

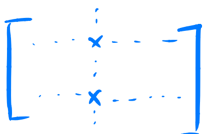
Recall that I_n is generated by

$$x_{i,j}^2$$

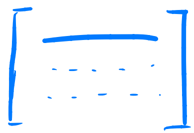
$$x_{i,j}\cdot x_{i,j'}$$



$$x_{i,j}\cdot x_{i',j}$$



$$x_{i,1}+x_{i,2}+\cdots+x_{i,n}$$



$$x_{1,j}+x_{2,j}+\cdots+x_{n,j}$$

Graded Module Structure

Recall that I_n is generated by

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$x_{i,j} \cdot x_{i,j'}$

$x_{i,j} \cdot x_{i',j}$

$x_{i,1} + x_{i,2} + \cdots + x_{i,n}$

$x_{1,j} + x_{2,j} + \cdots + x_{n,j}$

Q: what is the (graded) $\mathfrak{S}_n \times \mathfrak{S}_n$ representation structure of $\mathbb{C}[X_{n \times n}]/I_n$?

Theorem (Rhoades)

$$\left(\mathbb{C}[x_{n\times n}]/I_n\right)_d \cong \bigoplus_{\substack{\lambda \vdash n \\ \lambda_1 = n-d}} V^{\lambda} \otimes V^{\lambda}$$

Where does this In come from?

Where does this I_n come from?

Orbit Harmonics!

Orbit Harmonics

$Z \subseteq \mathbb{C}^n$ finite locus of points

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$\operatorname{gr} I(Z)$ associated graded ideal

$$(*) \quad \mathbb{C}[Z] \cong \mathbb{C}[x_1, \dots, x_n] / I(Z)$$

$$\cong R(Z) := \mathbb{C}[x_1, \dots, x_n] / \operatorname{gr} I(Z)$$

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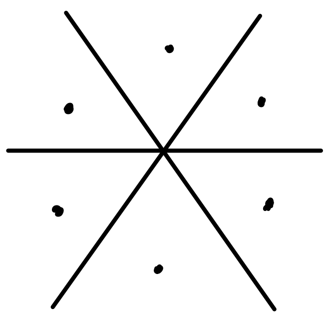
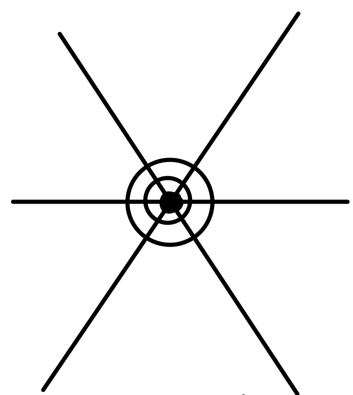
$(*)$ $\mathbb{C}[Z] \cong \mathbb{C}[x_1, \ldots, x_n] / I(Z) \cong R(Z)$



Z

$\xrightarrow[\text{harmonics}]{\text{Orbit}}$



$R(Z)$

Orbit Harmonics

point locus Z	$R(Z)$
regular $\mathfrak{S}_n$-orbit	Coinvariant ring (Kostant)
$\mathfrak{S}_n$-orbit w/ stabilizer $\mathfrak{S}_\mu$	Tanisaki quotient R_μ (Garsia-Procesi)
Ordered set partition locus	Delta-conjecture coinvariant ring (Haglund–Rhoades-Shimozono)
interior lattice points of zonotopes	Rings yielding Donaldson–Thomas invariants (Reineke – Rhoades – Tewari)
λ – tableau	Garsia-Haiman module V_λ (Haiman)

$\mathfrak{S}_n$ as matrices

$$w = [4, 1, 2, 5, 3] \in \mathfrak{S}_5$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \in Mat_{n \times n}(\mathbb{C})$$

Theorem (Rhoades)

$$I_n = \operatorname{gr} I(\mathfrak{S}_n)$$

Note that $I(\mathfrak{S}_n)$ contains

$$x_{ij}^2 - x_{ij}$$

$$x_{ij} \cdot x_{ij'}$$

$$x_{ij} \cdot x_{i'j}$$

$$x_{i,1} + x_{i,2} + \cdots + x_{i,n} - 1$$

$$x_{1,j} + x_{2,j} + \cdots + x_{n,j} - 1$$

What other matrix loci can we consider?

Colored Permutations

$$\mathfrak{S}_{n,r} := \mathbb{Z}_r \wr \mathfrak{S}_n$$

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One-line notation $\quad w = [4^2\ 2^0\ 5^1\ 3^0\ 1^2] \in \mathfrak{S}_{5,3}$

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Matrix form

$$\begin{bmatrix} 0 & 0 & 0 & q^2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q \\ 0 & 0 & 1 & 0 & 0 \\ q^2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$q = e^{2\pi i/3}$

Colored Permutations

Conjugacy classes of $\mathfrak{S}_{n,r}$ are labelled by r-partitions of n:

$$\underline{\lambda} = (\lambda^0, \lambda^1, \dots, \lambda^{r-1})$$

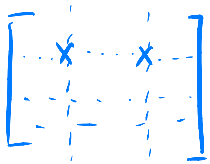
s.t. $$\sum_{i=0}^{r-1} |\lambda^i| = n$$

Theorem (L.)

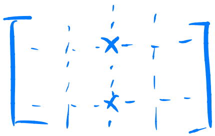
$\operatorname{gr} I(\mathfrak{S}_{n,r}) \subseteq \mathbb{C}[X_{n\times n}]$ is generated by

$$x_{i,j}^{r+1}$$

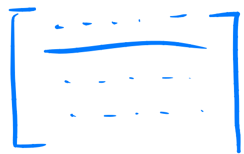
$$x_{i,j}\cdot x_{i,j'}$$



$$x_{i,j}\cdot x_{i',j}$$



$$x_{i,1}^{r}+x_{i,2}^{r}+\cdots+x_{i,n}^{r}$$



$$x_{1,j}^{r}+x_{2,j}^{r}+\cdots+x_{n,j}^{r}$$

$$w = [7^0\ 3^1\ 6^2\ 1^0\ 8^0\ 4^1\ 2^0\ 5^2] \in \mathfrak{S}_{8,3}$$

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$$C_0(w) = \{(1,7), (4,1), (5,8), (7,2)\}$$

$$C_1(w) = \{(2,3), (6,4)\}$$

$$C_2(w) = \{(3,6), (8,5)\}$$

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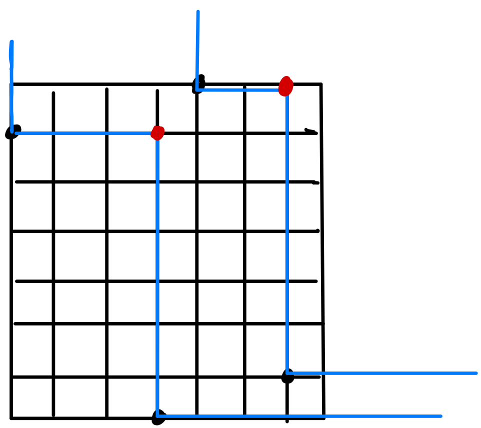


$$S(C_0(w)) = \{(4,7), (7,8)\}$$

$$w = [7^0\ 3^1\ 6^2\ 1^0\ 8^0\ 4^1\ 2^0\ 5^2] \in \mathfrak{S}_{8,3}$$

$$C_1(w) = \{(2,3),(6,4)\}$$

$$C_2(w) = \{(3,6),(8,5)\}$$

$$S(C_0(w)) = \{(4,7),(7,8)\}$$

$$S(w) = x_{2,3} \cdot x_{6,4} \cdot x_{3,6}^2 \cdot x_{8,5}^2 \cdot x_{4,7}^3 \cdot x_{7,8}^3$$

Theorem (L.)

$\{s(w) : w \in \mathfrak{S}_{n,r}\}$ descends to a basis of $R(\mathfrak{S}_{n,r})$. This is again the SUB w.r.t. the Toeplitz order.

Cor

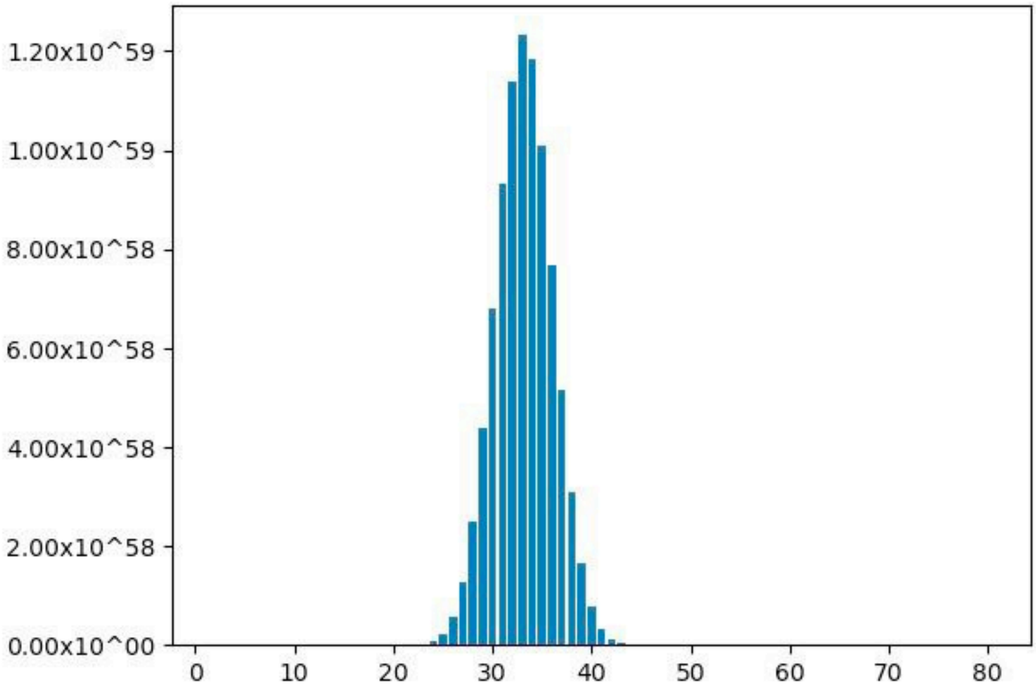
$$\mathrm{Hilb}(R(\mathfrak{S}_{n,r});q) = \sum_{w \in \mathfrak{S}_{n,r}} q^{r \cdot n - r \cdot \mathrm{lis}(C_0(w)) - \sum_{i=1}^{r-1} (r-i) \cdot |C_i(w)|}$$

Cor

$$\mathrm{Hilb}(R(\mathfrak{S}_{n,r});q) = \sum_{w \in \mathfrak{S}_{n,r}} q^{r \cdot n - r \cdot \mathrm{lis}(C_0(w)) - \sum_{i=1}^{r-1} (r-i) \cdot |C_i(w)|}$$

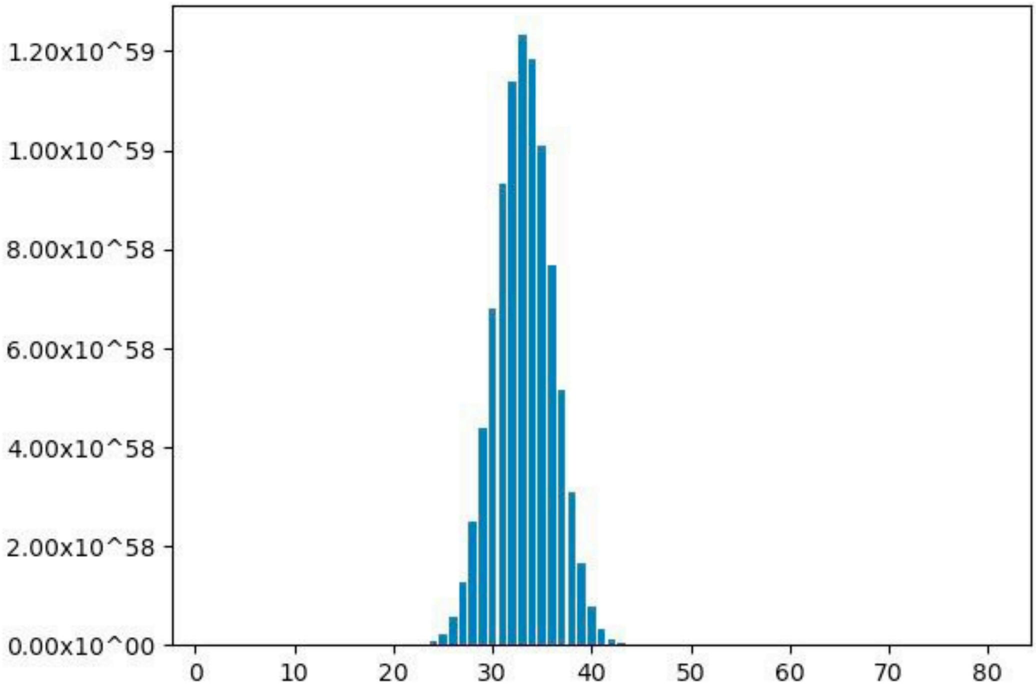
$$= \sum c_{n,r,k}\, q^{rn-k}$$

Distribution



$\{c_{n,r,k}\}$ when $r=2$ and $n=40$.

Distribution



Q: distribution as $n \to \infty$?

$\{c_{n,r,k}\}$ when $r=2$ and $n=40$.

Graded Structure

$$\mathbb{C}[\mathfrak{S}_{n,r}] \cong \mathbb{C}[\lambda_{n\times n}] / I(\mathfrak{S}_{n,r}) \cong \underset{\substack{\uparrow \\ \text{graded}}}{R(\mathfrak{S}_{n,r})}$$

as $\mathfrak{S}_{n,r} \times \mathfrak{S}_{n,r}$ modules

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as $\mathfrak{S}_{n,r}\times\mathfrak{S}_{n,r}$ modules

Theorem (L.)

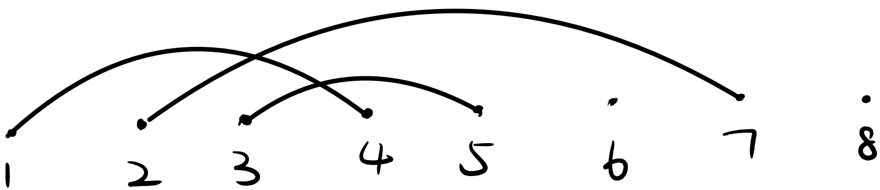
$$(R(\mathfrak{S}_{n,r}))_d \cong \bigoplus_{\underline{\lambda} \vdash_r n} \mathrm{End}(V^{\underline{\lambda}})$$

$$r\cdot\lambda^0_1 + \sum_{i=1}^{r-1} i\cdot|\lambda^i| = rn-d$$

Matching Locus

$\mathcal{M}_n \subseteq \mathfrak{S}_n$

$w = (1\ 4)(2\ 7)(3\ 5)(6)(8) \in \mathcal{M}_8$



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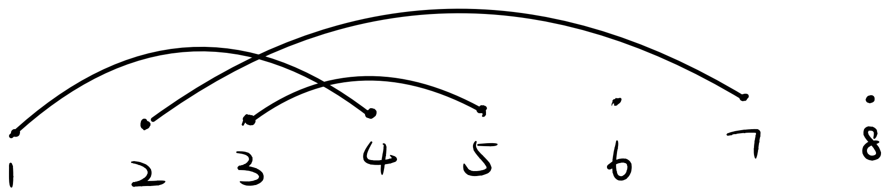
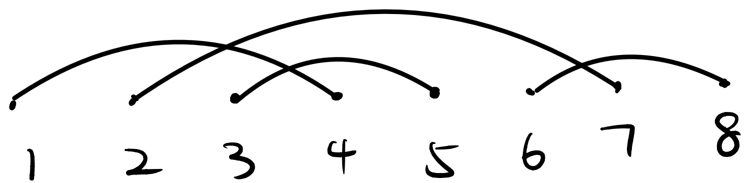


Symmetric permutation matrices

Perfect Matching Locus

$PM_n \subseteq M_n \subseteq \mathfrak{S}_n$

$w = (1\ 4)(2\ 7)(3\ 5)(6\ 8) \in PM_8$



Perfect Matching Locus

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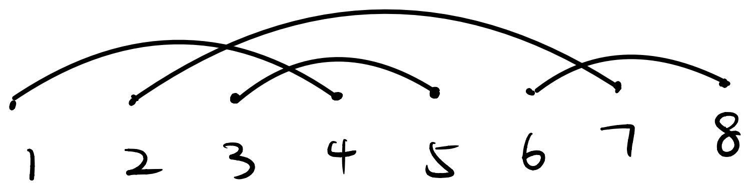

Symmetric permutation matrices w/ diagonal entries 0.

$\mathfrak{S}_n$ acts on $\mathcal{M}_n$ & $P\mathcal{M}_n$ by conjugation

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$$\mathrm{Frob}(\mathbb{C}[P\mathcal{M}_n]) = s_{n/2}[s_2]$$

plethysm

$$= \sum_{\substack{\lambda \vdash n \\ \text{all parts even}}} s_\lambda$$

Theorem (L., Ma, Rhoades, Zhu)

$\operatorname{gr} I(\mathcal{M}_n) \subseteq \mathbb{C}[x_{n\times n}]$ is generated by

$$x_{i,j}^2$$

$$x_{i,j} \cdot x_{i,j'}$$

$$x_{i,j} \cdot x_{i',j}$$

$$x_{i,1} + x_{i,2} + \cdots + x_{i,n}$$

$$x_{1,j} + x_{2,j} + \cdots + x_{n,j}$$

$$x_{i,j} - x_{j,i} \qquad \begin{bmatrix} \ddots & & x & \\ & \ddots & & \\ x & & \ddots & \\ & & & \ddots \end{bmatrix}$$

$$w = (1\ 4)(2\ 7)(3\ 5)(6)(8) \in \mathcal{M}_8$$

$$m(w) = x_{1,4}\, x_{2,7}\, x_{3,5}$$

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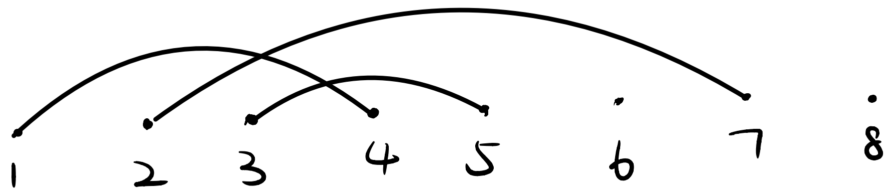


$m(w) = x_{1,4}\, x_{2,7}\, x_{3,5}$

Theorem (L., Ma, Rhoades, Zhu)

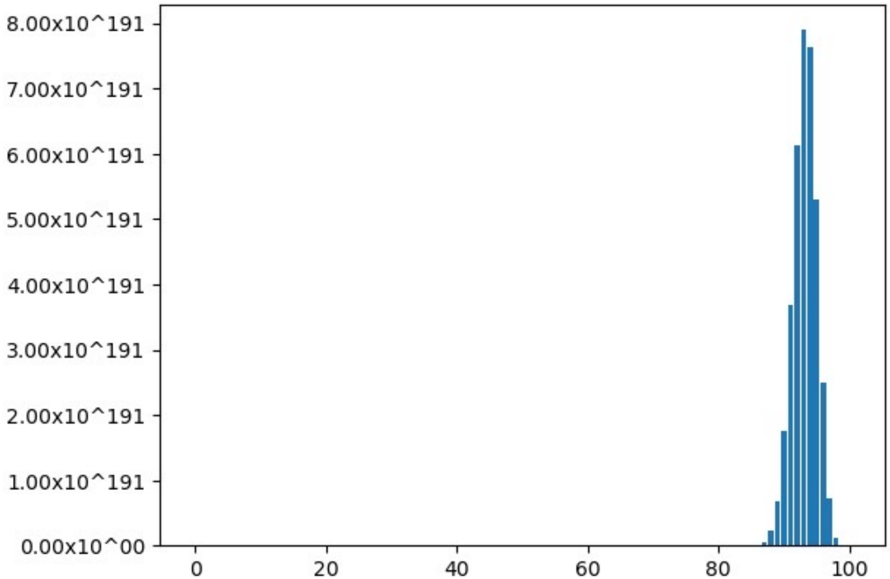
$\{m(w) : w \in \mathcal{M}_n\}$ descends to a vector space basis of $R(\mathcal{M}_n)$.

Cor

$$\mathrm{Hilb}(R(\mathcal{M}_n);q) = \sum_{d=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2d} (2d-1)!!\; q^d$$

Cor

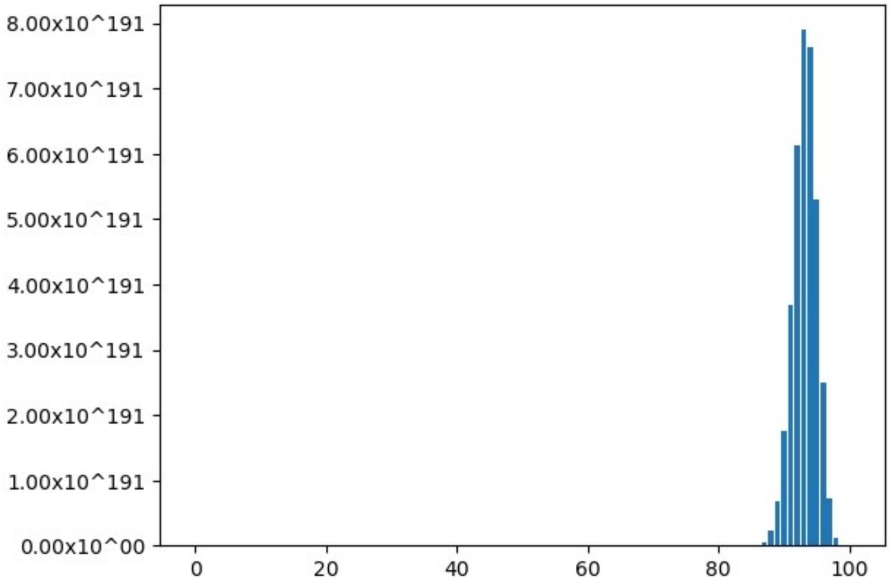
$$\mathrm{Hilb}(R(\mathcal{U}_n);q) = \sum_{d=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2d} (2d-1)!! \, q^d$$



Coefficients of $\mathrm{Hilb}(R(\mathcal{U}_n);q)$ when $n=200$.

Cor

$$\mathrm{Hilb}(R(\mathcal{U}_n);q) = \sum_{d=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2d} (2d-1)!!\, q^d$$



$\mathrm{Hilb}(R(\mathcal{U}_n);q)$ is log-concave and top-heavy

Graded Structure

Theorem (L., Ma, Rhoades, Zhu)

$$\mathrm{Frob}\Big((R(\mathcal{M}_n))_d \Big) = S_d[S_2] \cdot S_{n-2d}$$

Theorem (L., Ma, Rhoades, Zhu)

$\mathrm{gr}\, I(PM_n) \subseteq \mathbb{C}[x_{n\times n}]$ is generated by

$$x_{i,j}^2$$

$$x_{i,j} \cdot x_{i,j'}$$

$$x_{i,j} \cdot x_{i',j}$$

$$x_{i,1} + x_{i,2} + \cdots + x_{i,n}$$

$$x_{1,j} + x_{2,j} + \cdots + x_{n,j}$$

$$x_{i,j} - x_{j,i}$$

$$x_{i,i} \qquad \begin{bmatrix} \ddots & & \\ & \times & \\ & & \ddots \end{bmatrix}$$

Theorem (L., Ma, Rhoades, Zhu)

$$(R(PM_n))_d \cong \bigoplus_{\substack{\lambda \vdash n \\ \lambda \text{ even} \\ \lambda_1 = n-2d}} V^\lambda$$

We do not have a basis for $R(PM_n)$

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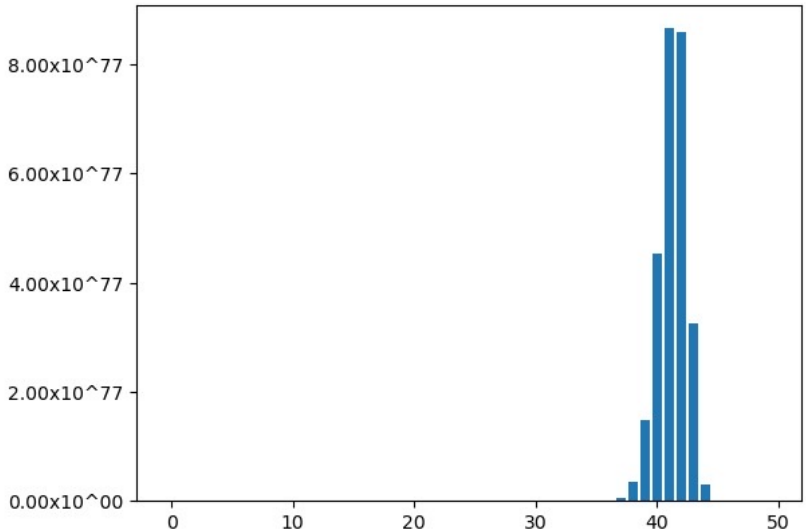
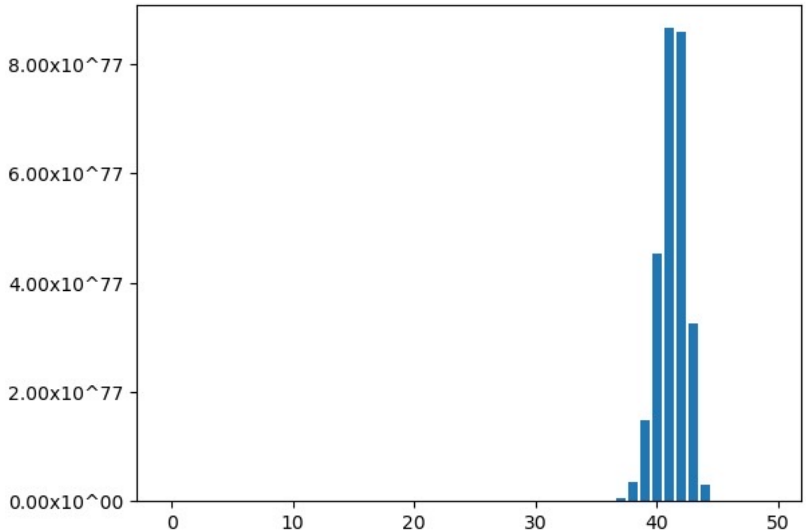
Cor $$\mathrm{Hilb}(R(PM_n);q) = \sum_{w \in PM_n} q^{\frac{n - \mathrm{lds}(w)}{2}}$$

Distribution



Hilb(R(PUn);q) when $n=100$

Distribution



Baik–Rains as $n \to \infty$, the coefficients of $\mathrm{Hilb}(R(PU_n); q)$ converges to the Tracy-Widom distribution of random GOE matrices.

Future Directions

Equivariant log-concavity

$V = \bigoplus_d V_d$ graded G-representation.

G-equivariant log-concave :

$$\exists \varphi : V_{d-1} \otimes V_{d+1} \hookrightarrow V_d \otimes V_d$$

that commutes with the action of G.

Conjecture (Rhoades)

$R(\mathfrak{S}_n)$ is $\mathfrak{S}_n \times \mathfrak{S}_n$-equivariant log-concave.

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$R(\mathfrak{S}_n)$ is $\mathfrak{S}_n \times \mathfrak{S}_n$-equivariant log-concave.

Conjecture (L., Ma, Rhoades, Zhu)

$R(M_n)$ and $R(PM_n)$ are $\mathfrak{S}_n$-equivariant log-concave.

Other Matrix Loci

- $G(r,p,n)$ $\quad (\mathfrak{S}_{n,r} = G(r,1,n))$

- Weyl groups H_3 and F_4

- Other cycle types in $\mathfrak{S}_n$
  (e.g. 1 big cycle, derangements, etc)

- Contingency tables (Oh–Rhoades)

Thanks for Listening !

Arxiv 2306.08718
2401.07850
2409.06175

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