Splicing skew shaped Positroid varieties

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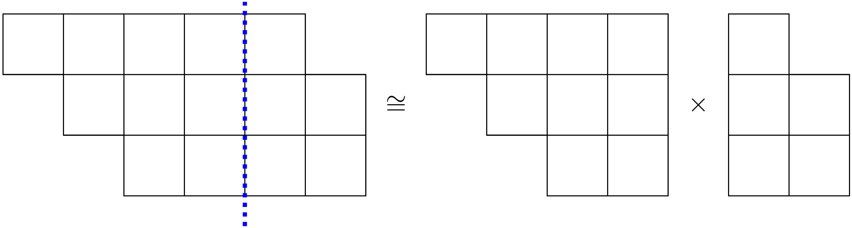
Summary

- skew shape $\lambda/\mu \rightsquigarrow$ skew shaped positroid variety $S^\circ_{\lambda/\mu}$

Given λ/μ, we have a operation of cutting this shape along a column. We are going to make sense of this geometrically.



Theorem (Gorsky–K. -Scroggin–Simental '25)

For certain open set $U \subseteq S^\circ_{\lambda/\mu}$, we have a splicing map for skew shaped positroid variety $S^\circ_{\lambda/\mu}$:

$$U \cong S^\circ_{\lambda_L/\mu_L} \times S^\circ_{\lambda_R/\mu_R}.$$

Skew shaped positroid variety

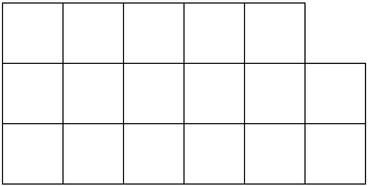
Skew diagram

λ, μ are partitions with $\mu_i \leq \lambda_i$ for all $1 \leq i \leq k$.

Definition

The **skew diagram** λ/μ is the set-theoretic difference of the Young diagrams of λ and μ.

Example: $\lambda = (6, 6, 5)$ **and** $\mu = (2, 1)$



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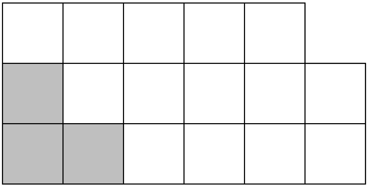

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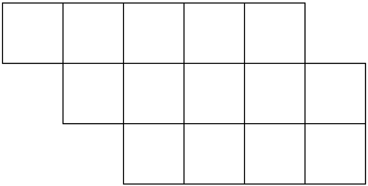

Geometry Set up

Fix integers $0 < k < n$.

- Grassmannian $Gr(k,n) := \{V \subseteq \mathbb{C}^n : \text{dim}(V) = k\}$.
- $V \in Gr(k,n)$ is represented with a full rank $k \times n$ matrix A whose rows span V.
- Plücker coordinates $\Delta_I(A)$:= maximal minor of A located in column set $I \subseteq \{1,2,\ldots,n\}$ with $|I| = k$.

Example

Take $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, then row space$(A) =$ span$(e_2, e_4) \in Gr(2,4)$.

$$\Delta_{1,2}(A) = \Delta_{1,3}(A) = \Delta_{1,4}(A) = \Delta_{2,3}(A) = \Delta_{3,4}(A) = 0, \quad \Delta_{2,4}(A) = -1.$$

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Skew shaped positroid variety

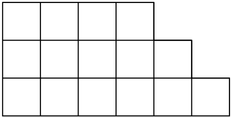
Skew shaped positroid variety is one example of positroid varieties, introduced by Knutson–Lam–Speyer in 2013.

Definition

Skew shaped positroid variety $S^{\circ}_{\lambda/\mu} := C_{w_\mu} \cap C^{w_\lambda}$.

Example $\lambda = (6,6,5)$ and $\mu = (2,1)$

$$C_{w_\mu} = \text{row span}\left(\begin{pmatrix} * & * & * & * & \textcircled{1} & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & * & \textcircled{1} & 0 & 0 \\ * & * & * & * & 0 & * & 0 & * & \textcircled{1} \end{pmatrix}\right)$$



$$C^{w_\lambda} = \text{row span}\left(\begin{pmatrix} \textcircled{1} & 0 & * & 0 & * & * & * & * & * \\ 0 & \textcircled{1} & * & 0 & * & * & * & * & * \\ 0 & 0 & 0 & \textcircled{1} & * & * & * & * & * \end{pmatrix}\right)$$

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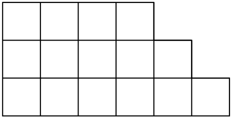

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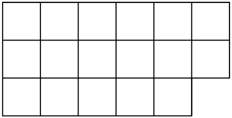
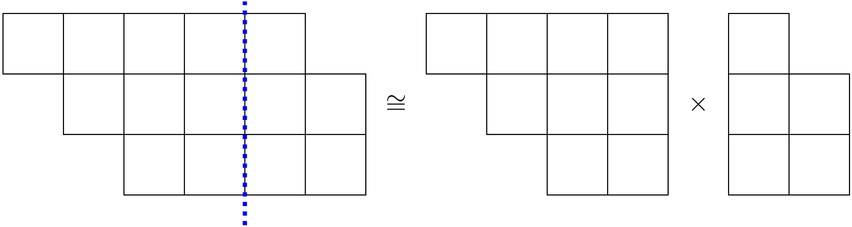


Main theorem preview



Theorem (Gorsky-K.-Scroggin-Simental, 2025)

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Labeling boxes in a skew diagram

For $\square \in \lambda/\mu$, the $\boldsymbol{\mu}_\square$ is the **smallest** shape containing μ and $\square$.

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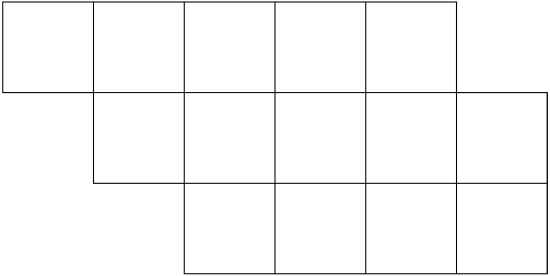

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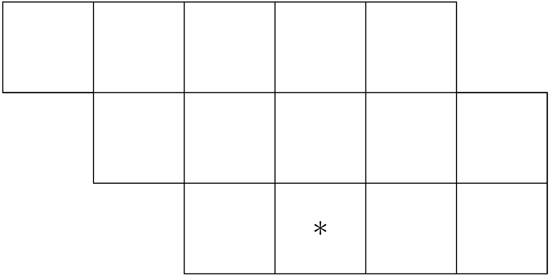
For $\square \in \lambda/\mu$, the $\boldsymbol{\mu}_\square$ is the **smallest** shape containing μ and $\square$.

Step 1: Pick one $\square \in \lambda/\mu$.



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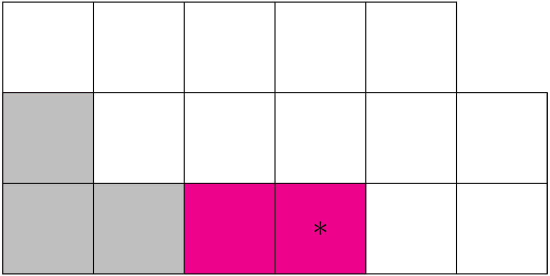
Step 2: Find $\mu_\square$.



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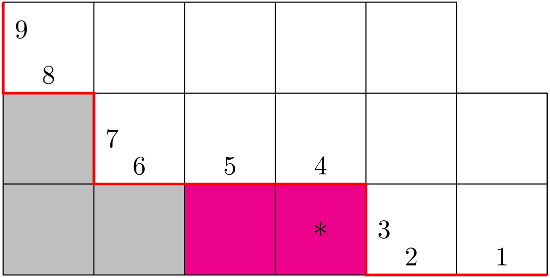
For $\square \in \lambda/\mu$, the $\boldsymbol{\mu}_\square$ is the **smallest** shape containing μ and $\square$.
$I'(\square)$:= the k-element subset consists of labels in the vertical steps of boundary of $\mu_\square$.

Step 3: Label the boundary in red.



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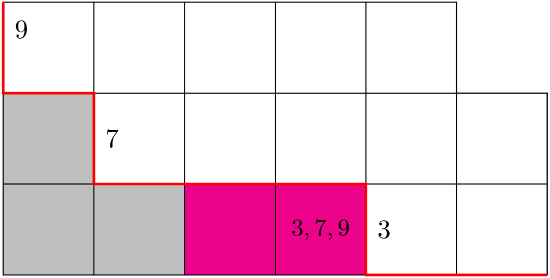
Step 4: Label $\square$ with vertical step labelings.

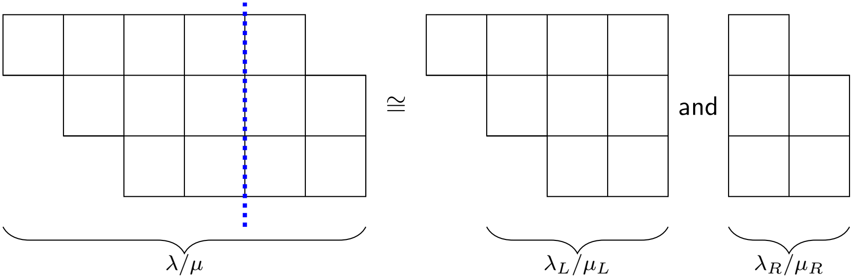


Labeling boxes in a skew diagram

5, 7, 8	5, 6, 7	4, 5, 6	3, 4, 5	2, 3, 4	
	5, 6, 9	4, 5, 9	3, 4, 9	2, 3, 9	1, 2, 9
		4, 7, 9	3, 7, 9	2, 7, 9	1, 7, 9

Main theorem: Part 1

- Choose a column of λ.
- Decompose λ/μ along the ath column, and get two skew diagrams λ_L/μ_L and λ_R/μ_R.



Note that we can pick any columns of λ/μ.

Main theorem: Part 1

Choose a column of λ.

- Decompose λ/μ along the ath column, and get two skew shapes λ_L/μ_L and λ_R/μ_R.
- Define
  $U = \{V \in S^{\circ}_{\lambda/\mu} : \Delta_{I'(\square)}(V) \neq 0 \quad \text{for all } \square \text{ in the } a\text{th column of } \lambda/\mu\}$.

Theorem (Gorsky–K. –Scroggin–Simental '25)

We have an explicit isomorphism

$$U \cong S^{\circ}_{\lambda_L/\mu_L} \times S^{\circ}_{\lambda_R/\mu_R}$$

Example: Finding U

$5,7,8$	$5,6,7$	$4,5,6$	$3,4,5$	$2,3,4$	
	$5,6,9$	$4,5,9$	$3,4,9$	$2,3,9$	$1,2,9$
		$4,7,9$	$3,7,9$	$2,7,9$	$1,7,9$

$$U \cong \{V \in S^{\circ}_{(6,6,5)/(2,1)} : \Delta_{3,7,9}(V) \neq 0, \Delta_{3,4,9}(V) \neq 0, \Delta_{3,4,5}(V) \neq 0\}$$

Cluster algebra

Cluster algebras, briefly

- Start with a quiver with some cluster variables labeling its vertices. This is called a **seed**.
- A *mutation* at a vertex produces a new quiver and a new cluster variable.
- Mutate in all possible directions. Get lots of cluster variables.
- Cluster algebra = $\mathbb{C}$[cluster variables].

Building up on the work of Leclerc '14 and Serhiyenko–Sherman-Bennett–Williams '19, Galashin and Lam proved the following:

Theorem (Galashin–Lam '23)

Any **positroid variety** *is a cluster variety, meaning that* $\mathbb{C}[S^{\circ}_{\lambda/\mu}]$ *is a cluster algebra.*

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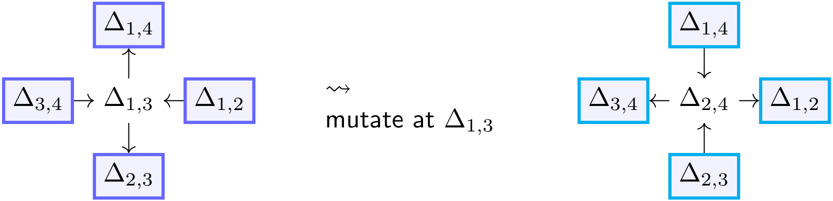

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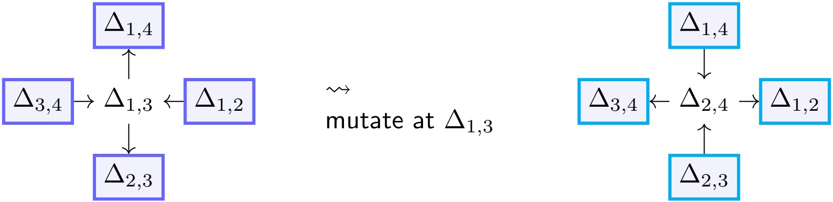

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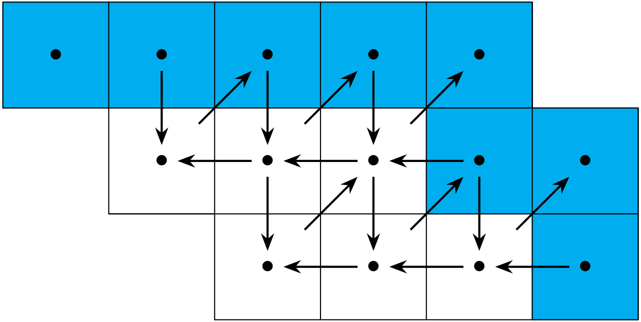
Seed preview



Boxes correspond to vertices in a quiver and arrows are shown in the figure.

What are the *initial* cluster variable at each vertex?

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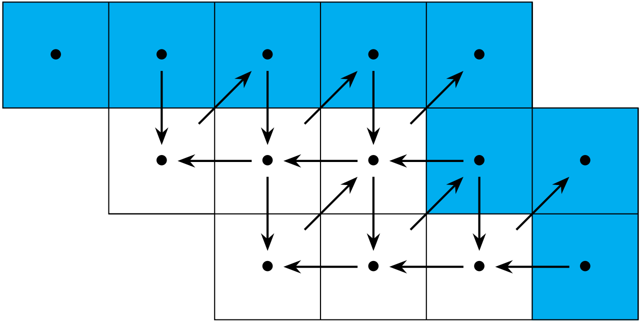

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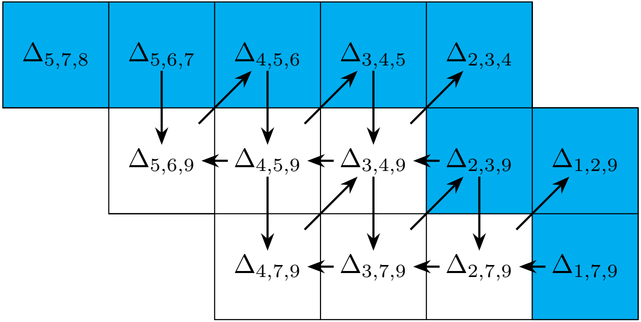
Cluster structure for $S^\circ_{\lambda/\mu}$

Proposition (Gorsky–K. –Scroggin–Simental '25)

The skew diagram λ/μ provides a seed for $S^\circ_{\lambda/\mu}$, where initial cluster variables are Plücker coordinates $\Delta_{I'(\square)}$.



Why quasi-cluster isomorphism?

Definition (Fraser '16)

The **quasi-cluster isomorphism** $f : X \to Y$ is a map that sends cluster variables to cluster variables, up to well-behaved multiplication of units in cluster algebra.

Concretely, we need to find a seed for X and a seed for Y with the same quiver where

seed for X seed for Y

The **quasi-cluster isomorphism** preserves essential geometrical data arising from a cluster structure, for example,

- $\{$cluster monomials$\} \subseteq$ theta basis [Gross–Hacking–Keel–Kontsevich],
- totally positive part of a variety,
- image of a special torus called the *cluster tori*.

Main theorem: Part 2

The open subset U inherits a cluster structure from $S^{\circ}_{\lambda/\mu}$.

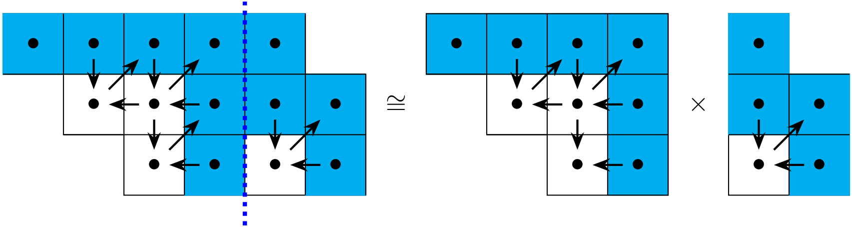
Theorem (Gorsky–K. –Scroggin–Simental '25)

Splicing isomorphism

$$U \cong S^{\circ}_{\lambda_L/\mu_L} \times S^{\circ}_{\lambda_R/\mu_R}$$

is a quasi-cluster isomorphism.

Example



Thank you!

FPSAC 2025 Soyeon

Further Direction

Further direction: Splicing Braid varieties

Braid group Br_n : like S_n, but $s_i^2 \neq e$. For example, $\beta = s_1 s_2 s_1 s_3 s_4 \in Br_4^+$.

Braid variety $X(\beta)$ vastly **generalizes** positroid varieties, double Bott–Samelson varieties, open Richardson varieties and etc.

Theorem (Casals–Gorsky–Gorsky–Le–Shen–Simental, Galashin–Lam–Sherman-Bennett–Speyer)

Braid varieties are cluster varieties.

Given β and some additional choices, we define the specific open subset $U \subset X(\beta)$ and two braids $\tilde{\beta}_1$ and $\tilde{\beta}_2$ and prove the following:

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We have an isomorphism

$$X(\tilde{\beta}_1) \times X(\tilde{\beta}_2) \cong U.$$

Conjecture: This isomorphism is a **quasi-cluster isomorphism**.

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Appendix: Splicing braid varieties

Splicing braid varieties continued

$$
\begin{array}{ccccccccccccc}
\mathcal{F}^{\mathrm{std}} & \xrightarrow{s_{i_1}} & \mathcal{F}^{1} & \xrightarrow{s_{i_2}} & \cdots & \xrightarrow{s_{i_{r_1}}} & \mathcal{F}^{r_1} & \xrightarrow{s_{i_{r_1+1}}} & \cdots & \xrightarrow{s_{i_r}} & \mathcal{F}^{\mathrm{ant}} \\
\Vert & & & & & & & & & & \Vert \\
\mathcal{F}^{\mathrm{std}} & \xleftarrow{s_{a_{\ell(w_0)}}} & \widetilde{\mathcal{F}}^{\ell(w_0)-1} & \longleftarrow & \cdots & \longleftarrow & \mathcal{F}(w_0 w) & \xleftarrow{s_{a_{\ell(w)}}} & \cdots & \xleftarrow{s_{a_1}} & \mathcal{F}^{\mathrm{ant}}
\end{array}
$$

Splicing braid varieties continued

$$
\begin{array}{ccccccccccccc}
\mathcal{F}^{\mathrm{std}} & \xrightarrow{s_{i_1}} & \mathcal{F}^{1} & \xrightarrow{s_{i_2}} & \cdots & \xrightarrow{s_{i_{r_1}}} & \mathcal{F}^{r_1} & \xrightarrow{s_{i_{r_1+1}}} & \cdots & \xrightarrow{s_{i_r}} & \mathcal{F}^{\mathrm{ant}} \\
\Vert & & & & & & & & & & \Vert \\
\mathcal{F}^{\mathrm{std}} & \xleftarrow{s_{a_{\ell(w_0)}}} & \widetilde{\mathcal{F}}^{\ell(w_0)-1} & \longleftarrow & \cdots & \longleftarrow & \mathcal{F}(w_0 w) & \xleftarrow{s_{a_{\ell(w)}}} & \cdots & \xleftarrow{s_{a_1}} & \mathcal{F}^{\mathrm{ant}}
\end{array}
$$

Splicing braid varieties continued

$$
\begin{array}{ccccccccccccc}
\mathcal{F}^{\text{std}} & \xrightarrow{s_{i_1}} & \mathcal{F}^1 & \xrightarrow{s_{i_2}} & \cdots & \xrightarrow{s_{i_{r_1}}} & \mathcal{F}^{r_1} & \xrightarrow{s_{i_{r_1+1}}} & \cdots & \xrightarrow{s_{i_r}} & \mathcal{F}^{\text{ant}} \\
\| & & & & & & & & & & \| \\
\mathcal{F}^{\text{std}} & \xleftarrow{s_{a_{\ell(w_0)}}} & \widetilde{\mathcal{F}}^{\ell(w_0)-1} & \longleftarrow & \cdots & \longleftarrow & \mathcal{F}(w_0 w) & \xleftarrow{s_{a_{\ell(w)}}} & \cdots & \xleftarrow{s_{a_1}} & \mathcal{F}^{\text{ant}}
\end{array}
$$