Rook matroids and log-concavity of P-Eulerian polynomials

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(Joint work with Per Alexandersson)

Outline

- **Rook matroids:**
  1. Non-nesting rook polynomials.
  2. Matroids from restricted rook placements.
  3. Relation to: transversal matroids, lattice path matroids, positroids.
- **Neggers-Stanley conjecture:**
  1. Posets and P-Eulerian polynomials.
  2. Skew shapes to width two posets.

Rook matroids

Non-nesting rook placements

- Let $\mathrm{NN}_{\lambda/\mu}$ be the set of *non-nesting rook placements* on λ/μ; no rook can lie South-East of another.

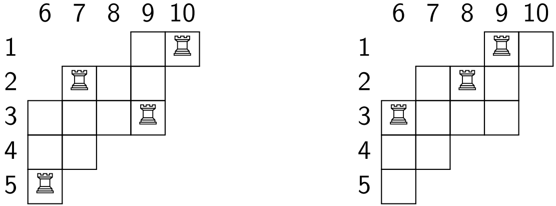
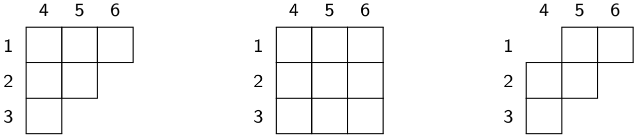


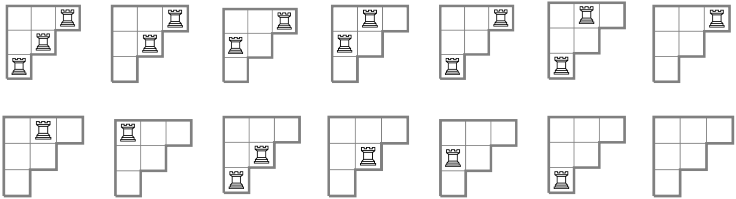
Figure: Left: nesting rook placement, right: non-nesting rook placement.

- **Motivation:** A combinatorial model for the Narayana numbers of type A, Narayana numbers of type B, the Fibonacci numbers.



Non-nesting rook polynomials

- Let λ/μ is a skew shape and $M_{\lambda/\mu}(t) = \sum_{k\geq 0} r_k(\lambda/\mu)t^k$ where $r_k(\lambda/\mu)$ is the number of non-nesting rook placements of size k.



- $\lambda = (n, n-1, \dots, 1) \implies M_\lambda(t) = N_{n+1}(t)$, Narayana polynomial.
- $\lambda = \underbrace{(a, \dots, a)}_{b \text{ times}} \implies M_\lambda(t) = \sum_{k\geq 0} \binom{a}{k}\binom{b}{k} t^k.$

Non-nesting rook polynomials

Theorem (Heilmann–Lieb '72, Nijenhuis '76)

For any board B, the rook polynomial $R_B(t)$ is real-rooted.

- **Tool:** Stable polynomials = multivariate analog of real-rootedness.

Theorem (Alexandersson, J. '24+)

For any skew shape λ/μ, the non-nesting rook polynomial $M_{\lambda/\mu}(t)$ is ultra-log-concave. Moreover, there exists a skew shape α/β such that $M_{\alpha/\beta}(t)$ is not real-rooted.

- $f(t) = \sum_{k=0}^{n} a_k t^k$ is ultra-log-concave if

$$\left(\frac{a_k}{\binom{n}{k}}\right)^2 \geq \frac{a_{k-1}}{\binom{n}{k-1}} \cdot \frac{a_{k+1}}{\binom{n}{k+1}} \quad \text{for all } 1 \leq k \leq n-1.$$

- **Tool:** Lorentzian polynomials = multivariate analog of log-concavity.

Non-nesting rook placements

- Skew shape λ/μ has r rows and c columns.
- Identify $\sigma \in \mathrm{NN}_{\lambda/\mu}$ by $R(\sigma) \cup C(\sigma)$ where

$$R(\sigma) = \{i \in [r] : \sigma \text{ has a rook in row } i\}$$
$$C(\sigma) = \{j \in [r+1, r+c] : \sigma \text{ has } \textbf{no} \text{ rook in column } j\}$$

(a) $R(\sigma) \cup C(\sigma) = \{2, 4, 5, 9, 10\}$

(b) $R(\sigma) \cup C(\sigma) = \{1, 2, 3, 7, 10\}$

Rook matroids

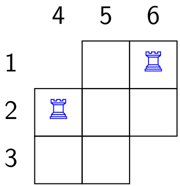
$M = (E, \mathcal{B})$ is a *matroid* with bases $\mathcal{B}$ if $\mathcal{B}$ is a non-empty collection of subsets of E satisfying:

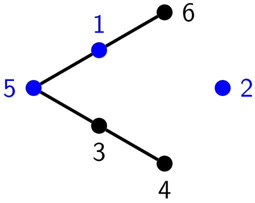
- For all $B_1, B_2 \in \mathcal{B}$, if $a \in B_1$ there exists $b \in B_2 \setminus B_1$ such that $(B_1 \setminus a) \cup b \in \mathcal{B}$.

Theorem (Alexandersson, J. '24+)

If λ/μ has r rows and c columns, then $\mathcal{R}_{\lambda/\mu} = ([r+c], \mathcal{B})$ is a matroid, where

$$\mathcal{B} = \{R(\sigma) \cup C(\sigma) : \sigma \in \mathrm{NN}_{\lambda/\mu}\}.$$





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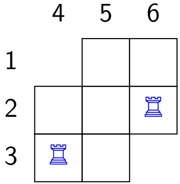

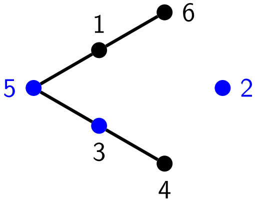

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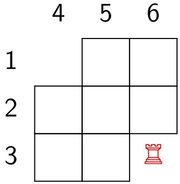
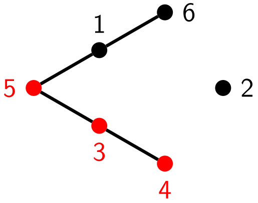




Log-concavity of the non-nesting rook polynomial

Theorem (Alexandersson, J. '24+)

For any skew shape λ/μ, the non-nesting rook polynomial $M_{\lambda/\mu}(t)$ is ultra-log-concave with no internal zeros.

- Proof sketch:

$$M_{\lambda/\mu}(t) = \sum_{\sigma \in \mathrm{NN}_{\lambda/\mu}} \prod_{i \in R(\sigma)} x_i \prod_{j \in C(\sigma)} y_j \Big|_{\mathbf{x}=(t,t,\ldots,t),\mathbf{y}=(1,1,\ldots 1)}.$$

- RHS is the basis-generating polynomial $P_M(\mathbf{z}) = \sum_{B \in \mathcal{B}} \prod_{i \in B} z_i$ of the rook matroid M on λ/μ.
- For every matroid M, $P_M(\mathbf{z})$ is Lorentzian (Brändén-Huh, '20).
- $M_{\lambda/\mu}$ is ultra-log-concave with no internal zeros.

Structure of rook matroids

Theorem (Alexandersson, J. 2024+)

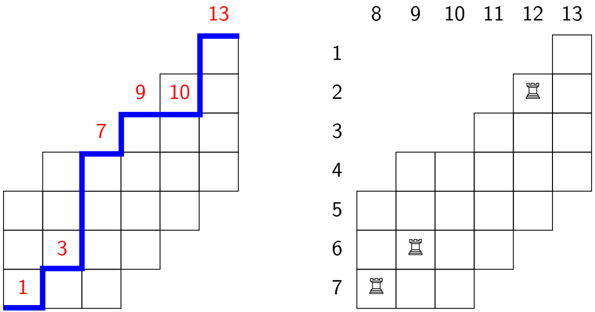
Rook matroids are

- *(Fundamental) transversal matroids,*
- *Positroids,*
- *Closed under duals and direct sums, but not under minors.*

- **Transversal:** every rook placement has a non-nesting representative.
- **Positroid:** rook-theoretic interpretation of Grassmann necklaces.
- **Closed under duals:** Conjugate the skew shape.
- **Not minor-closed:** Fundamental transversal matroids rarely are.

Matroids from skew shapes

- Given a skew shape λ/μ, consider a lattice path L contained in λ/μ.
- Identify L by its set of East steps.



Lattice paths contained in $\lambda/\mu \longleftrightarrow$ Non-nesting rook placements on λ/μ.

Relation to lattice path matroids

Theorem (Alexandersson, J. '24+)

Let λ/μ be a skew shape and $\mathcal{R}_{\lambda/\mu}$ and $\mathcal{P}_{\lambda/\mu}$ respectively be the rook and lattice path matroid on λ/μ. Then:

- $\mathcal{R}_{\lambda/\mu} \cong \mathcal{P}_{\lambda/\mu}$ *if and only if* $332/1$ *is* ***not*** *a subshape of* λ/μ.
- *Nevertheless, we always have equality of Tutte polynomials:*

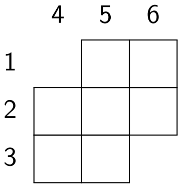
$$T(\mathcal{R}_{\lambda/\mu}; x, y) = T(\mathcal{P}_{\lambda/\mu}; x, y).$$

Theorem (Bonin, de Mier '25+)

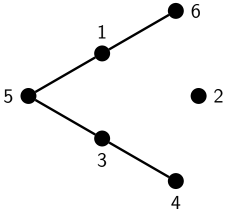
The universal valuative invariant $\mathcal{G}$ is equal on $\mathcal{R}_{\lambda/\mu}$ and $\mathcal{P}_{\lambda/\mu}$ for every skew shape λ/μ.

- Subshape: skew shape obtained from λ/μ by deleting rows/columns.
- The if direction follows because...

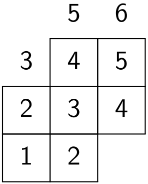
Q_6 and 332/1



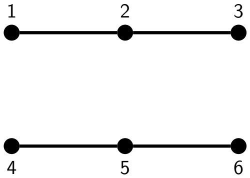
(a) $\mathcal{R}_{332/1}$



(b) Q_6



(c) $\mathcal{P}_{332/1}$



(d) $U_{2,4} \oplus_2 U_{2,4}$

Figure: $\mathcal{R}_{332/1}$ and $\mathcal{P}_{332/1}$ are not isomorphic.

Neggers–Stanley conjecture

P-Eulerian polynomials

- If (P,ω) is a labeled poset on n elements, its *Jordan–Hölder set* is given by

$$\mathcal{L}(P,\omega) = \{\sigma \in \mathfrak{S}_n : i \prec j \implies \omega(i) \text{ appears before } \omega(j) \text{ in } \sigma\}.$$

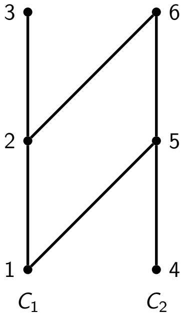


Figure: (P,ω) of width two, ω natural.

$$\mathcal{L}(P,\omega) = \{412356, 124356, \ldots\}$$

The *(P,ω)-Eulerian polynomial* is

$$W_{P,\omega}(t) = \sum_{\sigma \in \mathcal{L}(P,\omega)} t^{\mathrm{des}(\sigma)}.$$

e.g. on the left:

$$W_P(t) = t^3 + 6t^2 + 6t + 1$$

Neggers–Stanley conjecture

Conjecture (Neggers '78, Stanley '86)

Let (P, ω) be a labeled poset. Then $W_{P,\omega}(t)$ is real-rooted.

- For $P = n$-step ladder, $W_P(t) = \sum_{k=1}^{n+1} \frac{1}{n+1} \binom{n+1}{k} \binom{n+1}{k-1} t^{k-1}$.
- For $P =$ Disjoint union of chains of length a and b,

$$W_P(t) = \sum_{k \geq 0} \binom{a}{k} \binom{b}{k} t^k.$$

- **False** for *non*-naturally labeled width two poset (Brändén '04).
- **False** for naturally labeled width two poset (Stembridge '06).

Brenti's conjecture

Conjecture (Brenti '89)

Let (P, ω) be a labeled poset. Then $W_{P,\omega}(t)$ is log-concave with no internal zeros.

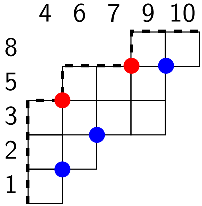
- W_P is symmetric and unimodal when P is naturally labeled and graded (Reiner–Welker '05, Brändén '07).

Theorem (Alexandersson, J. '24+)

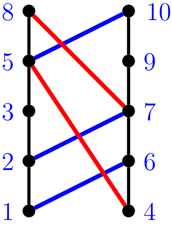
Let P be a naturally labeled poset of width two. Then $W_P(t)$ is ultra-log-concave with no internal zeros.

- **Proof idea:** Use non-nesting rook placements!

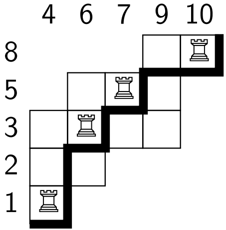
Poset – skew shape correspondence



(a) Skew shape λ/μ.



(b) Poset P.



(c) Rooks $\leftrightarrow$ paths $\leftrightarrow$ linear extensions

Linear extensions of thin $P \longleftrightarrow$ Non-nesting rook placements on λ/μ:

$$W_P(t) = M_{\lambda/\mu}(t).$$

Stembridge's counterexample to Neggers–Stanley implies $M_{\lambda/\mu}$ is not always real-rooted.

Related combinatorial objects

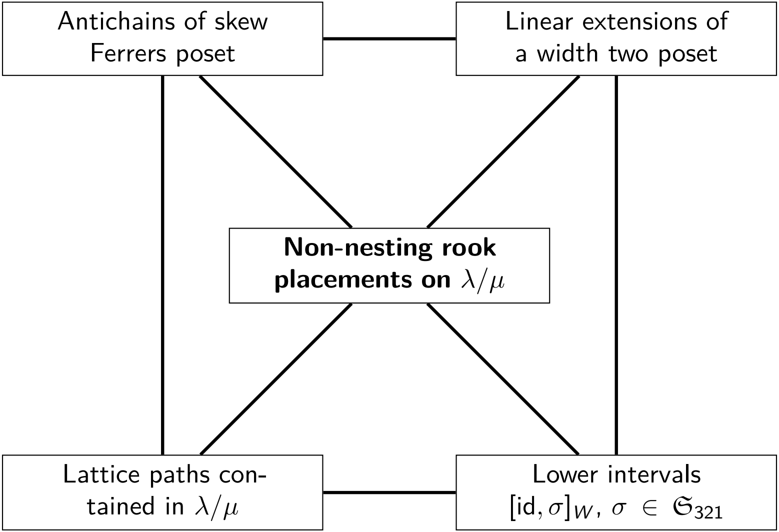


Figure: Non-nesting rook placements and friends.

Open ends

- **Algebraic:** Equivariant log-concavity for non-nesting rooks?
  1. Li (2022) did this for graph matchings.
  2. Find the right group action on $\mathrm{NN}_{\lambda/\mu}$ that preserves # of rooks.
  3. Actions on linear extensions of posets.

- **Dynamical:** Interpret classical operators on $\mathcal{L}(P)$ in terms of rooks.
  1. Promotion, rowmotion, Bender–Knuth toggles, modified Foata–Strehl.

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- For more on rook matroids:

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