On the Stanley–Stembridge conjecture

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Stanley–Stembridge conjecture

Stanley–Stembridge (1993) :
Stated several conjectures on immanants of Jacobi–Trudi matrices.

Stanley (1995) :
Introduced the *chromatic symmetric functions* for any graphs and reformulated one of the above conjectures.

Conjecture (Stanley–Stembridge 1993)

The chromatic symmetric functions for incomparability graphs of $(3+1)$-free posets are e-positive.

This is Problem 21 in the Stanley's survey "Positivity problems and conjectures in algebraic combinatorics".

Unit interval orders

Guay-Paquet (2013) :
It is enough to prove the e-positivity for $(3+1)$-free and $(2+2)$-free posets, i.e., for *unit interval orders*.

$$\#\{\text{unit interval orders of size } n\} = n\text{-th Catalan number}$$

$\Longrightarrow$ related to many other combinatorial objects such as

- Dyck paths,
- Hessenberg functions,
- 312-avoiding permutations,
- etc.

Conjugate Hessenberg functions

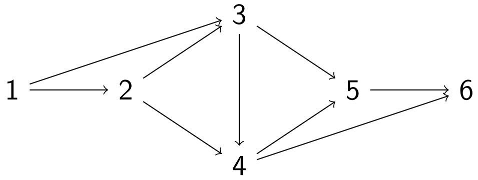
For our purpose, it is convenient to use the set of *conjugate Hessenberg functions* for a parametrization of unit interval orders defined by

$$\mathbb{E}_n := \{\mathsf{e} : [n] \to \mathbb{Z}_{\geq 0} \mid 0 \leq \mathsf{e}(i) < i, \mathsf{e}(i) \leq \mathsf{e}(i+1)\}.$$

An element $\mathsf{e} \in \mathbb{E}_n$ corresponds to a *unit interval graph* Γ_{e} with

vertices : $[n] := \{1, \ldots, n\}$
edges : $\{i \to j \mid \mathsf{e}(j) < i < j\}$.

For example, $\mathsf{e} = (0, 0, 0, 1, 2, 3) \in \mathbb{E}_6$ corresponds to



Chromatic quasisymmetric functions

Shareshian–Wachs (2012) :
Introduced a q-analogue of chromatic symmetric functions called *chromatic quasisymmetric functions* for any labeled graphs.

$\Gamma = ([n], \mathsf{Edge})$: labeled graph.
$\kappa : [n] \to \mathbb{Z}_{>0}$: proper coloring for $\Gamma \overset{\text{def}}{\Longleftrightarrow} \kappa(i) \neq \kappa(j)$ for $i \to j \in \mathsf{Edge}$.
$\mathsf{asc}(\kappa) := \#\{i \to j \in \mathsf{Edge} \mid \kappa(i) < \kappa(j)\}$.

Definition (Shareshian–Wachs 2012)

The *chromatic quasisymmetric function* $\mathbf{X}_\Gamma(q)$ of Γ is defined by

$$\mathbf{X}_\Gamma(q) := \sum_{\kappa} q^{\mathsf{asc}(\kappa)} \prod_{i=1}^{n} x_{\kappa(i)},$$

where κ runs over the set of all proper colorings of Γ.

Various incarnations

Proposition (Shareshian–Wachs)

If Γ is a unit interval graph, then $\mathbf{X}_\Gamma(q)$ is symmetric.

$\mathbf{X}_\Gamma(q)$ for unit interval graphs also appear in many places such as :

- plethystic substitutions of unicellular LLT polynomials for the corresponding Dyck paths by Carlsson–Mellit,
- cohomology rings of regular semisimple Hessenberg varieties of type A for the corresponding Hessenberg functions, conjectured by Shareshian–Wachs and proved by Brosnan–Chow and Guay-Paquet,
- traces of the Hecke algebras of type A at the Kazhdan–Lusztig basis elements for the corresponding 312-avoiding permutations by Clearman–Hyatt–Shelton–Skandera.

Refinements and the main results

Let us expand $\mathbf{X}_\Gamma(q)$ in terms of elementary symmetric functions :

$$\mathbf{X}_\Gamma(q) = \sum_{\lambda \vdash n} c_\lambda(\Gamma; q)\, e_\lambda(x).$$

Conjecture (Shareshian–Wachs 2012)

For any unit interval graph Γ, we have $c_\lambda(\Gamma; q) \in \mathbb{Z}_{\geq 0}[q]$.

Main Theorem (H. 2024)

For any unit interval graph Γ and $q \in \mathbb{R}_{>0}$, we have $c_\lambda(\Gamma; q) \geq 0$.
In particular, the Stanley–Stembridge conjecture holds.

Observations

Notations : $[m]_q \coloneqq \frac{1-q^m}{1-q}$, $[m]_q! \coloneqq \prod_{i=1}^m [i]_q$.

Proposition (H. 2025)

For any $\mathsf{e} \in \mathbb{E}_n$, we have

$$\sum_{\lambda \vdash n} p_\lambda(\Gamma_{\mathsf{e}}; q) = 1, \quad p_\lambda(\Gamma_{\mathsf{e}}; q) \coloneqq q^{|\mathsf{e}| - |\mathsf{e}_\lambda|} \frac{c_\lambda(\Gamma_{\mathsf{e}}; q)}{\prod_i [\lambda_i]_q!},$$

where we set $|\mathsf{e}| \coloneqq \sum_i^n \mathsf{e}(i)$ and $|\mathsf{e}_\lambda| = \sum_{i<j} \lambda_i \lambda_j$.

When $q = 1$, this follows easily by comparing the coefficients of $x_1 x_2 \cdots x_n$ and known to experts.

This and the Shareshian–Wachs conjecture suggest to consider $\{p_\lambda(\Gamma_{\mathsf{e}}; q)\}_{\lambda \vdash n}$ as a probability on the set of partitions of n.

Strategy of proof

Step 1 : Construct the desired probability directly (difficult).

Step 2 : Check that it coincides with $p_\lambda(\Gamma_{\mathsf{e}}; q)$ (easy).

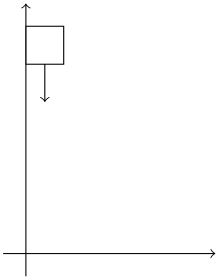
For Step 2, we use a characterization of the chromatic quasisymmetric functions given by Abreu–Nigro (2021), based on certain relations between $\mathbf{X}_\Gamma(q)$'s called the *modular laws*. These relations were first found by Guay-Paquet (2013) when $q = 1$.

Theorem (Abreu–Nigro 2021)

For unit interval graphs Γ, $\mathbf{X}_\Gamma(q)$'s are characterized by the modular laws, multiplicativity $\mathbf{X}_{\Gamma \cup \Gamma'}(q) = \mathbf{X}_\Gamma(q)\mathbf{X}_{\Gamma'}(q)$, and $\mathbf{X}_{\Gamma_n}(q) = [n]_q! e_n(x)$ where Γ_n is the complete graph with n vertices.

How to construct a probability?

Imagine that someone drops a box one by one to create Young diagrams :



t=0

We want to find transition probabilities (depending on $\mathsf{e} \in \mathbb{E}_n$) so that the probability of becoming λ at time n coincides with $p_\lambda(\Gamma_{\mathsf{e}}; q)$.

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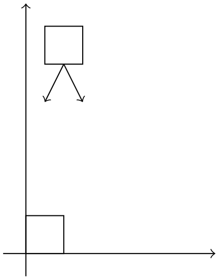

t=1

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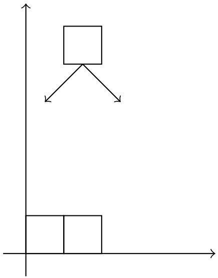
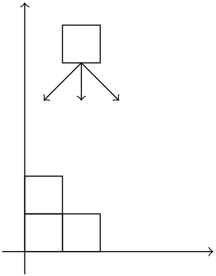


t=2

We want to find transition probabilities (depending on $\mathsf{e} \in \mathbb{E}_n$) so that the probability of becoming λ at time n coincides with $p_\lambda(\Gamma_{\mathsf{e}}; q)$.

How to construct a probability?

Imagine that someone drops a box one by one to create Young diagrams :



t=3

We want to find transition probabilities (depending on $\mathsf{e} \in \mathbb{E}_n$) so that the probability of becoming λ at time n coincides with $p_\lambda(\Gamma_{\mathsf{e}}; q)$.

First attempt

Define $\Phi : \mathbb{E}_n \to \mathsf{V}_n := \mathbb{Q}(q)\langle \lambda \mid \lambda \vdash n \rangle$ by

$$\Phi(\mathsf{e}) = \sum_{\lambda \vdash n} p_\lambda(\Gamma_{\mathsf{e}}; q)\lambda.$$

First assume for simplicity that the desired stochastic process is Markov, i.e., we assume that there exist linear maps $\Omega_r : \mathsf{V}_n \to \mathsf{V}_{n+1}$ so that

$$\Omega_r(\Phi(\mathsf{e})) = \Phi(\mathsf{e} \cup r)$$

if $\mathsf{e} \in \mathbb{E}_n$ and $\mathsf{e} \cup r := (\mathsf{e}(1), \ldots, \mathsf{e}(n), r) \in \mathbb{E}_{n+1}$.

For example, we should have

$$\begin{aligned} \Phi(0) &= \Box = \Omega_0(\emptyset), \\ \Phi(0,0) &= \Box\Box = \Omega_0\left(\Box\right), \\ \Phi(0,1) &= \begin{smallmatrix}\Box\\\Box\end{smallmatrix} = \Omega_1\left(\Box\right). \end{aligned}$$

$n = 2$

For $n = 2$, we obtain

$$\Phi(0,0,0) = \ydiagram{3} = \Omega_0(\ydiagram{2}), \quad \Phi(0,0,2) = \ydiagram{2,1} = \Omega_2(\ydiagram{2}),$$

$$\Phi(0,1,1) = \ydiagram{2,1} = \Omega_1\left(\ydiagram{1,1}\right), \quad \Phi(0,1,2) = \ydiagram{1,1,1} = \Omega_2\left(\ydiagram{1,1}\right),$$

$$\Phi(0,0,1) = \frac{1}{[2]_q}\ydiagram{2,1} + \frac{q}{[2]_q}\ydiagram{3} = \Omega_1(\ydiagram{2}).$$

We do not need a formula of $\Omega_0\left(\ydiagram{1,1}\right)$ for the reconstruction of Φ, but it might be natural to take

$$\Omega_0\left(\ydiagram{1,1}\right) = \ydiagram{2,1}.$$

$n = 3$

For $n = 3$, we obtain for example

$$\Phi(0,0,0,r) = \frac{[r]_q}{[3]_q}\,\ydiagram{1,3} + \frac{q^r[3-r]_q}{[3]_q}\,\ydiagram{4} = \Omega_r(\ydiagram{3}) \quad (0 \le r \le 3),$$

$$\Phi(0,0,1,1) = \frac{[2]_q}{[3]_q}\,\ydiagram{1,3} + \frac{q^2}{[3]_q}\,\ydiagram{4} = \frac{1}{[2]_q}\,\Omega_1\left(\ydiagram{1,2}\right) + \frac{q}{[2]_q}\Omega_1(\ydiagram{3}),$$

$$\Longrightarrow \ \Omega_1\left(\ydiagram{1,2}\right) = \ydiagram{1,3},$$

$$\Phi(0,0,1,2) = \frac{1}{[2]_q}\,\ydiagram{2,2} + \frac{q}{[3]_q}\,\ydiagram{1,3} + \frac{q^3}{[2]_q[3]_q}\,\ydiagram{4}$$

$$= \frac{1}{[2]_q}\,\Omega_2\left(\ydiagram{1,2}\right) + \frac{q}{[2]_q}\Omega_2(\ydiagram{3}) \ \Longrightarrow \ \Omega_2\left(\ydiagram{1,2}\right) = \ydiagram{2,2},$$

$$\Phi(0,0,2,2) = \ydiagram{2,2} = \Omega_2\left(\ydiagram{1,2}\right).$$

Last two equalities suggest that this approach works well to some extent.

Failures

However, we will soon arrive at a contradiction :

$$\Phi(0,0,2,2) = \ydiagram{2,2} = \Omega_2\left(\ydiagram{1,2}\right),$$
$$\Phi(0,1,1,2) = \frac{1}{[2]_q}\ydiagram{1,1,2} + \frac{q}{[2]_q}\ydiagram{1,3} \overset{?}{=} \Omega_2\left(\ydiagram{1,2}\right).$$

We interpret this as a failure of Markov property for the stochastic process since we have $\Phi(0,0,2) = \Phi(0,1,1)$ but they are obtained as

$$\Phi(0,0,2) : \ydiagram{1} \to \ydiagram{2} \to \ydiagram{1,2},$$
$$\Phi(0,1,1) : \ydiagram{1} \to \ydiagram{1,1} \to \ydiagram{1,2}.$$

I.e., the desired stochastic process depends on the “past states”.

Markovization

Idea : We extend the state space by remembering all the past states in order to make the process "more Markov".

We realize this idea by writing down the present time in the added box, i.e., by considering the standard Young tableaux.

$$\begin{array}{rcl} \widetilde{\mathsf{V}}_n := \mathbb{Q}(q)\langle T \mid T \in \mathsf{SYT}(\lambda), \lambda \vdash n\rangle & \xrightarrow{\pi} & \mathsf{V}_n \\ T & \mapsto & \lambda \end{array}$$

Problem

Find linear maps $\widetilde{\Omega}_r : \widetilde{\mathsf{V}}_n \to \widetilde{\mathsf{V}}_{n+1}$ such that

- we have $\Phi(\mathsf{e}) = \pi\,\widetilde{\Omega}_{\mathsf{e}(n)}\widetilde{\Omega}_{\mathsf{e}(n-1)}\cdots\widetilde{\Omega}_{\mathsf{e}(1)}(\emptyset)$ for any $\mathsf{e} \in \mathbb{E}_n$,
- $\widetilde{\Omega}_r(T)$ is a linear combination of standard Young tableaux obtained by adding $\boxed{n+1}$ on a top of some column of T.

Experimental results

Surprisingly, one can determine $\widetilde{\Omega}_r(T)$ *uniquely* and *consistently* for many T experimentally. For example, we may avoid the above inconsistency by

$$\widetilde{\Omega}_2\left(\ytableaushort{3,12}\right) = \ytableaushort{34,12}, \quad \widetilde{\Omega}_2\left(\ytableaushort{2,13}\right) = \frac{1}{[2]_q}\ytableaushort{4,2,13} + \frac{q}{[2]_q}\ytableaushort{2,134}.$$

Usually, formulas for $\widetilde{\Omega}_r(T)$ are simple and look like

$$\widetilde{\Omega}_r(T) = \frac{[r]_q}{[m]_q}T' + \frac{q^r[m-r]_q}{[m]_q}T'',$$

but we encounter more difficult formulas such as

$$\widetilde{\Omega}_4\left(\ytableaushort{46,1235}\right) = \frac{[3]_q}{[2]_q[4]_q}\ytableaushort{7,46,1235} + \frac{q}{[2]_q^2}\ytableaushort{467,1235} + \frac{q^2[3]_q}{[2]_q[4]_q}\ytableaushort{46,12357}.$$

Experimental observations

Similar formulas occur for

$$\begin{array}{|c|c|}\hline 4&6\\\hline\end{array}\atop\begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline 4&5\\\hline\end{array}\atop\begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline 3&6\\\hline\end{array}\atop\begin{array}{|c|c|c|c|}\hline 1&2&4&5\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline 3&5\\\hline\end{array}\atop\begin{array}{|c|c|c|c|}\hline 1&2&4&6\\\hline\end{array}.$$

We observe that the places of boxes $\boxed{i}$ with $i > 4$ are the same in this example. By looking at other examples of $\widetilde{\Omega}_r(T)$ focusing on the places of boxes $\boxed{i}$ with $i > r$, we observe that the transition rules only depend on the binary sequence ("*Maya diagram*") $\delta^{(r)}(T) = (\delta_i)_{i\in\mathbb{Z}}$ determined by

- looking at T from above and record the numbers in the boxes,
- replacing the numbers with $> r$ by 1 and by 0 otherwise,
- completing it by adding 1^∞ to the left and 0^∞ to the right.

For the above examples, we obtain

$$\delta^{(4)}(T) = (1^\infty, 0, 1, 0, 1, 0^\infty).$$

Experimental observations

Similar formulas occur for

$$\begin{ytableau} 4 & 6 \\ 1 & 2 & 3 & 5 \end{ytableau}, \quad \begin{ytableau} 4 & 5 \\ 1 & 2 & 3 & 6 \end{ytableau}, \quad \begin{ytableau} 3 & 6 \\ 1 & 2 & 4 & 5 \end{ytableau}, \quad \begin{ytableau} 3 & 5 \\ 1 & 2 & 4 & 6 \end{ytableau}.$$

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For the above examples, we obtain

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Guess the general formulas

We further observe that the new boxes are allowed to fall only at the places of leftmost consecutive sequences of 0 in $\delta^{(r)}(T)$. Let us write $f_c(T)$ the standard Young tableau obtained by adding $\boxed{n+1}$ at c-th column of T.

For example, when $\delta^{(r)}(T) = (1^\infty, 0^a, 1^b, 0^\infty)$, we obtain

$$\widetilde{\Omega}_r(T) = \frac{[a]_q}{[a+b]_q} f_1(T) + \frac{q^a [b]_q}{[a+b]_q} f_{a+b+1}(T).$$

Experiments suggest that if $\delta^{(r)}(T) = (1^\infty, 0^a, 1^b, 0^c, 1^d, 0^\infty)$, we have

$$\widetilde{\Omega}_r(T) = \frac{[a]_q [a+b+c]_q}{[a+b]_q [a+b+c+d]_q} f_1(T) + \frac{q^a [b]_q [c]_q}{[a+b]_q [c+d]_q} f_{a+b+1}(T) + \frac{q^{a+c} [b+c+d]_q [d]_q}{[a+b+c+d]_q [c+d]_q} f_{a+b+c+d+1}(T).$$

Now it is easy to guess the general formulas without further experiments.

Definition of $\widetilde{\Omega}_r$

For any binary sequence $\delta = (\delta_i)_{i \in \mathbb{Z}}$, we write

$$W(\delta) := \{i \in \mathbb{Z} \mid \delta_i = 0, \delta_{i-1} = 1\} \quad \text{(leftmost white boxes)}$$
$$R(\delta) := \{i \in \mathbb{Z} \mid \delta_i = 1, \delta_{i-1} = 0\} \quad \text{(leftmost red boxes)}$$

Definition (H. 2024, reformulation by Guay-Paquet)

We define a linear map $\widetilde{\Omega}_r : \widetilde{\mathsf{V}}_n \to \widetilde{\mathsf{V}}_{n+1}$ by

$$\widetilde{\Omega}_r(T) := \sum_{c \in W(\delta^{(r)}(T))} \frac{\prod_{i \in R(\delta^{(r)}(T))} [i - c]_q}{\prod_{j \in W(\delta^{(r)}(T)) \setminus \{c\}} [j - c]_q} f_c(T).$$

One can check that the coefficients are nonnegative and sum to 1.

Probabilistic formula

Now we can state our main results.

Theorem (H. 2024)

For any conjugate Hessenberg function $\mathsf{e} \in \mathbb{E}_n$, we have

$$\Phi(\mathsf{e}) = \pi\,\widetilde{\Omega}_{\mathsf{e}(n)}\widetilde{\Omega}_{\mathsf{e}(n-1)}\cdots\widetilde{\Omega}_{\mathsf{e}(1)}(\emptyset).$$

Corollary

For any unit interval graph Γ and $q \in \mathbb{R}_{>0}$, the chromatic quasisymmetric function $\mathbf{X}_\Gamma(q)$ specialized at $q \in \mathbb{R}_{>0}$ is e-positive. In particular, the Stanley–Stembridge conjecture holds.

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- Generalize the probabilistic formula to more general labeled graphs with symmetric chromatic quasisymmetric functions.
  (Probabilistic models can be considered more generally.)

Thank you!