Symmetries of periodic and free boundary q-Whittaker measures

Jimmy He

The Ohio State University

joint work with Michael Wheeler

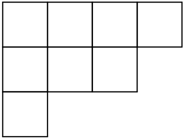
Summary

This talk is about two things:

- Identities of symmetric functions (q-Whittaker and Hall–Littlewood functions)
- A different perspective on symmetric functions (connections to probability/statistical physics)

Random partitions

Schur measures



$\lambda = (4, 3, 1), \qquad l(\lambda) = 3.$

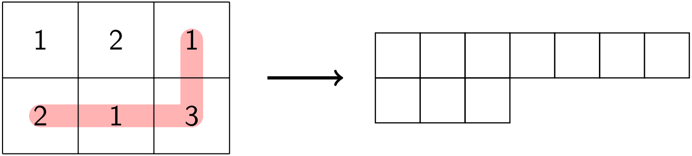
- Many models of random partitions defined using symmetric functions.
- Define the Schur functions by

$$s_\lambda(x_1, \ldots, x_n) = \frac{\det \left(x_i^{\lambda_j + n - j} \right)_{i,j=1}^{n}}{\prod_{i<j}(x_i - x_j).}$$

  This is a symmetric polynomial in the x_i's.
- The Schur measure (Okounkov '01) with parameters $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ is probability measure on λ proportional to $s_\lambda(x_1, \ldots, x_n)s_\lambda(y_1, \ldots, y_m)$. Normalization constant given by Cauchy identity

$$\sum_\lambda s_\lambda(x_1, \ldots, x_n)s_\lambda(y_1, \ldots, y_m) = \prod_{i,j} \frac{1}{1 - x_i y_j}.$$

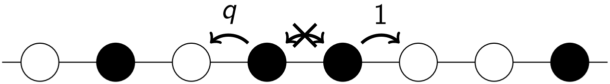
Connections to statistical physics



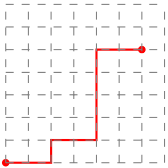
- For many choices of the x_i and y_j, the Schur measure is connected to models coming from statistical physics.
- The Robinson–Schensted–Knuth correspondence sends an $n \times m$ array of integers to a pair of tableau of the same shape λ. If the integers are geometric random variables of parameter $x_i y_j$, then λ is distributed as the Schur measure.
- Define a model called last passage percolation. Consider a random field $X_{i,j}$ of independent geometric random variables of parameter p. The passage time from $(0,0)$ to (n,m) is given by $\max_\gamma \sum_i X_{\gamma_i}$, where γ is an up-right path. Under RSK this corresponds to λ_1.

Macdonald measures

- Can define Macdonald measure using the Macdonald polynomials $P_\lambda(x; q, t)$ analogously to the Schur measures (Borodin–Corwin '14). They have been used to study many models coming from statistical physics.
- When $q = 0$, this is called the Hall–Littlewood measure, related to the six vertex model and particle systems like the ASEP.



- When $t = 0$, this is called q-Whittaker measure, related to polymer models like the log–gamma polymer.



$$Z(6,5) = \sum_\gamma \prod_i X_{\gamma_i}$$

Periodic measures on partitions

Periodic Schur measure

What is a periodic measure on partitions?

- Borodin '07 defined the periodic Schur measure, with the probability of λ proportional to $\sum_\mu u^{|\mu|} s_{\lambda/\mu}(x_1, \dots, x_n) s_{\lambda/\mu}(y_1, \dots, y_m)$. Here, μ is a partition contained in λ, and $s_{\lambda/\mu}$ is a skew Schur function.
- Normalization constant given by

$$\sum_{\lambda,\mu} u^{|\mu|} s_{\lambda/\mu}(x_1, \dots, x_n) s_{\lambda/\mu}(y_1, \dots, y_m) = \frac{1}{(u; u)_\infty} \prod_{i,j} \frac{1}{(x_i y_j; u)_\infty},$$

  where $(x; q)_\infty = \prod_{i \geq 0}(1 - q^i x)$ is the q-Pochhammer symbol.
- This measure turns out to be related to a quasi-periodic version of last passage percolation. One considers a cylinder tiled by $n \times m$ arrays of geometric random variables, but where powers of u are accumulated in the parameters as well.

Periodic q-Whittaker measure

- We can define a probability measure on partitions called the periodic q-Whittaker measure by setting the probability of λ proportional to

$$\sum_{\mu} u^{|\mu|} P_{\lambda/\mu}(x_1, \ldots, x_n; q, 0) Q_{\lambda/\mu}(y_1, \ldots, y_n; q, 0).$$

  The $Q_{\lambda/\mu}$ are a scalar multiple of the $P_{\lambda/\mu}$ (skew Macdonald polynomial).

- The normalization constant is

$$\frac{1}{(u;u)_\infty} \prod_{i,j} \frac{1}{(x_i y_j; u, q)_\infty},$$

  where $(x; q, u)_\infty = \prod_{i,j=0}^{\infty}(1 - x u^i q^j)$. This is a version of the Cauchy identity.

Symmetries of periodic q-Whittaker measures

Theorem (H.–Wheeler '23)

The expression

$$\sum_{\mu,\lambda:\lambda_1 \le k} \frac{u^{|\mu|}}{(q;q)_{n-\lambda_1}} P_{\lambda/\mu}(x_1,\dots,x_n;q,0) Q_{\lambda/\mu}(y_1,\dots,y_m;q,0)$$

is symmetric in the parameters u and q.

- Generalizes an identity of Imamura–Mucciconi–Sasamoto '23, who showed the special case when one parameter is 0.
- The IMS proof is bijective, would be interesting to see if it could be extended to this case.

Probabilistic meaning

Theorem (H.–Wheeler '23)

The expression

$$\sum_{\mu,\lambda:\lambda_1\leq k} \frac{u^{|\mu|}}{(q;q)_{k-\lambda_1}} P_{\lambda/\mu}(x_1,\dots,x_n;q,0)Q_{\lambda/\mu}(y_1,\dots,y_m;q,0)$$

is symmetric in the parameters u and q.

- The expression has a probabilistic interpretation: If χ is a q-geometric random variable, meaning $\mathbb{P}(\chi \leq n) = \frac{(q;q)_\infty}{(q;q)_n}$, then the expression is equivalent to $\mathbb{P}(\lambda_1 + \chi \leq k)$.

Contour integral formulas

Theorem (H.–Wheeler '23)

The expression

$$\sum_{\mu,\lambda:\lambda_1\le k} \frac{u^{|\mu|}}{(q;q)_{n-\lambda_1}} P_{\lambda/\mu}(x_1,\dots,x_n;q,0)\,Q_{\lambda/\mu}(y_1,\dots,y_m;q,0)$$

equals

$$\Phi_n(u,q)\oint_C \frac{dz_1}{2\pi i z_1}\cdots\oint_C \frac{dz_k}{2\pi i z_k}\prod_{i,j}(1+z_i^{-1}x_j)\prod_{i,j}(1+z_i y_j)\widetilde{\Delta}(z;q,u),$$

where C is a circle centered at 0,

$$\Phi_n(u,q)=\frac{(1-qu)^k}{n!(1-u)^k(1-q)^k},$$

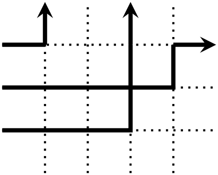
and

$$\widetilde{\Delta}(z;q,u)=\prod_{i\neq j}\frac{(1-quz_iz_j^{-1})(1-z_iz_j^{-1})}{(1-qz_iz_j^{-1})(1-uz_iz_j^{-1})}.$$

Stochastic six vertex model

$\mathbf{p}_{i,j}$ | $1-\mathbf{p}_{i,j}$ | $t\mathbf{p}_{i,j}$ | $1-t\mathbf{p}_{i,j}$ | 1 | 1

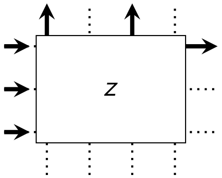
$$\mathbf{p}_{i,j} = \frac{1-x_i y_j}{1-t x_i y_j}$$



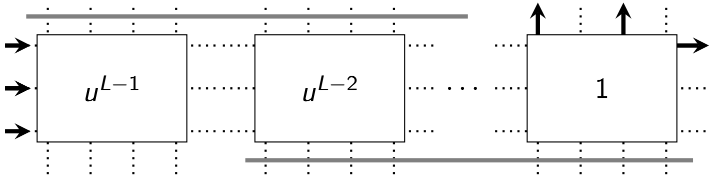
- Stochastic six vertex model is probability distribution on configurations of arrows traveling edges of a grid.
- Model depends on global parameter t, and row/column parameters $x_1 \ldots, x_n$ and $y_1 \ldots, y_m$.
- Can be sampled one vertex at a time. Probabilities at vertex (i,j) depend on t and $x_i y_j$ (called the spectral parameter). Come from R-matrix of $U_q(\widehat{\mathfrak{sl}_2})$.

Quasi-periodic boundary conditions

We represent a single $m \times n$ 6VM (with spectral parameter scaled by z) by



We define quasi-periodic 6VM of length L by

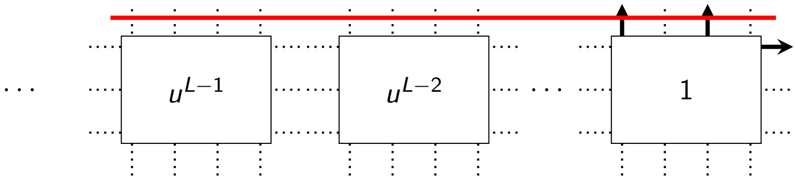


Gray line indicates identification of edges. Can send $L \to \infty$, and we do so.

Periodic Hall–Littlewood measure

- Analogous to periodic q-Whittaker measure, we define the periodic Hall–Littlewood measure by setting probability of λ proportional to
$$\sum_{\mu} u^{|\mu|} P_{\lambda/\mu}(x_1, \ldots, x_n; 0, t) Q_{\lambda/\mu}(y_1, \ldots, y_m; 0, t).$$
- Normalization constant is
$$\frac{1}{(u;u)_\infty} \prod_{i,j} \frac{(tx_iy_j;u)_\infty}{(x_iy_j;u)_\infty}.$$

Quasi-periodic 6VM and periodic HL measure



Theorem (H.–Wheeler '23)

The distribution for the number of times arrows cross the red line plus χ equals that of $l(\lambda)$, where λ follows the periodic HL measure and χ is u-geometric (with parameters in both models matching).

- Proof uses vertex models, Yang–Baxter equation.
- When $u \to 1$, model becomes stationary periodic stochastic six vertex model, but observable in theorem blows up.

Free boundary measures

Free boundary measures

- So far, we've discussed various measures related to the Cauchy identity. We now turn to the Littlewood identity, which turns out to correspond to models with either one or two open boundaries.
- The classic example is the Pfaffian Schur measure, for which λ has probability proportional to $s_\lambda(x_1, \dots, x_n)\mathbf{1}_{\lambda' \text{ even}}$ (here λ' means rows and columns in picture are swapped). Normalization constant given by the Littlewood identity

$$\sum_{\lambda:\lambda' \text{ even}} s_\lambda(x_1, \dots, x_n) = \prod_{i<j} \frac{1}{1 - x_i x_j}.$$

- Generalized by Betea–Bouttier–Nejjar–Vuletić '18 to free boundary Schur measure, where probability of λ proportional to $\mathbf{1}_{\lambda' \text{ even}} \sum_{\mu' \text{ even}} u^{|\mu|} s_{\lambda/\mu}(x_1, \dots, x_n)$, with normalization constant

$$\frac{1}{(u;u)_\infty} \prod_{i<j} \frac{1}{(x_i x_j; u)_\infty}$$

Free boundary q-Whittaker measure

- We can define a probability measure on partitions called the periodic q-Whittaker measure by setting the probability of λ proportional to

$$\sum_{\mu} u^{|\mu|/2} h^*_{\lambda'}(a, b; q) h_{\mu'}(c/\sqrt{u}, d/\sqrt{u}; q) P_{\lambda/\mu}(x_1, \dots, x_n; q, 0).$$

  The h^*_λ and h_μ are boundary weights, defined in terms of parameters a, b, c, d. If $a = b = c = d = 0$, restrict μ' and λ' to be even.

- The normalization constant is explicit and factorized (but complicated). When $a = b = c = d = 0$, it's

$$\frac{1}{(u; u)_\infty (uq; u, q)_\infty} \prod_{i<j} \frac{1}{(x_i x_j; u, q)_\infty}.$$

Symmetries of free boundary q-Whittaker measures

Theorem (H.–Wheeler '25+)

The expression

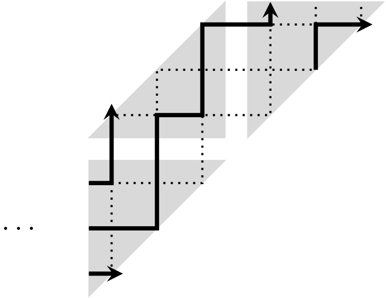
$$\sum_{\lambda,\mu:\lambda_1\le k} u^{|\mu|/2}\frac{h_{k-\lambda_1}(ab;q)h^*_{\lambda'}(a,b;q)}{(q;q)_{k-\lambda_1}}h_{\mu'}(c/\sqrt{u},d/\sqrt{u};q)P_{\lambda/\mu}(x;q,0)$$

is symmetric in the parameters u and q, and separately in a, b, c, d.

- As in periodic setting, we find explicit contour integral formulas, and this has a probabilistic interpretation.
- Generalizes previous identities of Imamura–Mucciconi–Sasamoto '23 (and extended in H. '24).

Free boundary Hall–Littlewood measure

- We can define the free boundary Hall–Littlewood measure analogously. Like in periodic setting, can be related to observables in quasi-open six vertex model. Here arrows enter from diagonal with weights depending on a, b, c, d.



A pair of curious identities

Theorem (H.–Wheeler '23)

$$\sum_{\lambda,\mu:\lambda_1\le k} \frac{(t;t)_{m_k(\mu)}}{(t;t)_{m_k(\lambda)}} u^{|\mu|} P_{\lambda/\mu}(x_1,\ldots,x_n;0,t) Q_{\lambda/\mu}(y_1,\ldots,y_m;0,t)$$

$$= \frac{1}{(u;u)_k} \left(\prod_{j=1}^{m} y_j^k\right) P_{k^m}(x_1,\ldots,x_n,y_1^{-1},\ldots,y_m^{-1};u,t).$$

Theorem (Finn–Vanicat '17)

$$\sum_{\lambda,\mu:\lambda_1\le 2k} u^{|\mu|/2} \frac{h_\lambda(a,b;t)}{h_{m_{2k}(\lambda)}(ab;t)} h^*_\mu(c/\sqrt{u},d/\sqrt{u};t) P_{\lambda/\mu}(x_1,\ldots,x_n;0,t)$$

$$= C_k(u,t,a,b,c,d) \left(\prod_{i=1}^{n} x_i^k\right) K_{k^n}(x_1,\ldots,x_n;u,t,a,b,c,d).$$

Thanks for listening!