Towards plethystic $\mathfrak{sl}_2$ crystals

Álvaro Gutiérrez (University of Bristol)

Our main problem

Let $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C}) = \{M \in \mathsf{Mat}_{2\times 2} \ : \ \mathsf{tr}(M) = 0\}$ with a Lie bracket.

Facts

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Clebsch–Gordan (1800s), Kashiwara (1990s)

$\mathfrak{sl}_2$ crystals

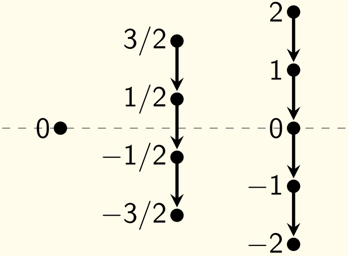
Any $\mathfrak{sl}_2$ representation V has a crystal:

A *crystal* is a vertex-weighted directed graph,

- if $x \longrightarrow y$ then $\mathrm{wt}(y) = \mathrm{wt}(x) - 1$,
- connected components are paths,
- it is weight-symmetric.

Connected components correspond to irreducible representations.

$$\mathrm{Sym}^0\,\mathbb{C}^2 \oplus \mathrm{Sym}^3\,\mathbb{C}^2 \oplus \mathrm{Sym}^4\,\mathbb{C}^2$$



A similar problem (ii)

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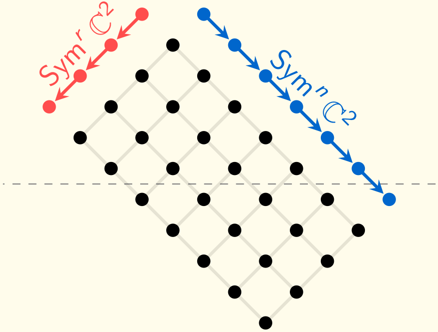

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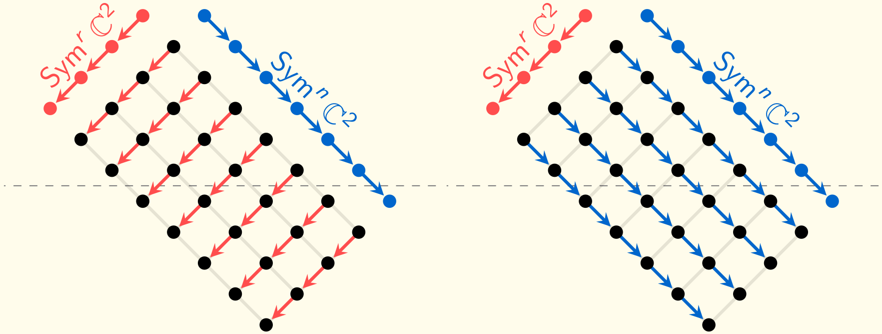

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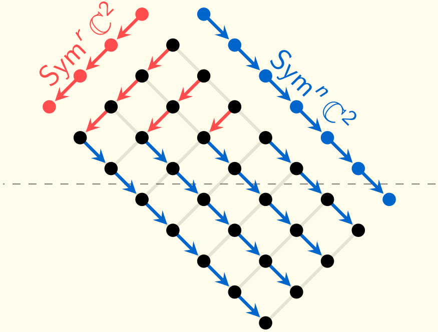

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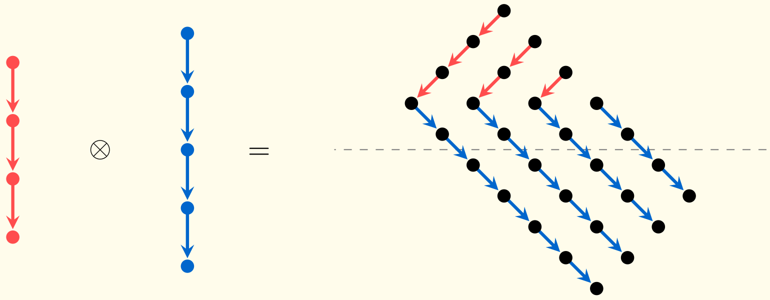

$$\mathrm{Sym}^3 \mathbb{C}^2 \otimes \mathrm{Sym}^4 \mathbb{C}^2 = \mathrm{Sym}^9 \mathbb{C}^2 \oplus \mathrm{Sym}^7 \mathbb{C}^2 \oplus \mathrm{Sym}^5 \mathbb{C}^2 \oplus \mathrm{Sym}^3 \mathbb{C}^2$$

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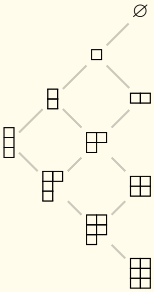
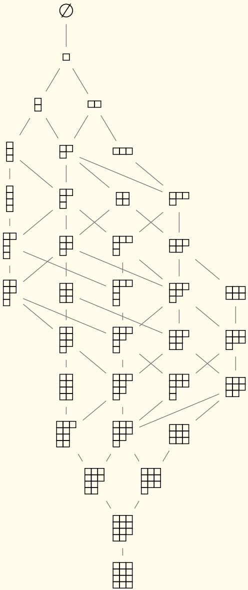
This is a ranked poset: Young's lattice.

$L(2,3) =$



Example: $L(3,4)$



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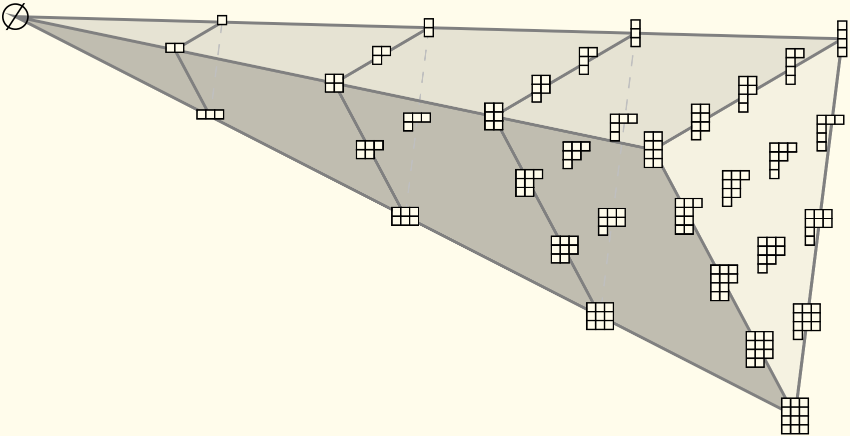

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Literature for Problem B

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Our strategy for Problem B

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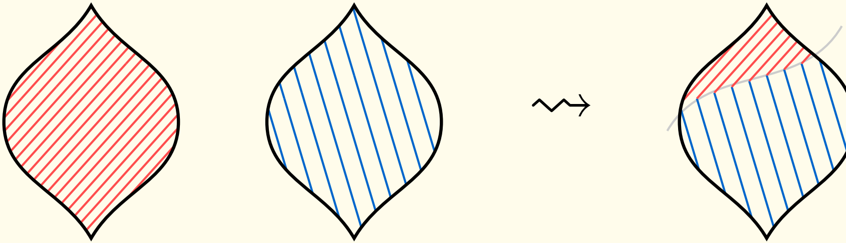

The top decomposition

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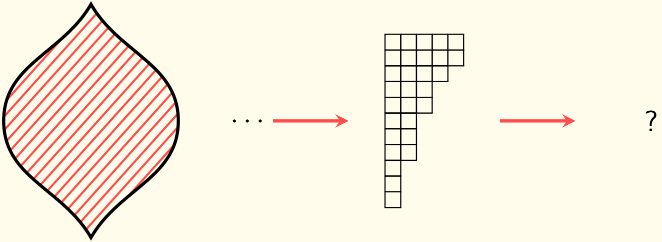

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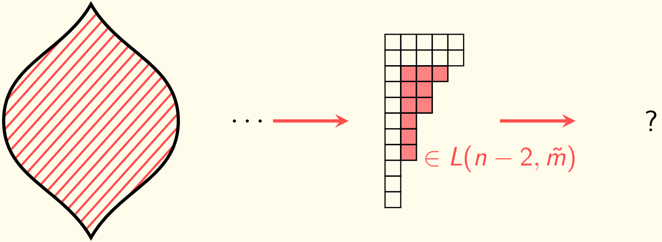

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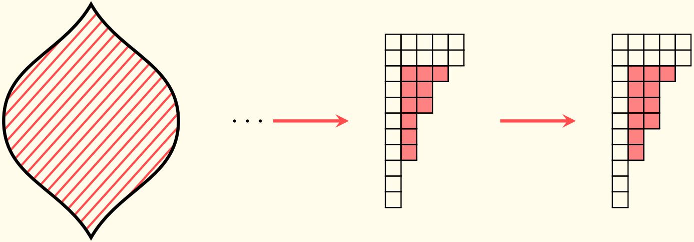

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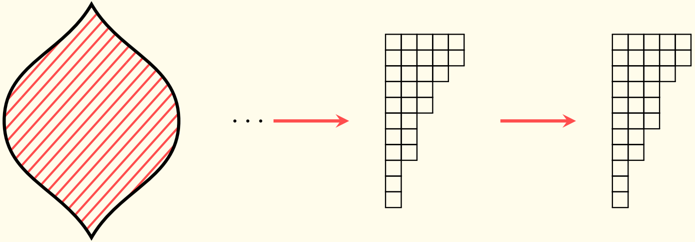

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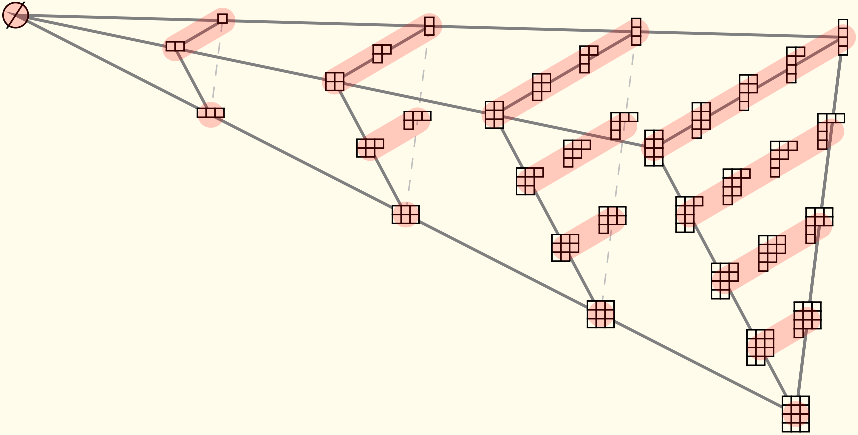

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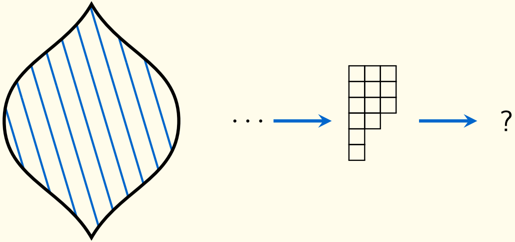

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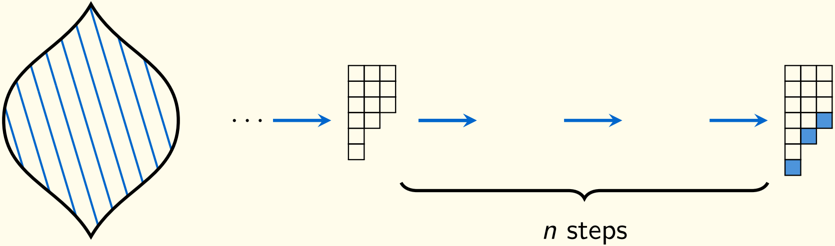

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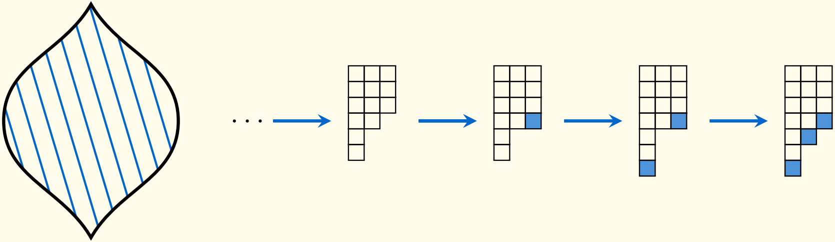

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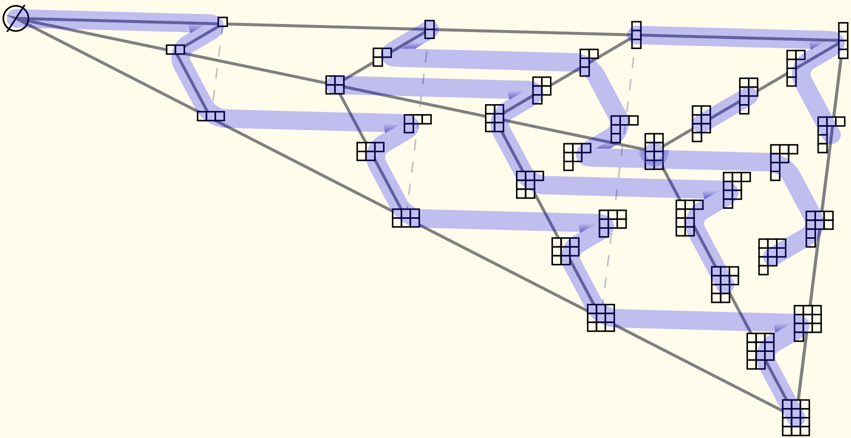

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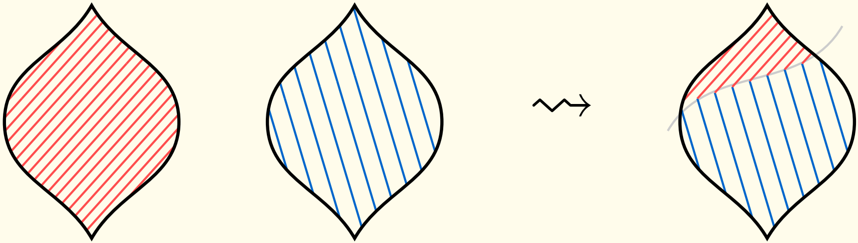

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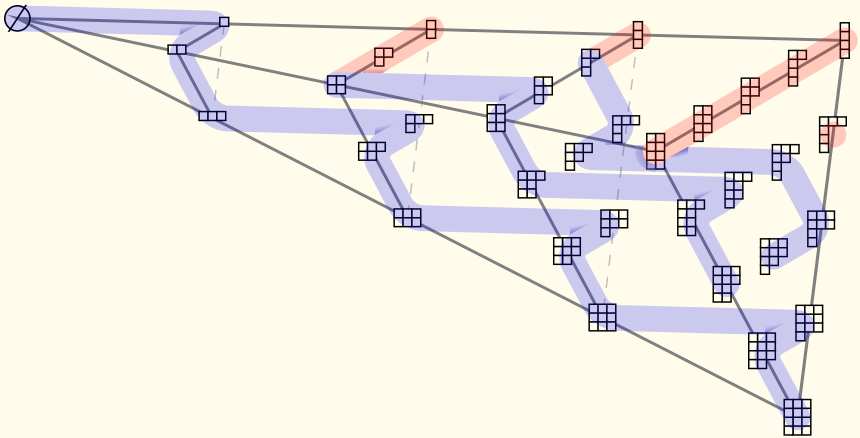

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- New combinatorial proof for a formula for the number of constituents of $\Lambda^2 \operatorname{Sym}^r \mathbb{C}^2$, $\Lambda^3 \operatorname{Sym}^r \mathbb{C}^2$, by Almkvist–Fossum'78
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What about $n = 5$ and beyond?

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