Lusztig's q-weight multiplicities and KR crystals

Hyeonjae Choi

Seoul National University

(joint work with Donghyun Kim and Seung Jin Lee)

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Symmetric function

For a partition λ, the Schur function s_λ is defined by

$$s_\lambda = \sum_T x^{\mathsf{wt}(T)}$$

where the sum is over all semistandard Young tableaux T of shape λ, and for a composition $\alpha = (\alpha_1, \alpha_2, \dots)$, we write $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$.

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The Schur function can be expressed as $s_\lambda = \sum_\mu K_{\lambda,\mu} m_\mu$, where m_μ is the monomial symmetric function and $K_{\lambda,\mu}$ is the Kostka number.

The Kostka number $K_{\lambda,\mu}$ counts the number of semistandard Young tableaux of shape λ and weight μ.

Kostka-Foulkes polynomial

The Kostka-Foulkes polynomials $K_{\lambda,\mu}(q)$, q-analogue of the Kostka number, are defined by

$$s_\lambda = \sum_\mu K_{\lambda,\mu}(q) P_\mu(x;q),$$

where $P_\mu(x;q)$ is the Hall-Littlewood polynomial.

Charge

The charge is a statistic on semistandard Young tableaux, introduced by Lascoux and Scḧtzenberger (1978). It provides a combinatorial formula for the Kostka-Foulkes polynomial:

$$K_{\lambda,\mu}(q) = \sum_{T} q^{\mathrm{charge}(T)}$$

where the sum is over all semistandard Young tableaux T of shape λ and weight μ.

Charge

We define the charge on a standard Young tableau T.

Let $\mathrm{Des}(T)$ be the set of integers i such that $i+1$ appears to the right of i in T. Then the charge is defined by $\sum_{i \in \mathrm{Des}(T)} (n-i)$.

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For example, consider the standard Young tableau

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 & \\ \cline{1-2} 5 & & \\ \cline{1-1}\end{array}.$$

Then $\mathrm{Des}(T) = \{2, 5\}$ and $\mathrm{charge}(T) = (6-2) + (6-5) = 5$.

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- Lusztig's q-weight multiplicities, which generalize the Kostka-Foulkes polynomials to other Lie types.

Weight multiplicity

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The weight multiplicity $\mathrm{KL}^{\mathfrak{g}}_{\lambda,\mu}$ is the dimension of the μ-weight space in the irreducible representation of $\mathfrak{g}$ with highest weight λ.
By the Weyl character formula,

$$\mathrm{KL}^{\mathfrak{g}}_{\lambda,\mu} = \sum_{w \in W} (-1)^{\ell(w)} [e^{w(\lambda+\rho)-(\mu+\rho)}] \prod_{\alpha \in R^+} \frac{1}{1-e^{\alpha}}$$

where W is Weyl group, R^+ is the set of positive roots, $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$, and $[e^{\beta}]f$ denotes the coefficient of e^{β} in f.

Lusztig's q-weight multiplicity

The q-analogue of $\mathrm{KL}^{\mathfrak{g}}_{\lambda,\mu}$ is defined by

$$\mathrm{KL}^{\mathfrak{g}}_{\lambda,\mu}(q) = \sum_{w \in W} (-1)^{\ell(w)} [e^{w(\lambda+\rho)-(\mu+\rho)}] \prod_{\alpha \in R^+} \frac{1}{1 - q e^{\alpha}}.$$

$\mathrm{KL}^{\mathfrak{g}}_{\lambda,\mu}(q)$ is Lusztig's q-weight multiplicity.

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- $\mathrm{KL}^{\mathfrak{g}}_{\lambda+(k^n),\mu+(k^n)}(q)$ stabilize for sufficiently large k. These are called stable KL polynomials.
- Combinatorial formulas for stable KL polynomials are known: for types B and C by Shimozono (2005) ; and for type D by Lecouvey and Shimozono (2007).

Type C

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- Lecouvey (2005) studied cyclage graphs on Kashiwara-Nakashima tableaux and defined charge to conjecture a combinatorial formula.
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For a partition μ, Lee conjectured a formula in terms of Killirov-Reshitikhin crystals, which was proved by C.-Kim-Lee (2024).

Combinatorial object

There are several known combinatorial models for $\mathrm{KL}_{\lambda,\mu}^{C_n}$:

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- King tableaux.
- Semistandard oscillating tableaux [Lee '23].

In this talk, we focus on semistandard oscillating tableaux.

Semistandard oscillating tableau

An oscillating horizontal strip (ohs) S is a triple of partitions (λ, μ, ν) such that both μ/λ and μ/ν are horizontal strips.

We define:

- $\mathrm{length}(S) = |\mu/\lambda| + |\mu/\nu|$,
- $I(S) = \lambda$, $F(S) = \nu$, and $c(S) = \mu_1$.

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A semistandard oscillating tableaux T is a sequence of ohs $(S_1, S_2, \cdots, S_n)$ satisfying:

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We define the following:

- $\mathrm{wt}(T) = (\mathrm{length}(S_1), \cdots, \mathrm{length}(S_n))$ and $c(T) = \max(c(S_i))$.
- $T \in \mathrm{SSOT}(\lambda, \mu)$ if $F(S_n) = \lambda$ and $\mathrm{wt}(T) = \mu$.
- $T \in \mathrm{SSOT}_{\leq g}(\lambda, \mu)$ if $T \in \mathrm{SSOT}(\lambda, \mu)$ and $c(T) \leq g$.

Example

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- $T_1 = ((\emptyset, \ydiagram{1}, \ydiagram{1}), (\ydiagram{1}, \ydiagram{1,1}, \ydiagram{1,1}), (\ydiagram{1,1}, \ydiagram{1,1}, \ydiagram{1}), (\ydiagram{1}, \ydiagram{1}, \emptyset))$,
- $T_2 = ((\emptyset, \ydiagram{1}, \ydiagram{1}), (\ydiagram{1}, \ydiagram{1}, \emptyset), (\emptyset, \ydiagram{1}, \ydiagram{1}), (\ydiagram{1}, \ydiagram{1}, \emptyset))$,
- $T_3 = ((\emptyset, \ydiagram{1}, \ydiagram{1}), (\ydiagram{1}, \ydiagram{2}, \ydiagram{2}), (\ydiagram{2}, \ydiagram{2}, \ydiagram{1}), (\ydiagram{1}, \ydiagram{1}, \emptyset))$.

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$\mathrm{SSOT}_{\leq 1}(\lambda,\mu)$ contains only T_1 and T_2, since $c(T_3) = 2$.

Type C object

Lee (2023) proved that for any $g \geq \lambda_1$,

$$\mathrm{KL}^{C_n}_{\lambda,\mu} = |\,\mathrm{SSOT}_{\leq g}(\hat{\lambda}, \hat{\mu})|,$$

where $\hat{\lambda} = (g - \lambda_n, \ldots, g - \lambda_1)$.

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We now investigate the energy function on the Kirillov–Reshetikhin crystal, which plays the role of the charge statistic on SSOT.

Kirillov-Reshetikhin crystal

The Kirillov-Reshetikhin crystals (KR crystals) are crystal bases $B^{r,s}$ for certain irreducible finite-dimensional modules $W_s^{(r)}$, called Kirillov-Reshetikhin modules (KR modules), over the quantized affine algebra $U_q'(\mathfrak{g})$.

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- For a partition μ, define $B_\mu = B^{\mu_n,1} \otimes \cdots \otimes B^{\mu_1,1}$.
- For a partition λ, let $\mathrm{HW}(B_\mu, \lambda)$ be the set of classical highest weight elements of weight λ^t in B_μ.
- By duality, we use KR crystals of type $B_N^{(1)}$ for sufficiently large N.

Kirillov-Reshetikhin crystal

There exists a natural bijection between $\mathrm{SSOT}(\lambda, \mu)$ and $\mathrm{HW}(B_\mu, \lambda)$ by recording the column indices.

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For $\mathrm{SSOT}_{\leq 1}((0,0,0,0),(1,1,1,1))$, we have:

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Energy function

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In general, the energy function is difficult to compute.

However, when $B = (B^{1,1})^{\otimes n}$, the energy function can be computed explicitly.

Energy function when standard case

Let $a_n \otimes a_{n-1} \otimes \cdots \otimes a_1$ be an element in $(B^{1,1})^{\otimes n}$ of type $B_N^{(1)}$.

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Let $a_n \otimes a_{n-1} \otimes \cdots \otimes a_1$ be an element in $(B^{1,1})^{\otimes n}$ of type $B_N^{(1)}$.

The local energy function $H(b,a)$ is defined by:

$$H(b,a) = \begin{cases} 2 & \text{if } a = 1 \text{ and } b = \overline{1} \\ 1 & \text{if } b \succ a \text{ and } (b,a) \neq (\overline{1}, 1) \\ 0 & \text{if } b \preceq a \end{cases}$$

under the order $1 \prec 2 \prec \cdots \prec \overline{2} \prec \overline{1}$.

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The energy function $\overline{D}$ is defined by $\overline{D} = \sum_{i=1}^{n-1} (n-i) H(a_{i+1}, a_i)$.

Type A case

For type A, the energy function coincides with the charge statistics [Nakayashiki-Yamada '97].

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For example, consider

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \leftrightarrow 3 \otimes 1 \otimes 2 \otimes 2 \otimes 1 \otimes 1$$

Since $3 \otimes 1 \otimes 2 \otimes 2 \otimes 1 \otimes 1$, we have $\overline{D}(T) = (6-5) + (6-2) = 5$, which exactly matches the charge of T.

Example

$\mathrm{SSOT}((0,0,0,0),(1,1,1,1)) = \{T_1, T_2, T_3\}$ where $T_1 = -1 \otimes -1 \otimes 1 \otimes 1$, $T_2 = -1 \otimes 1 \otimes -1 \otimes 1$, and $T_3 = -1 \otimes -2 \otimes 2 \otimes 1$.

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- $-1 \otimes -1 {\color{red}\otimes} 1 \otimes 1$ with $\overline{D}(T_1) = (4-2) \times 2 = 4$.
- $-1 {\color{red}\otimes} 1 \otimes -1 {\color{red}\otimes} 1$ with $\overline{D}(T_2) = (4-3) \times 2 + (4-1) \times 2 = 8$.
- $-1 {\color{blue}\otimes} -2 {\color{blue}\otimes} 2 {\color{blue}\otimes} 1$ with $\overline{D}(T_3) = (4-3) + (4-2) + (4-1) = 6$.

Main theorem

Theorem (C.-Kim-Lee, 2024)

$$\mathrm{KL}^{C_n}_{\lambda,\mu}(q) = \sum_{T \in \mathrm{SSOT}_{\leq g}(\hat{\lambda},\hat{\mu})} q^{\overline{D}(T)},$$

where $g \geq \lambda_1$.

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where $g \geq \lambda_1$.

As a corollary, we have $\mathrm{KL}^{C_n}_{\lambda+(1^n),\mu+(1^n)}(q) \geq \mathrm{KL}^{C_n}_{\lambda,\mu}(q)$,

since $\mathrm{SSOT}_{\leq g+1}(\hat{\lambda},\hat{\mu}) \supseteq \mathrm{SSOT}_{\leq g}(\hat{\lambda},\hat{\mu})$.

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When $g = 1$, we have $\lambda = (1, 1, 1, 1)$ and $\mu = (0, 0, 0, 0)$.

$$\mathrm{KL}^{C_n}_{(1,1,1,1),(0,0,0,0)}(q) = q^{\overline{D}(T_1)} + q^{\overline{D}(T_2)} = q^8 + q^4.$$

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When $g = 2$, we have $\lambda = (2,2,2,2)$ and $\mu = (1,1,1,1)$.

$$\mathrm{KL}^{C_n}_{(2,2,2,2),(1,1,1,1)}(q) = q^{\overline{D}(T_1)} + q^{\overline{D}(T_2)} + q^{\overline{D}(T_3)} = q^8 + q^6 + q^4.$$

Level-restricted q weight multiplicity

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The level-restricted q-weight multiplicity $\mathrm{KL}^{\mathfrak{g},lr}_{\lambda,\mu}(q)$ is defined by

$$\mathrm{KL}^{\mathfrak{g},lr}_{\lambda,\mu}(q) = \sum_{w \in W} (-1)^{\ell(w)} [e^{w(\lambda+\rho)-(\mu+\rho)}] \prod_{\alpha \in R_A} \frac{1}{1 - qe^{\alpha}} \prod_{\alpha \in R^+ \setminus R_A} \frac{1}{1 - e^{\alpha}}.$$

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We also proved the following formula

$$\mathrm{KL}^{C_n,lr}_{\lambda,\mu}(q) = \sum_{\substack{T\in\mathrm{SSOT}(\hat{\lambda},\hat{\mu})\\ c(T)\leq g}} q^{||\hat{\mu}||+\frac{|\hat{\mu}|-|\hat{\lambda}|}{2}-\overline{D}(\phi_r(T))}$$

using the row KR crystals $B^{1,\mu_n} \otimes \cdots \otimes B^{1,\mu_1}$ of type $C_N^{(1)}$.

Summary and future direction

We also investigate these multiplicities for other Lie types.

<table>
<tr><th></th><th colspan="2">Lusztig's q-weight multiplicity</th><th>l.r. q-weight multiplicity</th></tr>
<tr><td>type A</td><td colspan="3">Lascoux and Schützenberger (1978)</td></tr>
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- A natural next step is to fill in the missing entries.
- It would also be interesting to investigate the connection with rigged configurations.

Thanks for listening