Real Stability and Log Concavity are coNP-Hard

Tracy Chin

University of Washington

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What the heck is coNP???

For the purposes of this talk, you can mentally replace coNP-hard with NP-hard, or just "hard", and you will lose essentially nothing.

Stable Polynomials

Real Stable Polynomials

Definition

A polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is *real stable* if $f(ta + b) \in \mathbb{R}[t]$ is real-rooted for all $a \in \mathbb{R}^n_{>0}, b \in \mathbb{R}^n$.

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Most proofs using stable polynomials start with a polynomial known to be stable, then apply a series of stability-preserving operations.

Real Stability is Hard

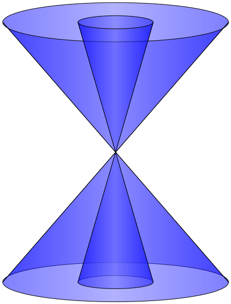
Theorem (C. 2024)

It is coNP-hard to decide if a homogeneous cubic polynomial is real stable.

Main Tool: Hyperbolic Polynomials

Definition

Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial and let $e \in \mathbb{R}^n$. We say that f is *hyperbolic with respect to* e if $f(e) > 0$ and $f(te + x) \in \mathbb{R}[t]$ is real-rooted for all $x \in \mathbb{R}^n$.



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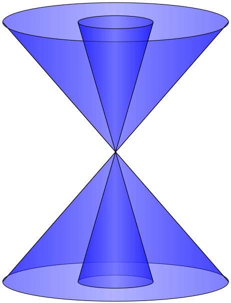

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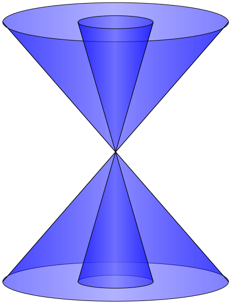


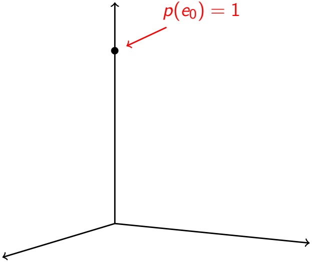
Theorem (Gårding, 1959)

If f *is hyperbolic with respect to* e*, then it is also hyperbolic with respect to every* a *in the connected component of* $\mathbb{R}^n \setminus V_{\mathbb{R}}(f)$ *containing* e*.*

Real Stability is coNP-Hard

Theorem (Saunderson, 2019)

Let $p(x_0, x) = x_0^3 - 3x_0\|x\|^2 + 2q(x)$. *Then* p *is hyperbolic with respect to* e_0 *if and only if* $\max_{\|x\|=1} |q(x)| \leq 1$.



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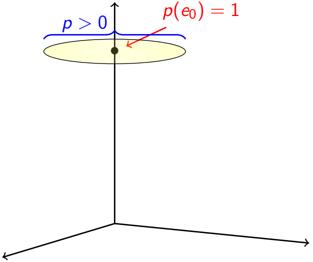

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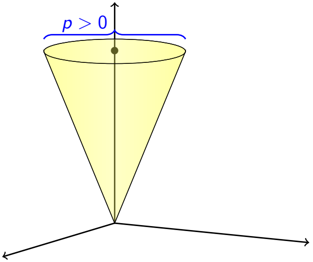

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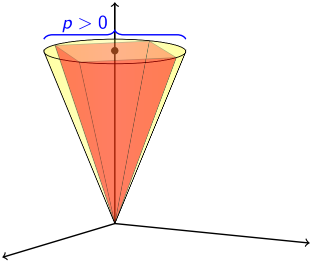

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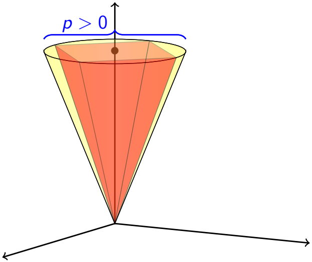

- p is hyperbolic with respect to e_0 if and only if it is hyperbolic with respect to every point in the red cone.
- After a change of variables, the red cone acts like $\mathbb{R}^{2n}_{\geq 0}$, so $\widetilde{p}$ is stable if and only if p is hyperbolic with respect to e_0.

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$$\max_{\|x\|^2+\|y\|^2=1} \sum_{ij\in E} x_i x_j y_{ij} = \sqrt{\frac{2}{27}}\sqrt{1-\frac{1}{\omega(G)}}.$$

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Corollary (C. 2024)

Testing whether a homogeneous polynomial of degree $d \geq 3$ is real stable is coNP-hard.

Log-Concave Polynomials

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It is coNP-hard to decide if a homogeneous polynomial of degree 4 is log concave.

Convexity is Hard

Theorem (Ahmadi et al., 2011)

Let b be a biquadratic form in $2n$ variables, and let

$$f = b(x;y) + \frac{n^2\gamma}{2}\left(\sum x_i^4 + \sum y_i^4 + \sum x_i^2 x_j^2 + \sum y_i^2 y_j^2\right).$$

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Theorem (Motzkin and Straus, 1965)

Let G be a graph and let $\omega(G)$ denote its clique number. Then

$$b_G(x,y) = -2k \sum_{ij \in E} x_i x_j y_i y_j - (1-k)\|x\|^2 \|y\|^2$$

is PSD if and only if $\omega(G) \leq k$.

Convexity to Log-Concavity

Theorem (Ahmadi et al., 2011; Motzkin and Straus, 1965)

Let G be a graph on n vertices, and let

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Theorem (C. 2024)

Suppose $f \in \mathbb{R}[x_1, \ldots, x_n]$ is a homogeneous quartic, and let $N > 0$ be at least as large as the largest coefficient of f. Define

$$g(x_1, \ldots, x_n, z) = N(x_1 + \cdots + x_n + z)^4 - f(x).$$

Then g is log concave if and only if f is convex on $\mathbb{R}^n_{\geq 0}$.

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$g = N(x_1 + \cdots + x_n + z)^4 - f(x)$ *is log concave if and only if* f *is convex on* $\mathbb{R}^n_{\geq 0}$.

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Note $\nabla^2 g = 12N(x_1 + \cdots + x_n + z)^2 \mathbb{1}\mathbb{1}^T - \nabla^2 f$

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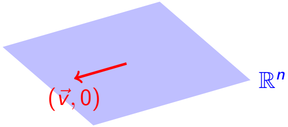

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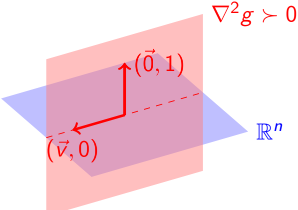

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Corollary

It is coNP-hard to test if a homogeneous polynomial of degree $d \geq 4$ is log concave.

Plot Twist!

Lorentzian Polynomials

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A homogeneous polynomial $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ is *Lorentzian* if for all $\alpha \in \mathbb{Z}^n_{\geq 0}$, $\partial^\alpha f$ is log concave.

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Lemma (Anari et al., 2024; Brändén and Huh, 2020)

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1. *For all $|\alpha| \leq d-2$, the graph with edges $\{ij : \partial_{ij}^2 f \neq 0\}$ is connected*
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- $O(n^{d-2})$ derivatives to check
- For $|\alpha| \leq d-2$, check connectivity of a graph $\rightarrow O(n^2 \cdot n^{d-2})$
- For $|\alpha| = d-2$, diagonalize the quadratic form $\rightarrow O(n^3 \cdot n^{d-2})$

Future Directions

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Real stability and log concavity are in the complexity class $\forall\mathbb{R}$ (universal theory of the reals). Are they $\forall\mathbb{R}$-complete?

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    - Operations that look like derivatives or other “nice” linear operators
- My undergrad CS professor was right

Thank you!

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If $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ *is homogeneous of degree* 3 *such that the graph* $\{ij : \partial_i \partial_j f \neq 0\}$ *is connected, then* f *is log concave if and only if* $\partial_1 f, \ldots, \partial_n f$ *are log concave.*

Degree Minimality

Theorem (C. 2024)

- *Real stability is coNP-hard for degree* $d \geq 3$.
- *Log concavity is coNP-hard for degree* $d \geq 4$.

In degree $d = 2$, real stable = strongly log-concave = log-concave.

Theorem (C. 2024)

If $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ *is homogeneous of degree* 3 *such that the graph* $\{ij : \partial_i \partial_j f \neq 0\}$ *is connected, then* f *is log concave if and only if* $\partial_1 f, \ldots, \partial_n f$ *are log concave.*

- $\nabla^2 f(x) = \sum_{i=1}^n x_i \nabla^2 \partial_i f(x)$
- $\partial_i f$ is quadratic, so $\nabla^2 \partial_i f$ is constant
- f log-concave $\Rightarrow \sum x_i \nabla^2 \partial_i f$ has one positive eigenvalue for all $x \in \mathbb{R}^n_{>0} \Rightarrow \nabla^2 \partial_i f$ has one positive eigenvalue