A new proof of an inverse Kostka matrix problem

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A combinatorial problem from an algebraic identity

Kostka matrix: $K = (K_{\lambda,\mu})_{\lambda,\mu \vdash n}$, where
$K_{\lambda,\mu} = \#\{SSYT \text{ shape } \lambda, \text{ content } \mu\}$.

Theorem (Eğecioğlu–Remmel '90)

$$(K^{-1})_{\mu,\lambda} = \sum_T \operatorname{sgn}(T),$$

where the sum is over the set of special rim hook tableaux of content μ and shape λ.

ER's proof is that their combinatorial interpretation satisfies the same recurrence as K^{-1}.

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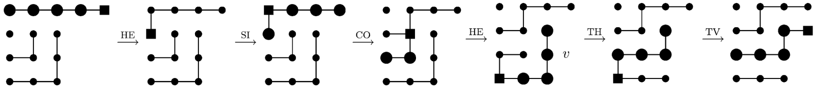
ER gave a combinatorial proof of $KK^{-1} = I$.

Problem (Eğecioğlu–Remmel '90)

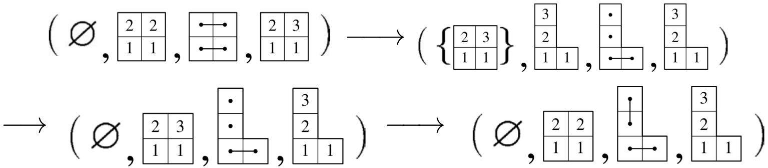
Show combinatorially that $K^{-1}K = I$.

Previous Work on $K^{-1}K = I$

Sagan–Lee (2006): Overlapping rooted special rim-hook tableaux (only for standard Young tableaux)



Loehr–Mendes (2006): Bijective matrix algebra method of converting ER proof of $KK^{-1} = I$



Our approach

Symmetric functions:
$\mathbf{Sym} = \bigoplus_{n\geq 0} \mathbf{Sym}_n = \mathbb{C}[h_1, h_2, \dots]$
$\mathbf{Sym}_n$ bases (partitions): homogeneous h_λ, Schur s_λ
$\mathbf{Sym}$ identity: $h_\mu = \sum_{\lambda \vdash n} K_{\lambda,\mu} s_\lambda$

$\mathbf{Sym}$ problem: Prove $K^{-1}K = I$ combinatorially.

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Noncommutative symmetric functions:

$\mathbf{NSym} = \bigoplus_{n\geq 0} \mathbf{NSym}_n = \mathbb{C}\langle H_1, H_2, \ldots\rangle$

$\mathbf{NSym}_n$ bases (compositions): homogeneous H_α, Immaculate[1] $\mathfrak{S}_\alpha$

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NSym problem: Prove $\tilde{K}^{-1}\tilde{K} = I$ combinatorially.

[1] Berg–Bergeron–Saliola–Serrano–Zabroki (FPSAC 2012 Nagoya, Japan)

Our approach

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Approach: Solve the **NSym** problem. Use it for the **Sym** problem.

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Jacobi–Trudi identities

Recall the Jacobi–Trudi identity : $s_\lambda = \det(h_{\lambda_i - i + j})$

$(h_0 = 1,\ h_{-k} = 0)$

$$s_{211} = \begin{vmatrix} h_2 & h_3 & h_4 \\ 1 & h_1 & h_2 \\ 0 & 1 & h_1 \end{vmatrix} = h_4 - h_{31} - h_{22} + h_{211}$$

(BBSSZ 2014) Immaculate Function[2] $\mathfrak{S}_\alpha := \mathfrak{det}(H_{\alpha_i - i + j})$

$$\mathfrak{S}_{121} = \begin{vmatrix} H_1 & H_2 & H_3 \\ H_1 & H_2 & H_3 \\ 0 & 1 & H_1 \end{vmatrix} = H_{121} - H_{13} - H_{211} + H_{31}$$

Note: If $\chi(H_i) := h_i$, then $\chi(\mathfrak{S}_\alpha) = s_\alpha$ $(H_0 = 1,\ H_{-k} = 0)$

[2] Their definition uses noncommuative Berenstein creation operators.

Immaculate Tableaux (BBSSZ 2014)

Immaculate tableau (IT) of shape α: a filling of the diagram of α by positive integers that has weakly increasing rows and strictly increasing **1st column**.

1	2	10		
2	8	9	9	21
6	7			
21				

Note: A semistandard Young tableau is an immaculate tableau.

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Note: A semistandard Young tableau is an immaculate tableau.

Noncommutative Kostka number $\tilde{K}_{\alpha,\beta}$: the number of immaculate tableaux of shape α and content β.

Noncommutative Kostka matrix $\tilde{K} = (\tilde{K}_{\alpha,\beta})$.

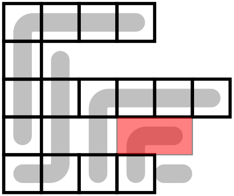
Theorem (BBSSZ 2014)

$$H_\beta = \sum_{\beta \vDash n} \tilde{K}_{\alpha,\beta} \mathfrak{S}_\alpha$$

NSym Kostka Matrix

$$\begin{array}{c c} & \begin{array}{cccc} 3 & 21 & 12 & 111 \end{array} \\ \begin{array}{c} 3 \\ 21 \\ 12 \\ 111 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ & \tilde{K} \end{array} \qquad \begin{array}{c c} & \begin{array}{cccc} 3 & 21 & 12 & 111 \end{array} \\ \begin{array}{c} 3 \\ 21 \\ 12 \\ 111 \end{array} & \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ & \tilde{K}^{-1} \end{array}$$

Tunnel Hook Coverings (Allen–Mason 2025)



Tunnel hook covering: a covering of a composition diagram by lattice paths (tunnel hooks) going down and left such that

1. there is a tunnel hook starting in each row (use next available cell if needed),
2. tunnel hooks may exit the diagram,
3. every time a tunnel hook covers a cell not in the diagram nor its starting row, it makes a negative cell later in that row, and
4. all negative cells are covered by tunnel hooks starting in that row.

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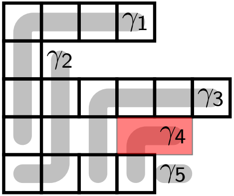

Let γ_i be the tunnel hook starting in row i

- shape: β shape of the diagram.
  $\beta = (4, 1, 6, 1, 4)$
- sign: $\mathrm{sgn}(T) = (-1)^k = (-1)^9 = -1$,
  $k = \#\{\text{rows crossed}\} = 9$
- content $\alpha = (\alpha_1, \alpha_2, \dots)$:
  $\Delta_i := \#\{\text{cells covered by } \gamma_i\} - \#\{\text{nondiagram cells covered in row } i\}$
  Ignore $\Delta_i = 0$ to make α
  $\alpha = (7, 4, 6, -1)$

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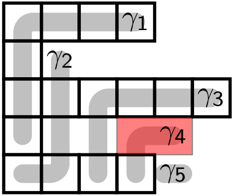

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$\text{THC}_{\alpha,\beta} = \{\text{tunnel hook coverings of content } \alpha \text{ and shape } \beta\}$

Theorem (Allen–Mason 2025)

$$\tilde{K}^{-1}_{\alpha,\beta} = \sum_{T \in \text{THC}_{\alpha,\beta}} \text{sgn}(T).$$

Proving $\tilde{K}^{-1}\tilde{K} = I$ combinatorially (Allen–C–Mason '25+)

$$\begin{array}{c} \\ \alpha=21 \\ \\ \\ \end{array}\begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{\tilde{K}^{-1}} \cdot \overset{\beta=12}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\tilde{K}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \boxed{0} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$0 = \sum_{\delta} \tilde{K}^{-1}_{\alpha,\delta}\tilde{K}_{\delta,\beta} = \sum_{(T,S)} \operatorname{sgn}(T),$$

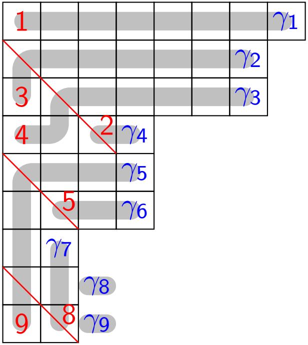
where the sum is over the set of (T, S) such that

- T is a tunnel hook covering of content α,
- S is an immaculate tableau of content β, and
- T and S have the same shape.

Problem

Construct a sign-reversing involution on this set of pairs (T, S).

Permutations and Tunnel Hook Coverings (AM 2023)



The j-th diagonal of a composition diagram are the cells $(1, j), (2, j + 1), \ldots$

The permutation $\pi = \pi(T)$ of a tunnel hook covering T is defined by $\pi_i = j$ if γ_i ends on diagonal j.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 3 & 4 & 2 & 9 & 5 & 8 & 6 & 7 \end{pmatrix}$$

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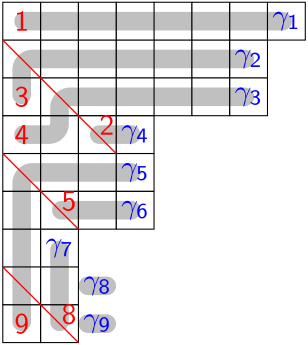

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If $\beta = (\beta_1, \ldots, \beta_\ell)$, $T \mapsto \pi(T)$ is a bijection $\bigsqcup_\alpha \mathrm{THC}_{\alpha,\beta} \to \mathfrak{S}_\ell$ s.t.

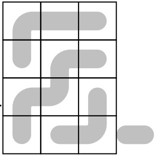
$$\mathrm{sgn}(\pi(T)) = \mathrm{sgn}(T)$$

Idea: sign-reversing involution $\leftrightarrow$ multiplying by transposition.

Let (T, S) be a pair of tunnel hook covering T and immaculate tableau S of the same shape. Let $\pi = \pi(T)$.

$T =$

$\pi = 2431$

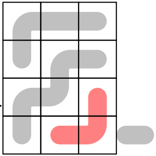


$S =$

1	1	2
2	5	5
3	4	5
4	4	4

Proving $\tilde{K}^{-1}\tilde{K} = I$ combinatorially (Allen–C–Mason '25+)

$T =$



$\pi = 2431$

$S =$

1	1	2
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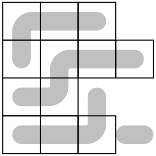
$m = 5$

$r = 3$

1. r: row with $\max(S) = m$ s.t. if m is in row i, then $\pi(i) \geq \pi(r)$
2. If $\pi(r) = r$ and m only appears in row r:
    1. Remove final row of T, S, induct, and reattach
3. Otherwise,
    1. S': move m to row $\pi^{-1}(\pi(r)+1) = 2$.
    2. T': $\pi(T') = (\pi(r), \pi(r)+1)\pi = 2341$

$T' =$



$S' =$

1	1	2	
2	5	5	5
3	4		
4	4	4	

Proving $\tilde{K}^{-1}\tilde{K} = I$ combinatorially (Allen–C–Mason '25+)

Since $\pi(T') = (\pi(r), \pi(r)+1)\pi$, we have $\mathrm{sgn}(T') = -\mathrm{sgn}(T)$.

Theorem (Allen–C–Mason '25+)

The map $\psi : \bigsqcup_\delta \mathrm{THC}_{\alpha,\delta} \times \mathrm{IT}_{\delta,\beta} \to \bigsqcup_\delta \mathrm{THC}_{\alpha,\delta} \times \mathrm{IT}_{\delta,\beta}$ *defined by* $\psi(T, S) = (T', S')$ *is a sign-reversing involution for any* $\alpha \neq \beta$.

When $\alpha = \beta$, $\bigsqcup_\delta \mathrm{THC}_{\alpha,\delta} \times \mathrm{IT}_{\delta,\alpha}$ *has exactly one element and is of of positive sign.*

Proving $\tilde{K}^{-1}\tilde{K} = I$ combinatorially (Allen–C–Mason '25+)

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We will use our involution for $\tilde{K}^{-1}\tilde{K} = I$ to construct an involution for $K^{-1}K = I$.

In the full paper, we have a combinatorial proof of $\tilde{K}\tilde{K}^{-1} = I$ that likewise can be used to construct a combinatorial proof of $KK^{-1} = I$.

Reduction to **Sym**–Two important notes

We will use our involution for $\tilde{K}^{-1}\tilde{K} = I$ to construct an involution for $K^{-1}K = I$.

Note 1

Every semistandard Young tableau is an immaculate tableau.

Let $dec(\alpha)$ denote the weakly decreasing rearrangement of α

Note 2

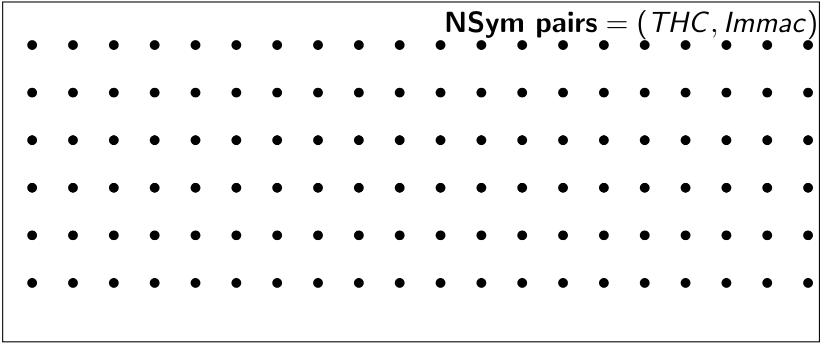
Since $\chi(H_\alpha) = h_{dec(\alpha)}$, we have

$$K^{-1}_{\lambda,\mu} = \sum_{\substack{\alpha \models n \\ dec(\alpha)=\lambda}} \sum_{T \in \mathrm{THC}_{\alpha,\mu}} \mathrm{sgn}(T).$$

Thus, tunnel hook covering provide a combinatorial interpretation of the (**Sym**) inverse Kostka matrix K^{-1}.

Combinatorial proof that $K^{-1}K = I$ (sketch) (A**C**M '25+)

For fix λ, μ, consider the collection of all pairs (T, S) where T is THC of content λ, S is immaculate tableau content μ, and S and T have the same shape (**NSym pairs**).



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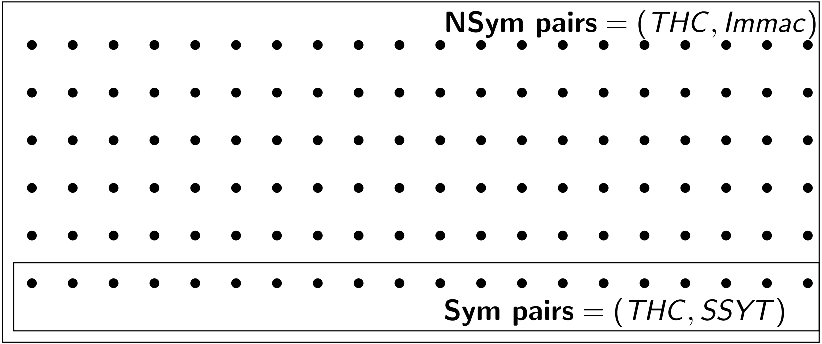
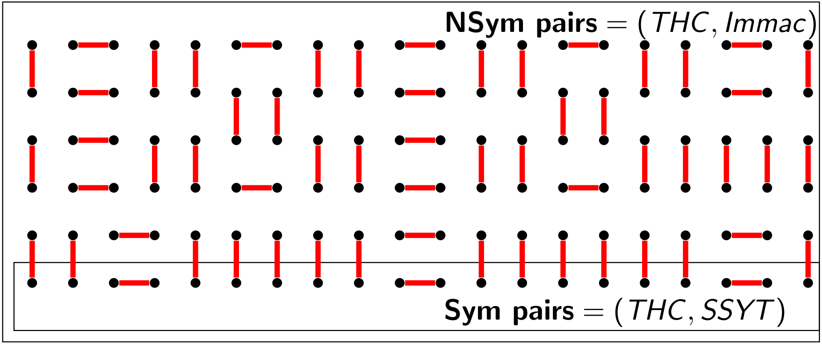
The set of $(THC, SSYT)$ pairs (**Sym pairs**) is contained in the set of **NSym pairs**.



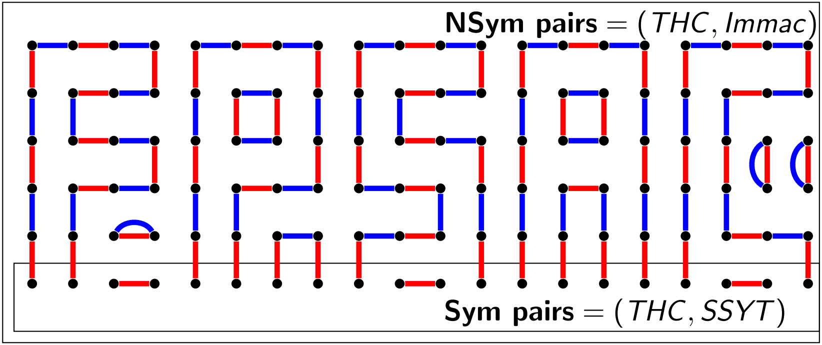
Our involution on **NSym pairs** is in red.

Note that this does not always take **Sym pairs** to **Sym pairs**.



Combinatorial proof that $K^{-1}K = I$ (sketch) (A**C**M '25+)

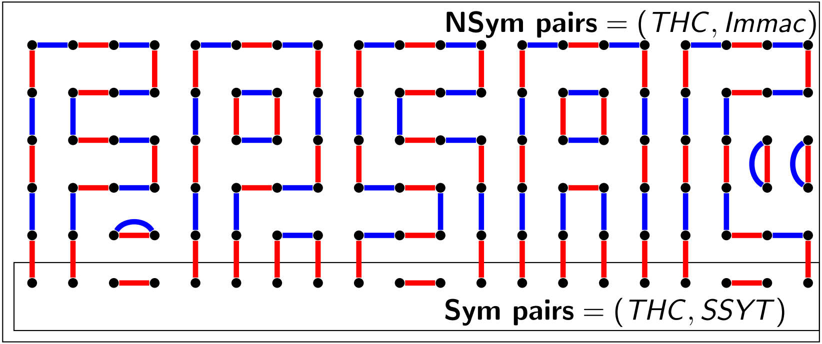
We introduce a new sign-reversing involution in blue on the set of **NSym pairs** that are not **Sym pairs** distinct from our involution



Combinatorial proof that $K^{-1}K = I$ (sketch) (ACM '25+)

Thus, the set of **NSym pairs** forms a (finite) graph.

- **Sym pairs** have degree 1 and the rest have degree 2.
- Thus, any component with **Sym** pair is a path starting and ending in **Sym**. This defines an involution on the **Sym pairs**.
- Since only the red involution applies to **Sym pairs**, these path have odd length, so it is sign-reversing. □



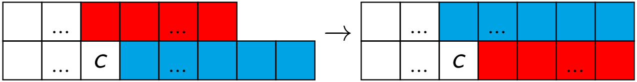
Blue involution

Let $E_{\lambda,\mu} =$ **NSym pairs** $\setminus$ **Sym pairs**
For $(T, S) \in E_{\lambda,\mu}$, say cell c of S is **bad** if either

- The cell above c is empty and c is not in the first row
- The cell above c contains a weakly larger element than in c

Since (T, S) has S immaculate, but not SSYT, it has a bad cell.

Let j be the leftmost column of S containing a bad cell and let i be the largest value such that row i contains a bad cell in column j. Swap the following cells to create S' where $c = (i, j)$.



Define T' to be THC with permutation
$\pi(T') = \pi(T)(\pi(i) - 1, \pi(i))$

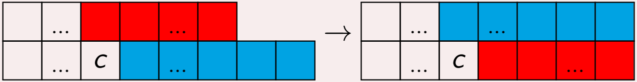
Blue involution - Inspiration

Abstract version of blue involution

A cell c of S is **bad** if either

- The cell above c is empty and c is not in the first row
- The cell above c contains a "larger" element than in c

Let j be the leftmost column of S containing a bad cell and let i be the largest value such that row i contains a bad cell in column j. Swap the following cells to create S' where $c = (i, j)$.

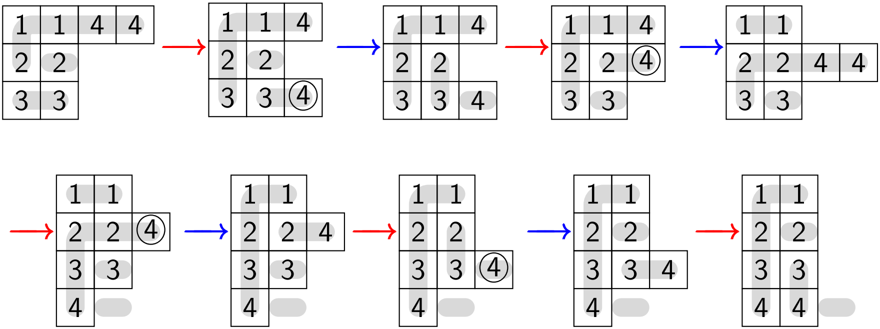


Spiritually the same involution that appears in work of Gessel–Viennot on combinatorial determinants (1989), Gasharov on chromatic symmetric functions (1996), and Shareshian–Wachs on chromatic quasisymmetric functions (2016), among other places.

Sign-reversing involution for $K^{-1}K = I$

Theorem (Allen–C.–Mason (2025+))

The described map on the **Sym pairs** *is a sign–reversing involution.*



Future Directions

1. Determine a "1 step" proof of $K^{-1}K = I$
2. Lift Schur function results to immaculate functions
   - Kostka polynomials
   - Littlewood–Richardson rule
3. Lift other **Sym** problems to **NSym** (where maybe the problem is easier)

Thanks!