A combinatorial formula for rank 2 scattering diagrams

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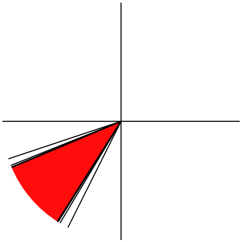
Scattering diagrams were originally developed in the context of mirror symmetry. They have since been found to be deeply related to

- positivity in cluster algebras
- dual canonical bases
- Gromov–Witten invariants
- representation theory (quiver moduli)



Their construction is complex and recursive, so computing them was notoriously difficult.

It turns out that scattering diagrams associated to cluster algebras have beautiful, and rather elementary, combinatorics. In particular, we show they can be computed using **tight gradings** on maximal Dyck paths.

Motivation: Cluster Algebras and Positivity

A **cluster algebra** has a distinguished set of generators, called the **cluster variables** $\{X_i\}_{i \in I}$. In a rank-r cluster algebra, all cluster variables can be obtained from r initial cluster variables via a combinatorial process called **mutation**.

These satisfy two nice properties:

The Laurent Phenomenon (Fomin-Zelevinsky '02)

Every cluster variable can be expressed as a Laurent polynomial with integer coefficients in the initial cluster variables.

Laurent Positivity (LS '15, GHKK '18; conj. by FZ '02)

This Laurent polynomial has *positive* coefficients.

- Proven for skew-symmetric cluster algebras by Lee–Schiffler
- Proven in full by Gross–Hacking–Keel–Kontsevich

Cluster Algebras and Scattering Diagrams

Gross–Hacking–Keel–Kontsevich constructed an associated **cluster scattering diagram** to every cluster algebra. This led to a manifestly positive expansion formula for cluster variables & the *theta basis*.

Most of the complexity of positivity is captured in the rank-2 case.

Definition

The rank-2 cluster algebra $\mathcal{A}(b, c)$ has cluster variables $\{X_k\}_{k\in\mathbb{Z}}$ obtained from initial variables X_1, X_2 via the mutation relation

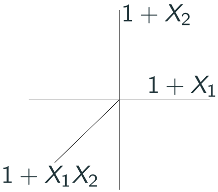
$$X_{k+1}X_{k-1} = \begin{cases} 1 + X_k^b & \text{for } k \text{ odd}, \\ 1 + X_k^c & \text{for } k \text{ even}. \end{cases}$$

Warning: Even in rank 2, computing the scattering diagram was difficult, as the recursive construction involves repeatedly solving systems of equations.

Scattering Diagrams in Rank 2

Definition

- A **wall** is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$, where $\mathfrak{d}$ is a line or ray of slope $\frac{p}{q}$ with a **wall-function** $f_{\mathfrak{d}} \in \mathbb{Z}[\![X_1^q X_2^p]\!]$ with constant term 1.
- A **scattering diagram** is a collection of walls.



If a path crosses $(\mathfrak{d}, f_{\mathfrak{d}})$, we get the **wall-crossing automorphism** $\rho_{\mathfrak{d}}(X_1^{m_1} X_2^{m_2}) = X_1^{m_1} X_2^{m_2} f_{\mathfrak{d}}^{(m_1, m_2)\cdot n}$, where n is a primitive normal to $\mathfrak{d}$.

Definition

A scattering diagram is **consistent** when the composition of wall-crossing automorphisms along any closed loop is the identity.

Generalized Cluster Scattering Diagrams

Fix polynomials P_1, $P_2 \in \mathbb{Z}_{\geq 0}[x]$ with constant term 1.

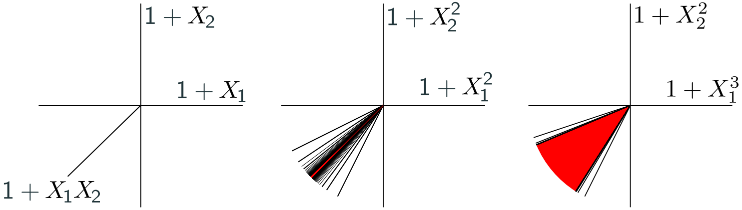
Definition (GHKK, Kontsevich-Soibelman)

The **generalized cluster scattering diagram** $\mathfrak{D}(P_1, P_2)$ is the minimal consistent completion of two walls

$$(x\text{-axis}, P_1(X_1)) \text{ and } (y\text{-axis}, P_2(X_2))$$

obtained by adding only rays.

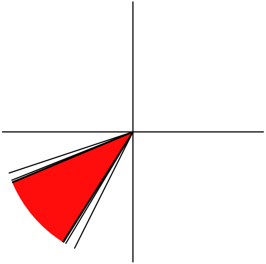
The usual cluster setting is when $P_1(x) = 1 + x^b$ and $P_2(x) = 1 + x^c$.



Finite type   Affine type   Wild type

The Badlands

In almost all cases, there is a full-dimensional cone called the **Badlands** that contains walls of every rational slope.



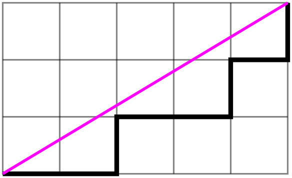
- It is the complement of the g-fan.
- All coefficients of the wall-functions are nonzero (Gräfnitz–Luo '23).
- There has been some work on the wall-functions of the limiting rays (Reading '20), the central ray (Reineke '12), and rays of slope $\frac{p}{q}$ where $\min(p, q) \leq 2$ (Akagi '23).

Other than this, very little was known about the wall-functions in the Badlands. We can now study these using Dyck path combinatorics.

Maximal Dyck Paths

Definition

The **maximal Dyck path** $\mathcal{P} = \mathcal{P}(m, n)$ is the sequence of unit north and east steps from $(0, 0)$ to (m, n) that is closest to the diagonal without crossing above it.



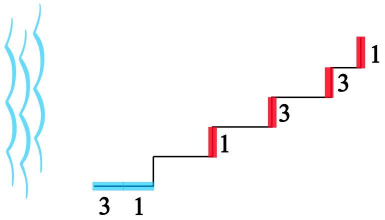
$\mathcal{P}(5, 3)$

A function from the set of edges on $\mathcal{P}$ to $\mathbb{Z}_{\geq 0}$ is called a **grading**.

Compatible Gradings

Let ω be a grading on a Dyck path $\mathcal{P}$. We associate a tiling as follows:

- Construct a tile of height $\omega(e)$ at each horizontal edge e, starting far from the ocean.
- If a new tile obstructs the ocean view of an old tile, build a copy of the old tile above.



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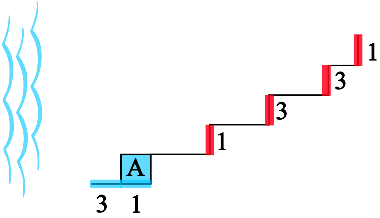

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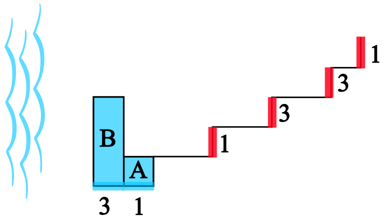

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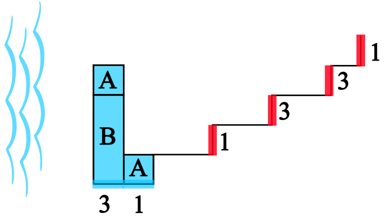

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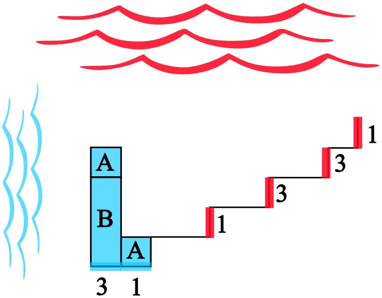

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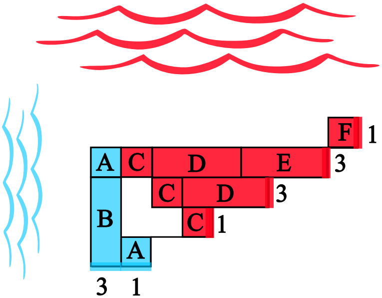

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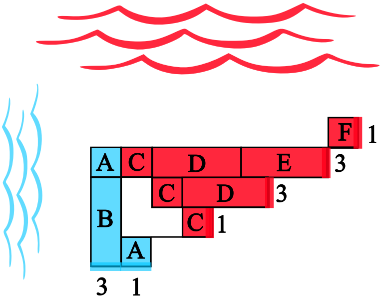


Definition (Lee–Schiffler, Lee–Li–Zelevinsky)

A grading ω is **compatible** if and only if the tiles are non-overlapping.

*May need to work on a cyclic shift of the Dyck path

Tight Gradings

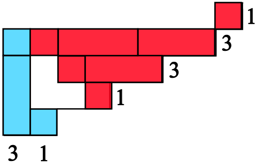
Let $\omega_{\mathbf{V}}$ be the total value of ω on vertical edges, similarly for $\omega_{\mathbf{H}}$.

Definition (B.–Lee–Mou)

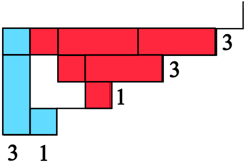
A compatible grading $\omega : \mathcal{P}(m,n) \to \mathbb{Z}_{\geq 0}$ is a **tight grading** if

$$n \cdot \omega_{\mathbf{V}} - m \cdot \omega_{\mathbf{H}} = \pm \gcd(\omega_{\mathbf{V}}, \omega_{\mathbf{H}}), \text{ and}$$

(i) all blue tiles have a red tile directly above, or
(ii) all red tiles have a blue tile to the left.



Not a tight grading



A tight grading with
$5 \cdot 7 - 9 \cdot 4 = -1$

The Tight Grading Formula

Definition (tight grading weight)

Weight each edge e by the coefficient of $x^{\omega(e)}$ in $\begin{cases} P_1 & \text{if vertical,} \\ P_2 & \text{if horizontal.} \end{cases}$

The **weight** of ω is the product of edge weights.

We can now give a combinatorial interpretation for all wall-function coefficients in a generalized cluster scattering diagram!

Let $\lambda_k(p,q)$ be the k-th wall-function coefficient of the wall of slope $\frac{q}{p}$.
Choose m, n such that $m \geq kp$, $n \geq kq$, and $np - mq = \pm 1$.

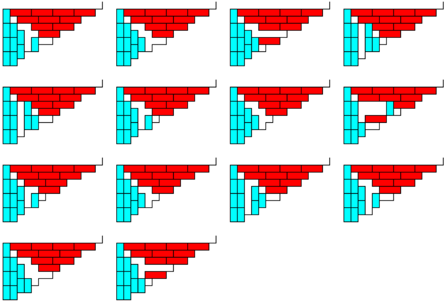
Theorem (B–Lee–Mou '24)

In $\mathfrak{D}(P_1, P_2)$, the wall-function coefficient $\lambda_k(p,q)$ is the weighted sum of tight gradings $\omega : \mathcal{P}(m,n) \to \mathbb{Z}_{\geq 0}$ with $\omega_{\mathbf{V}} = kp$ and $\omega_{\mathbf{H}} = kq$.

Wall-Function Tight Grading Example

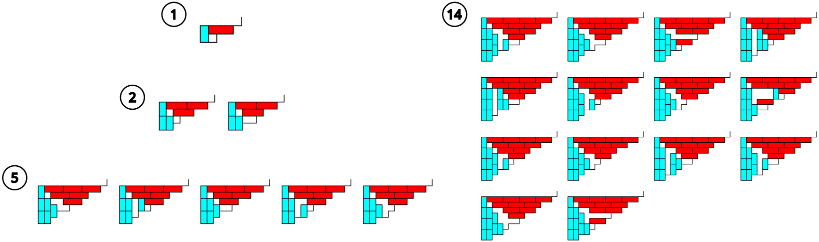
In $\mathcal{D}(1+x^3, 1+x^2)$, let's calculate $\lambda_4(3,2)$.

On the Dyck path $\mathcal{P}(14,9)$, we want to count the tight gradings ω with nonzero weight such that $\omega_{\mathbf{V}} = 12$ and $\omega_{\mathbf{H}} = 8$.



There are 14 tight gradings, hence $\lambda_4(3,2) = 14$.

In $\mathcal{D}(1+x^3, 1+x^2)$, the tight gradings corresponding to the central wall are enumerated by the Catalan numbers. **There is currently no known bijection to other Catalan objects!**



More generally, the tight gradings corresponding to the central wall in a cluster scattering diagram are enumerated by Raney (*aka* two-parameter Fuss-Catalan) numbers.

Applications of the Tight Grading Formula

As a direct application of our tight grading formula, we obtain explicit expressions for:

1. Euler characteristics of moduli spaces of framed stable representations on complete bipartite quivers
2. relative Gromov–Witten invariants on weighted projective planes

Our formula is also manifestly positive, yielding the following:

Corollary (B.–Lee–Mou)

The wall-function coefficients in any rank 2 generalized cluster scattering diagram are non-negative.

Laurent Positivity of Generalized Cluster Algebras

In classical cluster algebras, mutation relations are *binomial*.

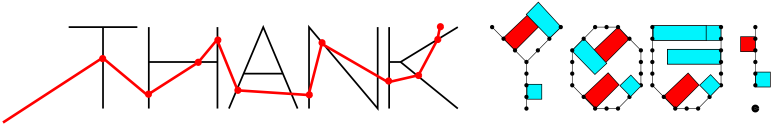
In **generalized cluster algebras**, mutation relations are *polynomial*.

- These were introduced by Chekhov and Shapiro in 2014, motivated by Teichmüller theory of surfaces with orbifold points.
- The generalized cluster scattering diagram $\mathcal{D}(P_1, P_2)$ can be used to study the rank-2 generalized cluster algebra $\mathcal{A}(P_1, P_2)$.

As a consequence of the scattering diagram positivity, we get

Theorem (B–Lee–Mou '24, conjectured by Chekhov–Shapiro '14)

Laurent positivity holds for generalized cluster algebras of any rank.



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[2] A. Burcroff, K. Lee, L. Mou. *Scattering diagrams, tight gradings, and generalized positivity.* PNAS, (2025).

[3] L. Chekhov, M. Shapiro. *Teichmüller spaces of Riemann surfaces with orbifold points of arbitrary order and cluster variables.* IMRN (2014).

[4] M. Gross, P. Hacking, S. Keel, M. Kontsevich. *Canonical bases for cluster algebras.* JAMS, (2018).

[5] K. Lee, R. Schiffler. *Positivity for cluster algebras.* Ann. of Math., (2015).