Spectrum of Random-to-Random Shuffling in the Hecke Algebra

FPSAC 2025: Sapporo, Japan

Sarah Brauner

Brown University → University of Pennsylvania

Joint work with:

Ilani Axelrod-Freed (MIT)

Judy Chiang (Minnesota)

Patty Commins (Minnesota → ICERM)

Veronica Lang (Michigan)

BIG IDEA: Study **EIGENVALUES** of

$\mathcal{R}_n(q)$ = q-RANDOM-TO-RANDOM,

an interesting but mysterious Markov chain in the Hecke algebra

using **COMBINATORIAL REPRESENTATION THEORY**

PROBABILISTIC MOTIVATION:

EIGENVALUES determine CONVERGENCE BEHAVIOR.

COMBINATORIAL MOTIVATION:

EIGENVALUES in $\mathbb{Z}_{\geq 0}$ AND $\mathbb{Z}_{\geq 0}[q]$

(Dieker-Saliola, 2018)

TODAY: (Axelrod-Freed, B, Chiang, Commins, Lang, 24)

WHY?! There must be underlying algebraic structure...

A SPOILER...

THEOREM (Axelrod-Freed, B, Chiang, Commins, Lang, 2024)

* For any $q \in \mathbb{C}$, all **EIGENVALUES** of

$$\mathcal{R}_n(q) := q\text{-RANDOM-TO-RANDOM}$$

are in $\mathbb{Z}_{\geq}[q]$ ← POLYNOMIALS in q with NON-NEGATIVE INTEGER COEFFICIENTS

ALGEBRAIC STRUCTURE: Eigenvectors of $\mathcal{R}_n(q)$ can be constructed inductively using representation theory

→ draw connections between $\mathcal{R}_n(q)$ and **JUCYS MURPHY ELEMENTS** of $\mathcal{H}_n(q)$ unlocking tools from "Okounkov-Vershik approach" to $\mathcal{H}_n(q)$

OUTLINE:

I. RANDOM-TO-RANDOM SHUFFLING

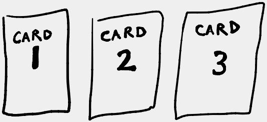
II. q- ANALOGUES

III. EIGENVALUES AND PROOF IDEAS

I. RANDOM-TO-RANDOM SHUFFLING

SCENARIO:

You have a deck of cards:

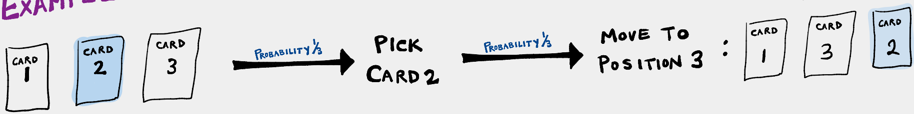


Shuffle via R_n: RANDOM-TO-RANDOM:

(1) pick a card at random

(2) move that card to a random position in the deck.

EXAMPLE:



NOTE: deck of n cards $\longleftrightarrow$ permutation of $\{1, 2, \ldots, n\}$ in $\mathfrak{S}_n$

QUESTION: How many times do you need to do this to have a "well-mixed" deck?

VIEW $\mathcal{R}_n$:= RANDOM-TO-RANDOM as a TWO STEP process:

$\|$

$\mathcal{B}_n^*$ = RANDOM-TO-BOTTOM ∘ BOTTOM-TO-RANDOM = $\mathcal{B}_n$

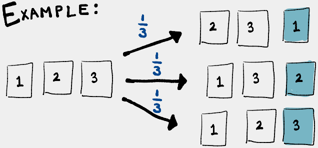
STEP 1   STEP 2

STEP 1:

Apply RANDOM-TO-BOTTOM := $\mathcal{B}_n^*$

* Pick card i with probability $\frac{1}{n}$
  ↳ move card i to the bottom

EXAMPLE:



Better to write:

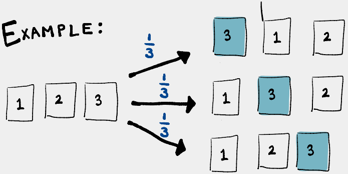
$$\mathcal{B}_3^*: 123 \longmapsto \frac{1}{3}\left(231 + 132 + 123\right)$$
$$+\ 0\left(213 + 312 + 321\right)$$

STEP 2:

Apply BOTTOM-TO-RANDOM := $\mathcal{B}_n$

* Pick position i with probability $\frac{1}{n}$
  ↳ move bottom card to position i

EXAMPLE:



Better to write:

$$\mathcal{B}_3: 123 \longmapsto \frac{1}{3}\left(321 + 132 + 123\right)$$
$$0\left(213 + 231 + 312\right)$$

EXAMPLE: What is $\mathcal{R}_3(123) = 123 \cdot \mathcal{R}_3 = 123 \cdot \mathcal{B}_3^* \cdot \mathcal{B}_3$?

STEP 1: Apply **RANDOM TO BOTTOM** to 123, i.e. $\mathcal{B}_3^*(123)$

$$\mathcal{B}_3^* : 123 \longmapsto \frac{1}{3}(123 + 132 + 231)$$

STEP 2: Apply **BOTTOM TO RANDOM** to $\mathcal{B}_3^*(123)$, i.e. $\mathcal{B}_3(\mathcal{B}_3^*(123))$

$$\begin{aligned}
&= \frac{1}{3^2}(123 + 132 + 312) \longleftarrow \frac{1}{3}\mathcal{B}_3(123)\\
&+ \frac{1}{3^2}(132 + 123 + 213) \longleftarrow \frac{1}{3}\mathcal{B}_3(132)\\
&+ \frac{1}{3^2}(231 + 213 + 123) \longleftarrow \frac{1}{3}\mathcal{B}_3(231)
\end{aligned}$$

$$= \frac{1}{9}(3 \cdot 123 + 2 \cdot 213 + 2 \cdot 132 + 1 \cdot 231 + 1 \cdot 312 + 0 \cdot 321)$$

UPSHOT: Coefficient of u is probability of obtaining u from w

We can rephrase this **ALGEBRAICALLY**:

* $\mathcal{B}_n^*$ and $\mathcal{B}_n$ act on permutations by POSITION = RIGHT MULTIPLICATION
  - ↳ Let ▇ = CYCLE NOTATION

STEP 1 RANDOM-TO-BOTTOM = $\mathcal{B}_n^* : \mathbb{C}[\mathfrak{S}_n] \longrightarrow \mathbb{C}[\mathfrak{S}_n]$

$$\mathcal{B}_n^* : \omega \longmapsto \omega \cdot \frac{1}{n}\Big(\underbrace{1}_{\text{moves bottom card to bottom}} + \underbrace{(n-1, n)}_{\text{moves card in position } n-1 \text{ to bottom}} + \underbrace{(n-2, n-1, n)}_{\text{moves card in position } n-2 \text{ to bottom}} + \cdots + \underbrace{(1, 2, \ldots, n)}_{\text{moves first card to bottom}} \Big)$$

STEP 2 BOTTOM-TO-RANDOM = $\mathcal{B}_n : \mathbb{C}[\mathfrak{S}_n] \longrightarrow \mathbb{C}[\mathfrak{S}_n]$

$$\mathcal{B}_n : \omega \longmapsto \omega \cdot \frac{1}{n}\Big(\underbrace{1}_{\text{moves bottom card to bottom}} + \underbrace{(n, n-1)}_{\text{moves bottom card to position } n-1} + \underbrace{(n, n-1, n-2)}_{\text{moves bottom card to position } n-2} + \cdots + \underbrace{(n, n-1, \ldots, 2, 1)}_{\text{moves bottom card to position } 1} \Big)$$

UPSHOT RANDOM-TO-RANDOM = $\mathcal{R}_n : \mathbb{C}[\mathfrak{S}_n] \longrightarrow \mathbb{C}[\mathfrak{S}_n]$

is the composition $\mathcal{R}_n := \underbrace{\mathcal{B}_n^*}_{\text{STEP 1}} \circ \underbrace{\mathcal{B}_n}_{\text{STEP 2}}$

We can encode $\mathcal{R}_n$ via a **PROBABILITY MATRIX**

* **ROWS** and **COLUMNS** are indexed by all "states" = permutations
* j^{th} column and i^{th} row: **PROBABILITY** of going from $\boxed{j}$ to $\boxed{i}$

EXAMPLE **PROBABILITY MATRIX: $\mathcal{R}_3$**

$$\mathcal{R}_3(123) = \frac{1}{9}\left(3\cdot 123 + 2\cdot 213 + 2\cdot 132 + 1\cdot 231 + 1\cdot 312 + 0\cdot 321\right)$$

$$\mathcal{R}_3 = \frac{1}{9} \begin{array}{c} \\ 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array} \begin{array}{c} \begin{array}{cccccc} 123 & 132 & 213 & 231 & 312 & 321 \end{array} \\ \left[\begin{array}{cccccc} 3 & 2 & 2 & 1 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \\ 2 & 1 & 3 & 2 & 2 & 1 \\ 1 & 0 & 2 & 3 & 1 & 2 \\ 1 & 2 & 0 & 1 & 3 & 2 \\ 0 & 1 & 1 & 2 & 2 & 3 \end{array}\right] \end{array}$$

* **SIMPLIFICATION:** Scale operators so matrices have entries in $\mathbb{Z}$

i.e. consider $n\cdot\mathcal{B}_n^*$, $n\cdot\mathcal{B}_n$ and $n^2\cdot\mathcal{R}_n$

↳ this scales eigenvalues by n^2

Recall our **INITIAL QUESTION:** how many times do we shuffle?

UPSHOT: The **EIGENVALUES** of our probability matrix tell us about convergence behavior of $\mathcal{R}_n$

GOAL: Determine **EIGENVALUES** of $\mathcal{R}_n$ $n! \times n!$ matrix

this is **HARD!**

* **2002:** Diaconis asks: What are eigenvalues of $\mathcal{R}_n$?
* **2002:** Uyemara-Reyes conjectures eigenvalues are in $\mathbb{Z}_{\geq 0}$
  ↳ proves some special cases
* **2018:** Dieker-Saliola **PROVE** eigenvalues of $\mathcal{R}_n$ in $\mathbb{Z}_{\geq 0}$.
* **2019:** Bernstein-Nestoridi compute cutoff behavior of $\mathcal{R}_n$

WHY STUDY RANDOM-TO-RANDOM?

(1) **CONNECTIONS** to many topics in **ALGEBRAIC COMBINATORICS**

* The Tsetlin library
* Gessel's fundamental quasisymmetric functions (Désarménien - Wachs)
* derangements of permutations (Reiner - Wachs)
* Solomon's Descent algebra (Bidigare-Hanlon-Rockmore, Brown)
* **NEW** in our work: Jucys Murphy elements

(2) **INTEGER EIGENVALUES** !?

(**NEW**: nice q-analog!)

(3) **METHODS:**

Representation theoretic techniques to study mixing times

II. q- ANALOGUES

IDEA: * Introduce a parameter q

↳ enriches and refines classical object.

* we recover classical objects at $q=1$

CLASSICAL OBJECT	q-ANALOGUE
$n \in \mathbb{Z}$	$[n]_q := \dfrac{1-q^n}{1-q} = \begin{cases} 1+q+q^2+\cdots+q^{n-1} & n>0 \\ 0 & n=0 \\ -q^{-1}-q^{-2}-\cdots-q^{n} & n<0 \end{cases}$ e.g. $[5]_q = 1+q+q^2+q^3+q^4$
$\mathbb{C}[\mathfrak{S}_n]$ the group algebra of the symmetric group	$\mathcal{H}_n(q)$ the Type A Iwahori Hecke algebra

Classical Object	q-Analogue
Define: $\mathbb{C}[\mathfrak{S}_n]$ is the algebra generated by $s_1, \dots, s_{n-1}$	**Define:** $\mathcal{H}_n(q)$ is the algebra generated by $T_1, \dots, T_{n-1}$
$s_i = (i, i+1)$	
such that for $1 \le i \le n-1$:	such that for $1 \le i \le n-1$:
(i) $s_i^2 = 1$	(i) $T_i^2 = (q-1)T_i + q$
Equivalently: $(s_i - 1)(s_i + 1) = 0$	Equivalently: $(T_i - q)(T_i + 1) = 0$
(ii) $s_i s_j = s_j s_i$ when $\lvert i-j \rvert \ge 2$	(ii) $T_i T_j = T_j T_i$ when $\lvert i-j \rvert \ge 2$
(iii) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$	(iii) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$
Linear Basis: $\{w \in \mathfrak{S}_n\}$	**Linear Basis:** $\{T_w : w \in \mathfrak{S}_n\}$
$w = \underbrace{s_{i_1} \cdots s_{i_\ell}}_{\text{reduced expression}}$	$T_w := T_{i_1} T_{i_2} \cdots T_{i_\ell}$

GOAL: Define random-to-random shuffling in $\mathcal{H}_n(q)$...

RECALL: After rescaling... (= cycle notation)

STEP 1: RANDOM TO BOTTOM: move random card to bottom of deck

$$\mathcal{B}_n^* : w \longmapsto w \cdot \left(1 + (n-1, n) + (n-2, n-1, n) + \cdots + (1, 2, \ldots, n)\right)$$

STEP 2: BOTTOM TO RANDOM: move bottom card to a random position

$$\mathcal{B}_n : w \longmapsto w \cdot \left(1 + (n, n-1) + (n, n-1, n-2) + \cdots + (n, n-1, \ldots, 2, 1)\right)$$

AND **RANDOM-TO-RANDOM:** move random card to random spot in deck

$$\mathcal{R}_n : w \longmapsto w \cdot \underbrace{\mathcal{B}_n^*}_{\text{STEP 1}} \circ \underbrace{\mathcal{B}_n}_{\text{STEP 2}}$$

$$\mathcal{R}_3 : 123 \longmapsto \left(3 \cdot 123 + 2 \cdot 213 + 2 \cdot 132 + 1 \cdot 231 + 1 \cdot 312 + 0 \cdot 321\right)$$

For ANY $q \in \mathbb{C}$, define

STEP 1 q- RANDOM TO BOTTOM: (= cycle notation)

$$B_n^*(q) : T_w \longmapsto T_w \cdot \left(1 + T_{(n-1,n)} + T_{(n-2,n-1,n)} + \cdots + T_{(1,2,\dots,n)}\right)$$

STEP 2 q- BOTTOM TO RANDOM:

$$B_n(q) : T_w \longmapsto T_w \cdot \left(1 + T_{(n,n-1)} + T_{(n,n-1,n-2)} + \cdots + T_{(n,n-1,\dots,1)}\right)$$

AND: q- RANDOM-TO-RANDOM:

$$R_n(q) : T_w \longmapsto T_w \cdot \underbrace{B_n^*(q)}_{\text{STEP 1}} \circ \underbrace{B_n(q)}_{\text{STEP 2}}$$

EXAMPLE: When $n=3$, writing permutations in one-line notation:

$$T_{123} \cdot R_3(q) = [3]_q \cdot T_{123} + q \cdot [2]_q \cdot T_{213} + [2]_q \cdot T_{132} + q\, T_{231} + q \cdot T_{132} + (q-1)\, T_{321}$$

IMPORTANTLY: $\mathcal{R}_n(q)$ defines a **MARKOV CHAIN** on $\mathcal{H}_n(q)$

where $q^{-1} \in (0,1]$ is a **PROBABILITY** ...

To realize $\mathcal{R}_n(q)$ as a Markov chain

(1) scale generators $\tilde{T}_w := q^{-\ell(w)} T_w$

(2) normalize by dividing by $[n]_q^2$

MOTIVATION: Bufetov showed that many interacting particle systems can be viewed as **RANDOM WALKS ON HECKE ALGEBRAS**

e.g.: ASEP, M-exclusion, TASEP, ASEP(q,j), stochastic vertex models, etc..

his work implies questions of convergence make sense, i.e. reversible stationary distribution exists

GOAL: Find **EIGENVALUES** of $\mathcal{R}_n(q)$

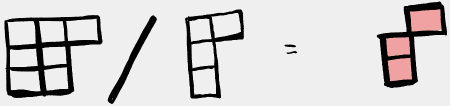
III. EIGENVALUES AND PROOF IDEAS

* For partitions $\mu \subset \lambda$, we say λ / μ is a **SKEW TABLEAU**

* Then λ / μ is a **HORIZONTAL STRIP** if each column of λ / μ has at most **ONE** box

EXAMPLE

(1)

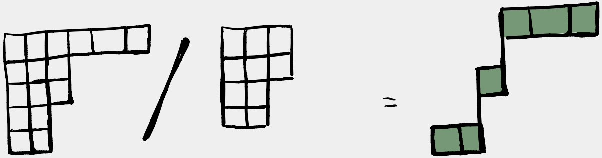


NOT a HORIZONTAL STRIP

$\lambda = (3, 2, 2)$ $\mu = (2, 1, 1)$

$= \lambda / \mu$

(2)



YES a HORIZONTAL STRIP

$\lambda = (6, 3, 3, 2, 2)$ $\mu = (3, 3, 2, 2)$

$= \lambda / \mu$

THEOREM (Axelrod-Freed, B, Chiang, Commins, Lang, 2024)

(i) Every **EIGENVALUE** of $R_n(q)$ is indexed by a pair of integer partitions λ, μ where

* $|\lambda| = n$
* $\mu \subset \lambda$
* λ / μ is a **HORIZONTAL STRIP**

→ **WRITE** $\mathcal{E}_{\lambda/\mu}(q)$

statistic on skew tableaux

(ii) $$\mathcal{E}_{\lambda/\mu}(q) = q^{|\lambda|}\,\mathrm{CONTENT}_q(\lambda/\mu) + \sum_{k=|\mu|+1}^{|\lambda|} q^{|\lambda|-k}[k]_q$$

In particular, $\mathcal{E}_{\lambda/\mu}(q) \in \mathbb{Z}_{\geq 0}[q]$.

EXAMPLES:

$\lambda = (n)$

and $\mu = \phi$

CLAIM: $\mathcal{E}_{\lambda/\mu}(q)$ simplifies to

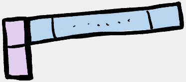
$$\mathcal{E}_{(n)/\phi}(q) = [n]_q \cdot [n]_q$$

When $q \in \mathbb{R}_{>0}$, this is the

LARGEST eigenvalue

$\lambda = (n-1, 1)$

and $\mu = (1,1)$



CLAIM: $\mathcal{E}_{\lambda/\mu}(q)$ simplifies to

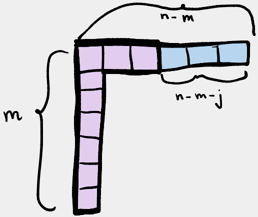
$$\mathcal{E}_{(n-1,1)/(1,1)}(q) = [n-2]_q \cdot [n+1]_q$$

When $q \in \mathbb{R}_{>0}$, this is the

SECOND LARGEST eigenvalue

MORE GENERALLY ...

for $\mu = (j, 1^m)$ and $\lambda = (n-m, 1^m)$

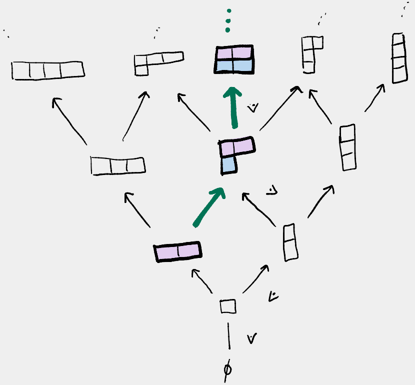


$$\mathcal{E}_{\lambda/\mu}(q) = [n-j-m]_q \cdot [n+j] .$$

Proof Idea:

Find eigenvalues of $\mathcal{R}_n(q)$ by recursively constructing **EIGENVECTORS**

INTUITION: Each eigenvector (and eigenvalue) is built by following a **PATH** in **YOUNG'S LATTICE**



$\mathcal{E}_{\lambda/\mu}(q)$

COMES FROM PATH

$\mu \longrightarrow \lambda$

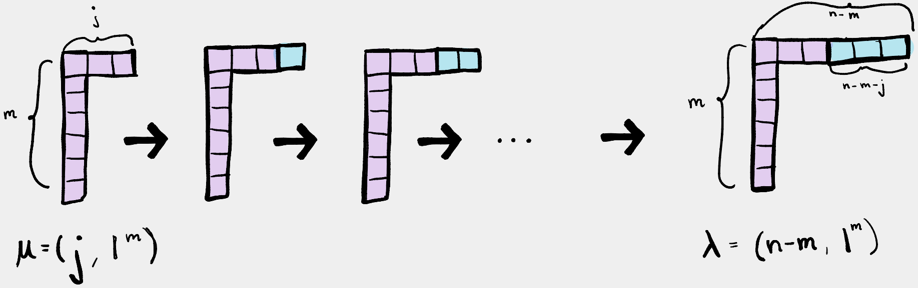
FOR EXPERTS: each partition corresponds to a Specht module

EXAMPLE:

Recall for $\mu = (j, 1^m)$ and $\lambda = (n-m, 1^m)$

$$\mathcal{E}_{\lambda/\mu}(q) = [n-j-m]_q \cdot [n+j]_q.$$

$\mathcal{E}_{\lambda/\mu}(q)$ COMES FROM THE PATH:



NEW METHOD: RELATE $R_n(q)$ to

$$\boxed{\begin{array}{l}\textbf{JUCYS MURPHY ELEMENTS} \\ J_k := \sum_{i \leq k} q^{i-k} T_{(i,k)} \quad \text{of } \mathcal{H}_n(q)\end{array}}$$

allows us to "move up" along Young's lattice

This **UNLOCKS** "Okounkov-Vershik approach" to representation theory and tools by Dipper, James, Young (**NEW** even when $q=1$)...

BENEFIT 1: Conceptually explains eigenvalue formula and simpler proofs than $q=1$

BENEFIT 2: Methods are robust to other contexts

* **K-RANDOM-TO-RANDOM** shuffling in the Hecke algebra (B, Commins, Grinberg, Saliola, 25)
* **DYADIC SHUFFLING** (B, Commins, Grinberg, Saliola, 25⁺)

THANK YOU!

CONTACT ME!

EMAIL: sarahbrauner@gmail.com

WEBSITE: sarahbrauner.com

* The CONTENT of λ/μ is

$$\text{CONTENT}(\lambda/\mu) := \sum_{(i,j)\in\lambda/\mu} (j - i)$$

column row

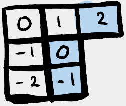
EXAMPLE

(1) $\lambda = (3,2,2)$

$\mu = (2,1,1)$

● $= \lambda/\mu$

column 3
row 1



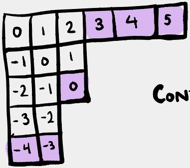
column 2
row 3

$$\text{CONTENT}(\lambda/\mu) = 2 + 0 + -1 = 1$$

(2) $\lambda = (6,3,3,2,2)$

$\mu = (3,3,2,2)$

● $= \lambda/\mu$



$$\text{CONTENT}(\lambda/\mu) = 5 + 3 + 4 + 0 + -3 + -4$$
$$= 5$$

THEOREM (Dieker–Saliola, 2018)

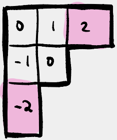
The eigenvalues of $\mathcal{R}_n$ are indexed by horizontal strips λ/μ:

$$\mathcal{E}_{\lambda/\mu} = \text{CONTENT}(\lambda/\mu) + \sum_{k=|\mu|+1}^{|\lambda|} k \in \mathbb{Z}_{\geq 0}$$

mysterious....

BIG DEAL!

EXAMPLE: $\lambda = (3,2,1)$ $\quad \mu = (2,2)$



horizontal strip λ/μ

$$\text{CONTENT}(\lambda/\mu) = 2 - 2 = 0$$

Then $\quad \mathcal{E}_{\lambda/\mu} = 0 + \sum_{k=5}^{6} k = 5 + 6 = 11.$

(To get normalized eigenvalues: divide by n^2).

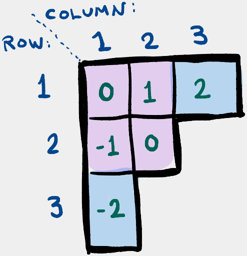
Given λ/μ, define its q-CONTENT:

$$\mathrm{CONTENT}_q(\lambda/\mu) := \sum_{\substack{(i,j) \in \lambda/\mu}} [j-i]_q \in \mathbb{Z}[q^{\pm 1}]$$

ROW i, COLUMN j

NOTE: $q^{|\lambda|}\,\mathrm{CONTENT}_q(\lambda/\mu) \in \mathbb{Z}[q]$.

EXAMPLE: $\lambda = (3,2,1)$ $\mu = (2,2)$:



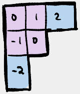
$$q^{|\lambda|}\,\mathrm{CONTENT}_q(\lambda/\mu) = q^6\left([2]_q + [-2]_q\right)$$
$$= q^6 + q^7 - q^5 - q^4.$$

Specializes to 0 when $q = 1$...

RECALL:

$$\mathcal{E}_{\lambda/\mu}(q) = q^{|\lambda|}\,\mathrm{CONTENT}_q(\lambda/\mu) + \sum_{k=|\mu|+1}^{|\lambda|} q^{|\lambda|-k}[k]_q$$

EXAMPLE: $\lambda = (3,2,1)$ $\mu = (2,2)$



:

$$q^{|\lambda|}\,\mathrm{CONTENT}_q(\lambda/\mu) = q^6\left([2]_q + [-2]_q\right)$$
$$= q^6 + q^7 - q^5 - q^4.$$

Then

$$\mathcal{E}_{\lambda\backslash\mu}(q) = q^6\left([2]_q + [-2]_q\right) + q\cdot[5]_q + q^0\cdot[6]_q$$
$$= q^6 + q^7 - \cancel{q^5} - \cancel{q^4} + q + q^2 + q^3 + q^4 + q^5 + 1 + q + q^2 + q^3 + \cancel{q^4} + \cancel{q^5}$$
$$= 1 + q + q^2 + q^3 + q^4 + q^5 + q^6 + q^7 + q + q^2 + q^3$$
$$= [8]_q + q[3]_q$$

DO ALL $\mathcal{R}_n(q)$- EIGENVECTORS ARISE IN THIS WAY?

YES: we construct an **EIGENBASIS** for each S^λ when $q \in \mathbb{R}_{>0}$...

↳ this will consist of eigenvectors coming from paths:

START OF PATH: μ. ⟶ **END OF PATH: λ**

CONSTRUCTED FROM $u \in S^\mu$ IN KERNEL OF $\mathcal{R}_{|\mu|}(q)$

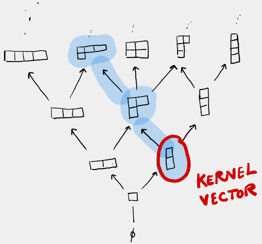
EIGENVECTOR v OF $\mathcal{R}_{|\lambda|}(q)$ IN S^λ

* For a given S^λ, **FIX λ, VARY μ**

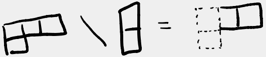
* **WARNING:** This requires understanding kernels of all $\mathcal{R}_j(q)$ for $j \leq n$ ⟶ there can be multiplicity!

EXAMPLE: $\lambda = (3,1) \ldots \dim\left(S^{(3,1)}\right) = 3$

Our EIGENBASIS will come from three paths:

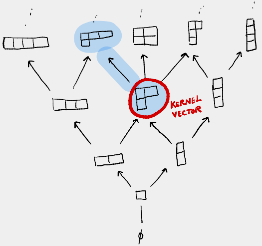


HORIZONTAL STRIP:



EIGENVALUE:

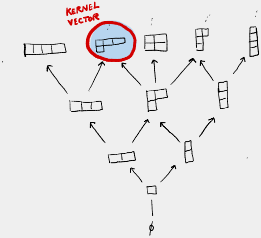
$\mathcal{E}$  $(q) = [2]_q [5]_q$



HORIZONTAL STRIP:

EIGENVALUE:

$\mathcal{E}$  $(q) = q^4 [2]_q + [4]_q$



HORIZONTAL STRIP:

EIGENVALUE:

$\mathcal{E}$  $(q) = 0$