Extended weak order for $\widetilde{S}_n$ and the lattice of torsion classes

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Motivation

Cambrian quotients (Reading, Reading–Speyer) are a machine for turning Coxeter group info into cluster algebra info

Coxeter groups $\Rightarrow$ Cluster algebras

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Weak order	$\Rightarrow$	Cambrian lattice

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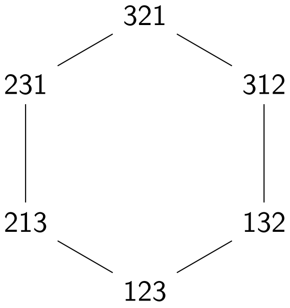
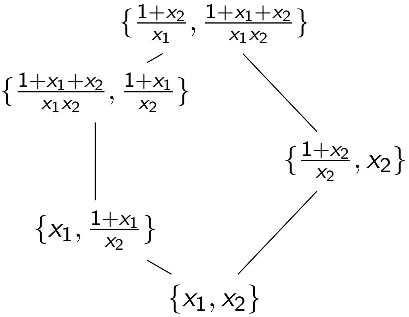
Weak order	$\Rightarrow$	Cambrian lattice
Weak order Hasse diagram	$\Rightarrow$	Ordered exchange graph



$\Rightarrow$



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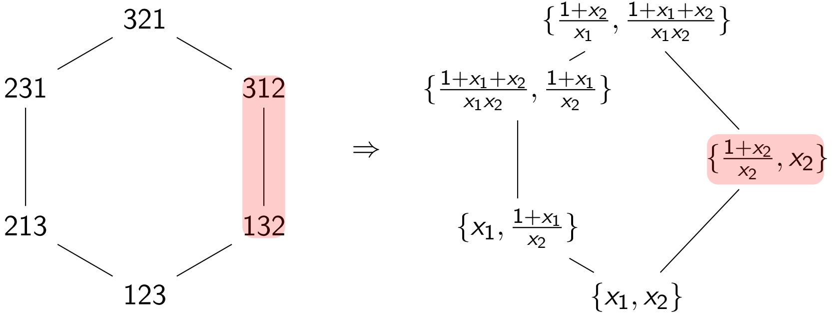

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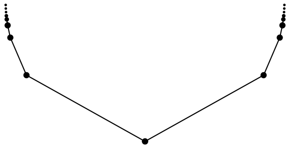
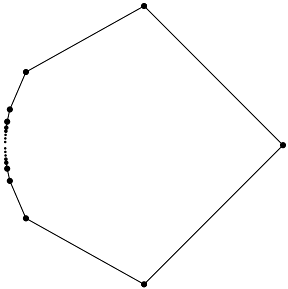
Weak order Hasse diagram $\overset{?}{\Longrightarrow}$ Ordered exchange graph



$\overset{?}{\Longrightarrow}$



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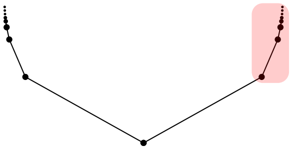

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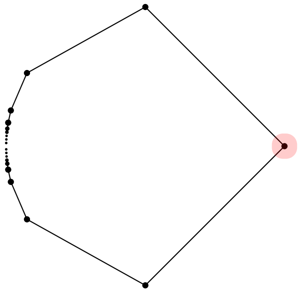

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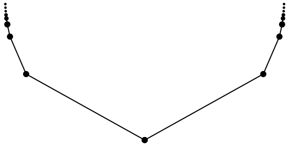
Weak order Hasse diagram $\Rightarrow$ Part of ordered exchange graph



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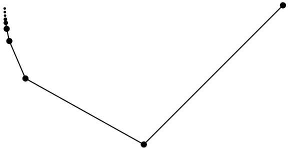

Motivation

Problem (Speyer, OPAC 2022)

Find a combinatorial lattice extending weak order with a Cambrian quotient giving the entire exchange graph.

Motivation

Two possible answers:

- Extended weak order of a Coxeter group W
  - Combinatorial
  - Explicit descriptions in affine type
  - If $|W| < \infty$, then Cambrian lattices describe cluster algebras
  - But not known to relate to cluster algebras if $|W| = \infty$
- Lattice of torsion classes for a preprojective algebra Π
  - Known to have quotients describing cluster algebras
  - But depends on a choice of field, not combinatorial
  - Harder to describe than the ordered exchange graph

Main result: in affine type, extended weak order is a "combinatorial skeleton" of torsion classes

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Corollary: Coxeter-theoretic models for affine cluster algebra exchange graphs

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Main result: in affine type, extended weak order is a "combinatorial skeleton" of torsion classes

Corollary: Coxeter-theoretic models for affine cluster algebra exchange graphs

Corollary: Complete description of torsion classes in type $\widetilde{A}$

Extended weak order

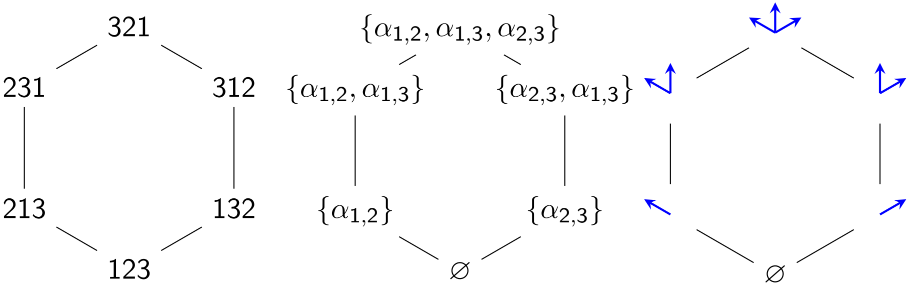
Weak order on S_n

Define $\alpha_{i,j} \coloneqq \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$. The **positive roots** of S_n are

$$\Phi^+ = \{\alpha_{i,j} \mid 1 \leq i < j \leq n\}$$

Definition

An **inversion** of π is a positive root $\alpha_{i,j}$ so that $\pi^{-1}(i) > \pi^{-1}(j)$.
Weak order puts $\pi \leq \pi'$ if $\mathrm{Inv}(\pi) \subseteq \mathrm{Inv}(\pi')$.



Affine symmetric group

Definition

The **affine symmetric group** $\widetilde{S}_n$ is the group of bijections $\widetilde{\pi} : \mathbb{Z} \to \mathbb{Z}$ such that:

- $\widetilde{\pi}(i+n) = \widetilde{\pi}(i) + n$ for all $i \in \mathbb{Z}$
- $\sum_{i=1}^{n}(\widetilde{\pi}(i) - i) = 0$

Let V have a basis $\alpha_0, \ldots, \alpha_n$ indexed by $\mathbb{Z}/n\mathbb{Z}$. Define $\widetilde{\alpha}_{i,j} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$. The **positive roots** of $\widetilde{S}_n$ are

$$\Phi^+ = \{\widetilde{\alpha}_{i,j} \mid i < j\}$$

A positive root $\widetilde{\alpha}_{i,j}$ is **real** if $i \not\equiv j \bmod n$.

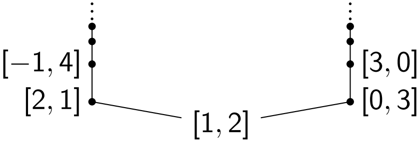
Definition

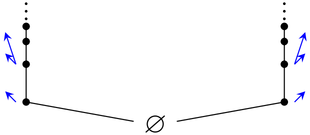
An **inversion** of $\widetilde{\pi}$ is a positive real root $\widetilde{\alpha}_{i,j} \in \Phi^+_{\mathrm{real}}$ so that $\widetilde{\pi}^{-1}(i) > \widetilde{\pi}^{-1}(j)$. **Weak order** puts $\widetilde{\pi} \le \widetilde{\pi}'$ if $\mathrm{Inv}(\widetilde{\pi}) \subseteq \mathrm{Inv}(\widetilde{\pi}')$.

Weak order for $\widetilde{S}_2$

We represent an affine permutation $\widetilde{\pi}$ via its **window notation**

$$[\widetilde{\pi}(1), \ldots, \widetilde{\pi}(n)]$$





Let W be a Coxeter group with positive roots Φ^+

Definition (Dyer)

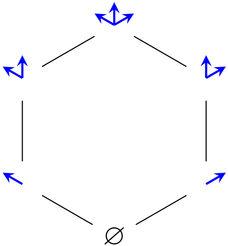
A set $B \subseteq \Phi^+$ is **biclosed** if it satisfies the following two properties for all $\alpha, \beta, \gamma \in \Phi^+$ such that $\gamma = a\alpha + b\beta$ with $a, b \geq 0$:

(Closed) If $\alpha \in B$ and $\beta \in B$, then $\gamma \in B$.

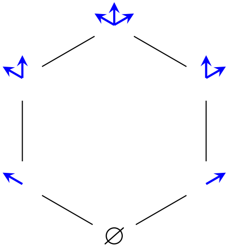
(Coclosed) If $\alpha \notin B$ and $\beta \notin B$, then $\gamma \notin B$.

The **extended weak order** of W is the poset of biclosed sets, ordered by containment.

Extended weak order for S_3



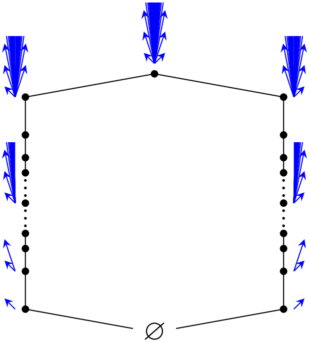
Extended weak order for S_3



Theorem (Dyer)

Finite biclosed sets are exactly the inversion sets of elements of W. Hence the poset of finite biclosed sets is isomorphic to weak order on W.

Extended weak order for $\widetilde{S}_2$



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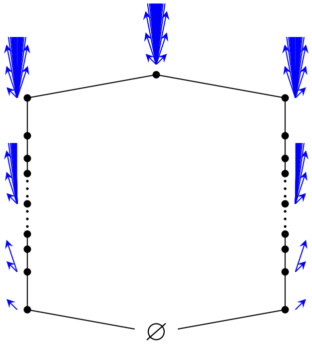
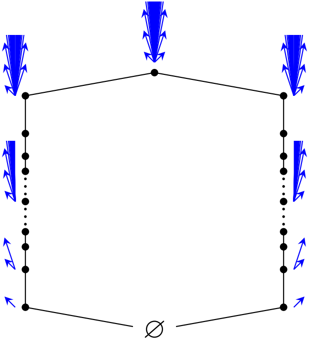


Conjecture (Dyer)

Extended weak order is a lattice for any Coxeter group.

Extended weak order for $\widetilde{S}_2$



Conjecture (Dyer)

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Theorem (B.–Speyer)

Extended weak order is a lattice for affine Coxeter groups.

A combinatorial model for extended weak order

Definition (B.–Speyer)

Fix $n \in \mathbb{N}$. A **translation invariant total order** (**TITO**) is a total order $(\prec)$ on $\mathbb{Z}$ so that

- For all $i, j \in \mathbb{Z}$, we have $i \prec j$ if and only if $i + n \prec j + n$, and
- For all $i \in \mathbb{Z}$, if $i + n \prec i$ then there exists a k with $i + n \prec k \prec i$.

Write TTot_n for the set of TITOs.
An **inversion** of $(\prec)$ is a positive root $\widetilde{\alpha}_{i,j}$ so that $j \prec i$.

Some TITOs in TTot_3:

$$\cdots \prec 9 \prec 8 \prec 7 \prec 6 \prec 5 \prec 4 \prec 3 \prec 2 \prec 1 \prec 0 \prec \cdots$$

$$\cdots \prec -2 \prec 1 \prec 4 \prec 7 \prec \cdots \prec \cdots \prec 0 \prec -1 \prec 3 \prec 2 \prec 6 \prec 5 \prec \cdots$$

$$\cdots \prec 0 \prec 3 \prec 6 \prec \cdots \prec \cdots \prec 7 \prec 4 \prec 1 \prec \cdots \prec \cdots \prec 8 \prec 5 \prec 2 \prec \cdots$$

A combinatorial model for extended weak order

TITOs can be encoded with window notation: e.g. in TTot_3 the notation $[1][3,2]$ encodes

$$\cdots \prec -2 \prec 1 \prec 4 \prec 7 \prec \cdots \prec \cdots \prec 0 \prec -1 \prec 3 \prec 2 \prec 6 \prec 5 \prec \cdots$$

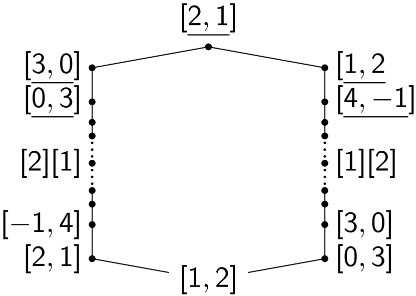
We write $[1][\underline{3,2}]$ to instead encode

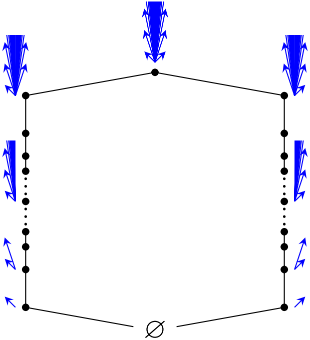
$$\cdots \prec -2 \prec 1 \prec 4 \prec 7 \prec \cdots \prec \cdots \prec 6 \prec 5 \prec 3 \prec 2 \prec 0 \prec -1 \prec \cdots$$

A combinatorial model for extended weak order

Theorem (B.–Speyer)

The map $(\prec) \mapsto \mathrm{Inv}(\prec)$ is a bijection from TTot_n to the extended weak order of $\widetilde{S}_n$.

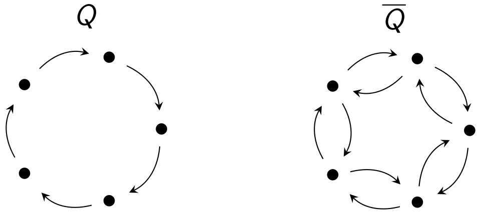




Torsion classes for preprojective algebras

Preprojective algebras

Q = quiver
$\overline{Q}$ = doubled quiver
k = algebraically closed
$k[\overline{Q}]$ = path algebra of doubled quiver



The **preprojective algebra** of Q is

$$\Pi_Q = k[\overline{Q}] / \sum_{a \in Q_1} (aa^* - a^*a)$$

Bricks

We represent modules via a vector space V_i on each node i of Q. The **dimension vector** is $\underline{\dim}\, M = \sum_{i \in Q_0} (\dim M^i)\alpha_i$.

Definition

A **brick** for Π is a module M so that $\mathrm{End}_\Pi(M) = k$.

Example: the A_2 quiver $Q = \bullet \xrightarrow{a} \bullet$.

$$\Pi = k[\bullet \underset{a^*}{\overset{a}{\rightleftarrows}} \bullet]/\langle aa^*, a^*a \rangle$$

$S_1 = k \quad 0$	$\underline{\dim}\, S_1 = \alpha_{1,2}$
$S_2 = 0 \quad k$	$\underline{\dim}\, S_2 = \alpha_{2,3}$
$P_1 = k \longleftarrow k$	$\underline{\dim}\, P_1 = \alpha_{1,3}$
$P_2 = k \longrightarrow k$	$\underline{\dim}\, P_2 = \alpha_{1,3}$

Bricks

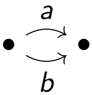
Theorem (Iyama–Reading–Reiten–Thomas, Dana–Speyer–Thomas, B., ...)

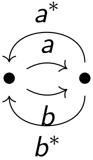
Interpret Q as a (generalized) Dynkin diagram for a root system Φ. Then any brick of Π_Q has $\underline{\dim}\, M \in \Phi^+$.

Definition

If M is a brick with $\underline{\dim}\, M$ is a real root, then M is **real**. Otherwise, M is **imaginary**.

Example: $\widetilde{A}_1$

$Q =$ 

$\Pi = k[$  $]/\langle aa^* - a^*a + bb^* - b^*b \rangle$

$$\delta = \alpha_1 + \alpha_2$$

Imaginary roots of Φ_Q are multiples of δ

Example of an imaginary brick:

Torsion classes

Definition

A **torsion class** for Π is a collection of (finite-dimensional) modules closed under isomorphisms, quotients, and extensions.

Fact

A torsion class is determined by the bricks it contains.

Example: A_2

$$Q = \bullet \xrightarrow{a} \bullet$$

$$\Pi = k[\, \bullet \underset{a^*}{\overset{a}{\rightleftarrows}} \bullet \,]/\langle aa^*, a^*a \rangle$$

$$S_1 = k \qquad 0$$

$$S_2 = 0 \qquad k$$

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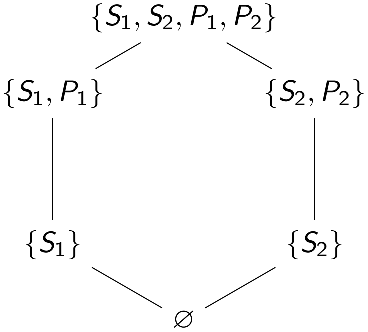
Tors(Π)



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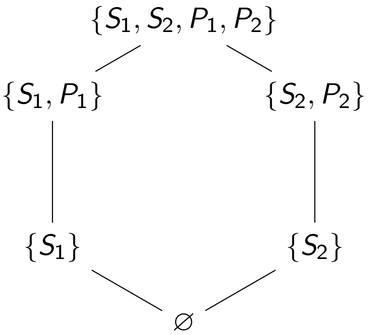
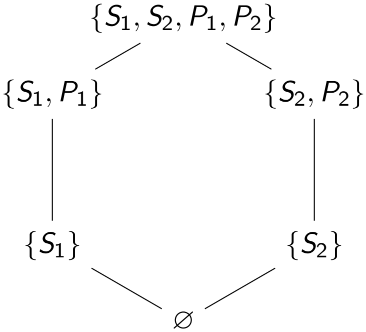
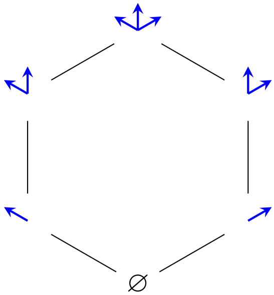


Theorem (Mizuno)

Let Q be a Dynkin quiver with (finite) Weyl group W. Then Tors(Π) is isomorphic to the weak order of W.

The isomorphism





Real torsion classes

Theorem (Demonet–Iyama–Reading–Reiten–Thomas)

Tors(Π) is a completely semidistributive lattice

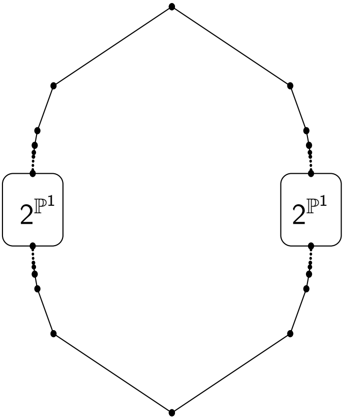
Given a torsion class $\mathcal{T}$, define

$$\underline{\dim}\,\mathcal{T} := \{\underline{\dim}\,B \mid B \in \mathcal{T} \text{ is a real brick}\}$$

Conjecture (Dana–Speyer–Thomas)

The map sending a torsion class $\mathcal{T}$ to $\underline{\dim}\,\mathcal{T} \subseteq \Phi^+_{\mathrm{real}}$ is a complete lattice quotient onto extended weak order.

Example: Tors($\Pi_{\widetilde{A}_1}$)



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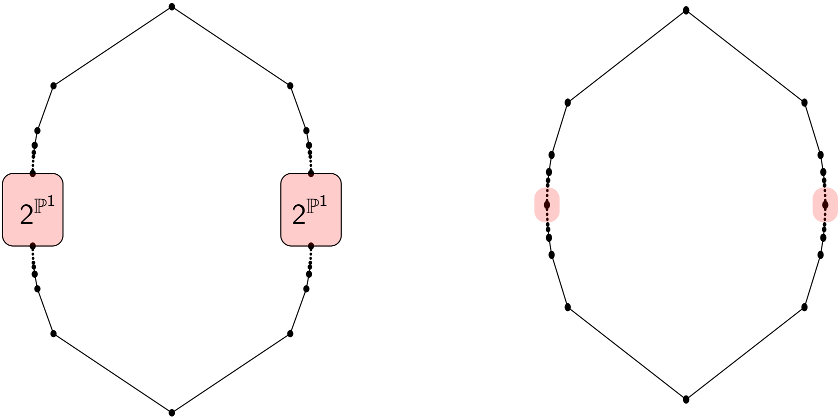


Main result

Theorem (B.)

If Q is an orientation of the extended Dynkin diagram of an affine Coxeter group W, then $\mathcal{T} \mapsto \underline{\dim}\,\mathcal{T}$ is a complete lattice quotient from torsion classes to extended weak order of W.

Corollary (B.)

There is a bijection between real bricks and completely join-irreducible elements of extended weak order.

Corollary (B.)

In type $\widetilde{A}$, there is an explicit parametrization of all torsion classes in terms of TITOs.

Application to cluster algebras

Quivers and cluster algebras

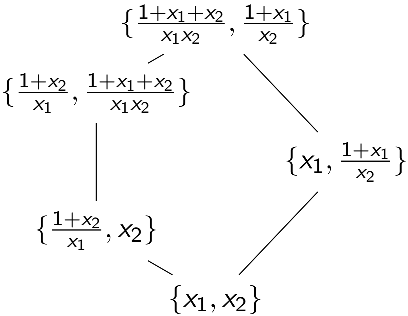
Associated to each (loopless 2-acyclic) quiver Q is a **cluster algebra** A_Q.

$$Q = \bullet \longrightarrow \bullet$$

Each node has a variable x_i attached

Clusters are sets of Laurent polynomials in $x_1, \ldots, x_n$ built from $\{x_1, \ldots, x_n\}$ using **mutation**

The **exchange graph** has vertices the clusters and edges the mutations



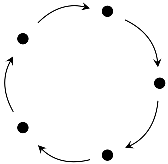
Extended weak order and exchange graphs

Theorem (B.)

Let Q be an orientation of an affine Dynkin diagram with Coxeter group W. Then there is a complete lattice quotient of extended weak order which contains the exchange graph of A_Q in its Hasse diagram.

Example: the oriented cycle

Consider the **oriented cycle**:

$Q =$ 

Then A_Q is a cluster algebra of type D; its clusters are counted by **type D Catalan numbers**. Its exchange graph is the edge graph of a **type D associahedron**.

Q is associated to the Coxeter group $\widetilde{S}_n$. The theorem says that the exchange graph is contained in the Hasse diagram of some quotient of extended weak order.

The affine Tamari lattice

Definition (B.–Defant)

A TITO ($\prec$) is **312-avoiding** if there are no integers $a < b < c$ with $c \prec a \prec b$.

The **affine Tamari lattice** ATam_n is the poset of 312-avoiding TITOs, ordered by containment of inversion sets.

Theorem (B.–Defant)

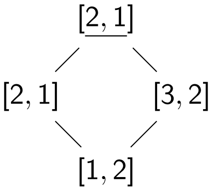
ATam_n is a quotient of the extended weak order of $\widetilde{S}_n$. The Hasse diagram of ATam_n is the exchange graph of A_Q.

ATam_n for $n = 1, 2$

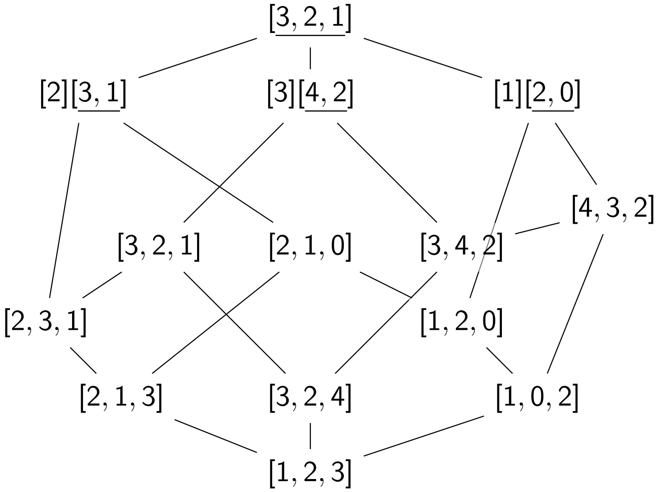
ATam_1

ATam_2

$[1]$



ATam$_3$



More to do!

Extended weak order is the source of many open conjectures. The results here are all special cases of the following:

Conjecture

Anything that works for weak order also works for extended weak order. Often it works better!

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Conjecture

Anything that works for weak order also works for extended weak order. Often it works better!

Can you find more examples?

Thank you!

TITOs to bricks

Each completely join-irreducible TITO has an **arc diagram**
E.g. with $[1][\underline{7,8,2}]$

$$\cdots \prec 1 \prec 5 \prec 9 \prec \cdots \prec 6 \prec 7 \prec 8 \prec 2 \prec 3 \prec 4 \prec \cdots$$