On multiple zeta functions with combinatorial structure

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Two Theories

Number Theory	$\Longleftrightarrow$	**Combinatorial Theory**
Zeta functions		Symmetric polynomials/functions
	$\Longleftarrow$	

Definition

Riemann zeta function $\zeta(s)$ is defined by the series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

where $s \in \mathbb{C}$ is a complex variable. This converges absolutely for $\Re(s) > 1$.

One of the interesting topics concerning the Riemann zeta function is to evaluate so-called “special values", that is to study the values of $\zeta(s)$ at $s = k$ with $k \in \mathbb{N}$, "$\zeta(k)$".

Definition(Euler)

The double zeta value $\zeta(k_1, k_2)$ is defined by

$$\zeta(k_1, k_2) = \sum_{0<m_1<m_2} \frac{1}{m_1^{k_1} m_2^{k_2}},$$

where $k_1 \in \mathbb{Z}_{>0}$ and $k_2 \in \mathbb{Z}_{>1}$.

For $k_1, k_2 \in \mathbb{Z}_{>1}$, Euler obtained

$$\zeta(k_1)\zeta(k_2) = \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2).$$

Definition

Multiple zeta functions of the Euler-Zagier type are defined by the series

$$\zeta(s_1, \ldots, s_n) = \sum_{m_1 < \cdots < m_n} \frac{1}{m_1^{s_1} \cdots m_n^{s_n}},$$

where $s_1, \ldots, s_n \in \mathbb{C}$. These series converge absolutely for $\Re(s_1), \ldots, \Re(s_{n-1}) \geq 1$ and $\Re(s_n) > 1$.

Multiple zeta values : For positive integers $k_1, \ldots, k_n$ with $k_n > 1$, $\zeta(k_1, \ldots, k_n)$ is called "multiple zeta values".

Definition

Multiple zeta-star functions of the Euler-Zagier type are defined by the series

$$\zeta^{\star}(s_1, \ldots, s_n) = \sum_{m_1 \leq \cdots \leq m_n} \frac{1}{m_1^{s_1} \cdots m_n^{s_n}},$$

where $s_1, \ldots, s_n \in \mathbb{C}$. These series converge absolutely for $\Re(s_1), \ldots, \Re(s_{n-1}) \geq 1$ and $\Re(s_n) > 1$.

Multiple zeta-star values : For positive integers $k_1, \ldots, k_n$ with $k_n > 1$, $\zeta^{\star}(k_1, \ldots, k_n)$ is called "multiple zeta-star values".

Basic symmetric polynomials/functions

- Elementary symmetric polynomial/function

$$e_n := \sum_{m_1 < \cdots < m_n} x_{m_1} \cdots x_{m_n}$$

- Complete symmetric polynomial/function

$$h_n : = \sum_{m_1 \leq \cdots \leq m_n} x_{m_1} \cdots x_{m_n}$$

These are speacial cases of Schur polynomials/functions.

Schur polynomial

- $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$: partition
  s.t. $\lambda_i \in \mathbb{Z}$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0$
- $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$: variables $(n \geq \ell)$

We define Schur polynomial associated with λ by

$$s_\lambda = s_\lambda(\boldsymbol{x}) = \frac{\det(x_j^{n-i+\lambda_i})}{\det(x_j^{n-i})}.$$

- $\det(x_j^{n-i}) = \prod_{1\leq i<j\leq n} (x_j - x_i)$ is Vandermonde determinant.
- **Example.** $\lambda = (2,1) = (2,1,0), \boldsymbol{x} = (x_1, x_2, x_3), n = 3$

$$s_\lambda(\boldsymbol{x}) = \frac{1}{\det(x_j^{3-i})} \det \begin{pmatrix} x_1^4 & x_2^4 & x_3^4 \\ x_1^2 & x_2^2 & x_3^2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{matrix} x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 \\ +x_1x_3^2 + x_2^2x_3 + x_2x_3^2 \end{matrix}$$

- Schur polynomials/functions have **tableau expression** .

Young diagram/tableau of shape λ

$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$: partition s.t. $\lambda_1 \geq \lambda_2 \geq \cdots$

- We identify a partition λ with its Young diagram $D_\lambda = \{(i,j) \in \mathbb{Z}^2 \,|\, 1 \leq i \leq \ell,\ 1 \leq j \leq \lambda_i\}$.
  Let λ' be the conjugate of λ, which is the partition whose Young diagram is the transpose of that of λ.

- Let X be a set. Young tableau $T = (t_{ij})$ of shape λ over X is a filling of D_λ obtained by putting $t_{ij} \in X$ into (i,j) box of D_λ.

  Example. $\lambda = (4,3,2)$, $t_{ij} \in X$.

$$D_\lambda = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & \\ \cline{1-3} & \\ \cline{1-2}\end{array} \qquad T = \begin{array}{|c|c|c|c|}\hline t_{11} & t_{12} & t_{13} & t_{14} \\ \hline t_{21} & t_{22} & t_{23} \\ \cline{1-3} t_{31} & t_{32} \\ \cline{1-2}\end{array}.$$

$T_\lambda(X)$: the set of all X-valued Young tableaux of shape λ.

- Semi-standard Young tableau $M = (m_{ij})$ of shape λ is a filling of D_λ obtained by putting $m_{ij} \in \mathbb{N}$ into (i,j) box of D_λ such that
  - the entries in each row are weakly increasing from left to right
  - the entries in each column are strictly increasing from top to bottom.

  We denote by $SSYT_\lambda$ the set of all semi-standard Young tableaux of shape λ.

Example. $\lambda = (4,3,2)$, $m_{ij} \in \mathbb{N}$, $M \in SSYT_\lambda$

$M =$

m_{11}	m_{12}	m_{13}	m_{14}
m_{21}	m_{22}	m_{23}	
m_{31}	m_{32}		

$$\begin{array}{ccccccc} m_{11} & \le & m_{12} & \le & m_{13} & \le & m_{14} \\ \wedge & & \wedge & & \wedge & & \\ m_{21} & \le & m_{22} & \le & m_{23} & & \\ \wedge & & \wedge & & & & \\ m_{31} & \le & m_{32} & & & & \end{array}$$

- $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$: partition
- $\boldsymbol{x} = (x_1, x_2, \cdots)$: variables

Schur function has **tableau expression**:

$$s_\lambda(\boldsymbol{x}) = \sum_{M \in SSYT_\lambda} \prod_{(i,j)\in D_\lambda} x_{m_{ij}},$$

Example. $\lambda = (2,1)$. Then

$$SSYT_\lambda = \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \ldots \right\}$$

$$s_\lambda(\boldsymbol{x}) = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + \cdots$$

Special cases of Schur polynomials/functions

- $\lambda = (\underbrace{1, 1, \ldots, 1}_{n})$.

  $e_n := s_{(1^n)}$: elementary symmetric polynomial

  $$= \sum_{m_1 < \cdots < m_n} x_{m_1} \cdots x_{m_n}$$

  $$\begin{array}{|c|}\hline m_1 \\ \hline \vdots \\ \hline m_n \\ \hline \end{array}$$

- $\lambda = (n)$.

  $h_n := s_{(n)}$: complete symmetric polynomial

  $$= \sum_{m_1 \le \cdots \le m_n} x_{m_1} \cdots x_{m_n}$$

  $$\begin{array}{|c|c|c|}\hline m_1 & \cdots & m_n \\ \hline \end{array}$$

[Definition] Schur multiple zeta function

- $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$: partition
- $\boldsymbol{s} = (s_{ij}) \in T_\lambda(\mathbb{C})$: variables

$$\boldsymbol{s} = \begin{array}{|c|c|}\hline s_{11} & s_{12} \\ \hline s_{21} & \\ \cline{1-1}\end{array}$$

Schur multiple zeta function associated with λ (introduced by Nakasuji, Phukusuwan and Yamasaki (2018))

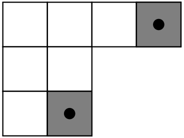
$$\zeta_\lambda(\boldsymbol{s}) = \sum_{(m_{ij}) \in SSYT_\lambda} \prod_{(i,j) \in D_\lambda} \frac{1}{m_{ij}^{s_{ij}}},$$

Example. $\lambda = (2, 1)$. Then

$$SSYT_\lambda = \left\{ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \\ \cline{1-1}\end{array}, \dots \right\}$$

$$\zeta_\lambda(\boldsymbol{s}) = \frac{1}{1^{s_{11}} 1^{s_{12}} 2^{s_{21}}} + \frac{1}{1^{s_{11}} 1^{s_{12}} 3^{s_{21}}} + \frac{1}{1^{s_{11}} 2^{s_{12}} 2^{s_{21}}} + \frac{1}{1^{s_{11}} 2^{s_{12}} 3^{s_{21}}} + \cdots.$$

For a partition $\lambda = (\lambda_1, \cdots, \lambda_\ell)$,
$C_\lambda \subset D_\lambda$: the set of all corners of λ.



Example.

$C_{(4,2,2)} = \{(1,4),\ (3,2)\}$. $\left(\boxed{\bullet} \in C_\lambda\right)$

$$W_\lambda := \left\{ \boldsymbol{s} = (s_{ij}) \in T_\lambda(\mathbb{C}) \;\middle|\; \begin{array}{l} \Re(s_{ij}) \geq 1 \text{ for } {}^\forall(i,j) \in D_\lambda \setminus C_\lambda \\ \Re(s_{ij}) > 1 \text{ for } {}^\forall(i,j) \in C_\lambda \end{array} \right\}.$$

$\zeta_\lambda(\boldsymbol{s})$ converges absolutely if $\boldsymbol{s} \in W_\lambda$.

$$\boldsymbol{s} = \begin{array}{|c|c|c|c|} \hline s_{11} & s_{12} & s_{13} & s_{14} \\ \hline s_{21} & s_{22} \\ \cline{1-2} s_{31} & s_{32} \\ \cline{1-2} \end{array} \in T_\lambda(\mathbb{C})$$

Skew type

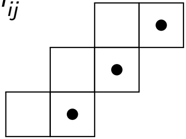
- Let λ and μ be partitions satisfying $\lambda \supset \mu$, that is $\lambda_i \geq \mu_i$ for all i.
- Skew Young diagram is the set difference between the two partitions, and a skew semi-standard Young tableau of shape λ/μ is a filling of the Young diagram λ/μ with positive integers such that the rows are weakly increasing and the columns are strictly increasing.
  Example. For $\lambda = (6, 3, 2, 2)$ and $\mu = (4, 1, 1)$,

				2	3
	1	5			
	3				
6	6				

- $\mathrm{SSYT}(\lambda/\mu)$: the set of all skew semi-standard Young tableaux of shape λ/μ.

Let $\boldsymbol{s} = (s_{ij}) \in T(\lambda/\mu, \mathbb{C})$. We generalize the definition of the Schur multiple zeta function to skew type as

$$\zeta_{\lambda/\mu}(\boldsymbol{s}) = \sum_{(m_{ij}) \in SSYT_{(\lambda/\mu)}} \prod_{(i,j) \in D_{(\lambda/\mu)}} \frac{1}{m_{ij}^{s_{ij}}},$$



.

$C(\lambda/\mu) \subset D(\lambda/\mu)$: set of all corners of λ/μ.

Lemma

$$W_{\lambda/\mu} :=$$

$$\left\{ (s_{ij}) \in T(\lambda/\mu, \mathbb{C}) \;\middle|\; \begin{array}{l} \Re(s_{ij}) \geq 1 \text{ for } {}^{\forall}(i,j) \in D(\lambda/\mu) \setminus C(\lambda/\mu) \\ \Re(s_{ij}) > 1 \text{ for } {}^{\forall}(i,j) \in C(\lambda/\mu) \end{array} \right\}.$$

Then, $\zeta_{\lambda/\mu}(\boldsymbol{s})$ converges absolutely if $\boldsymbol{s} = (s_{ij}) \in W_{\lambda/\mu}$.

- When $\lambda = (1)$ and $\mu = \emptyset$, $\boldsymbol{s} = (s) \in T_\lambda(\mathbb{C})$ $\boxed{m}$

$$\zeta_{(1)}(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \zeta(s).$$

- When $\lambda = (\underbrace{1, 1, \ldots, 1}_{n}) = (1^n)$, $\boldsymbol{s} = (s_{i1}) \in T_\lambda(\mathbb{C})$

m_{11}
$\vdots$
m_{n1}

$$\zeta_{(1^n)}(s_{11}, \ldots, s_{n1}) = \sum_{m_{11} < \cdots < m_{r1}} \frac{1}{m_{11}^{s_{11}} \cdots m_{n1}^{s_{n1}}} = \zeta(s_{11}, \ldots, s_{r1}).$$

- When $\lambda = (n)$, $\boldsymbol{s} = (s_{1j}) \in T_\lambda(\mathbb{C})$

m_{11}	m_{12}	$\cdots$	m_{1n}

$$\zeta_{(n)}(s_{11}, \ldots, s_{1n}) = \sum_{m_{11} \le \cdots \le m_{1n}} \frac{1}{m_{11}^{s_{11}} \cdots m_{1n}^{s_{1n}}} = \zeta^\star(s_{11}, \ldots, s_{1n}).$$

For $s \in \mathbb{C}$ with $Re(s) > 1$, we have

$$\zeta_\lambda(\{s\}^\lambda) = s_\lambda(1^{-s}, 2^{-s}, \ldots).$$

Proposition

Let $\lambda \vdash n$. Then, for $s \in \mathbb{C}$ with $\Re(s) > 1$, we have

$$\zeta_\lambda(\{s\}^\lambda) = \sum_{\mu \vdash n} \frac{\chi^\lambda(\mu)}{z_\mu} \prod_{i=1}^{\ell(\mu)} \zeta(\mu_i s).$$

Here, $z_\mu = \prod_{i \geq 1} i^{m_i(\mu)} m_i(\mu)!$ and $\chi^\lambda(\mu) \in \mathbb{Z}$ is the value of the character χ^λ attached to the irreducible representation of the symmetric group S_n of degree n corresponding to λ on the conjugacy class of S_n of the cycle type $\mu \vdash n$.

Application of combinatorial methods

Assumption(content-parametrize)

For $\boldsymbol{s} \in W_\lambda \subset T_\lambda(\mathbb{C})$ being variables for $\zeta_\lambda(\boldsymbol{s})$,

$$W_\lambda^{\mathrm{diag}} := \{\boldsymbol{s} \in W_\lambda | s_{i+n,j+n} = s_{i,j} \text{ for } {}^\forall n \in \mathbb{Z}\}$$

Notation

Assume that $\boldsymbol{s} = (s_{ij}) \in W_\lambda^{\mathrm{diag}}$. Let $s_{ij} = z_{j-i}$ by a given sequence $(z_k)_{k\in\mathbb{Z}}$.

Example. $\lambda = (5, 3, 3, 1)$

s_{11}	s_{12}	s_{13}	s_{14}	s_{15}
s_{21}	s_{22}	s_{23}		
s_{31}	s_{32}	s_{33}		
s_{41}				

$=$

z_0	z_1	z_2	z_3	z_4
z_{-1}	z_0	z_1		
z_{-2}	z_{-1}	z_0		
z_{-3}				

Schur polynomials / functions have determinant formula called **Jacobi-Trudi fomula**.

- $h_n = s_{(n)}$: complete symmetric polynomial
- $e_n = s_{(1^n)}$: elementary symmetric polynomial

Jacobi-Trudi formula

We have

$$s_\lambda = \det(e_{\lambda'_i - i + j})_{s \times s}$$

$$s_\lambda = \det(h_{\lambda_i - i + j})_{r \times r}$$

(Method of Proof)

- using the generating function for h_k and that for e_k
- using the **lattice path model** (combinatorial approach)

Theorem(N.-Phuksuwan-Yamasaki, 2018)

Assume that $\boldsymbol{s} = (s_{ij}) \in W_\lambda^{\mathrm{diag}}$. Put $s_{ij} = z_{j-i}$.

(1) Assume that $\Re(s_{i,\lambda'_i}) > 1$ for all $1 \le i < \lambda_1$. Then, we have

$$\zeta_\lambda(\boldsymbol{s}) = \det\left[\zeta(z_{j-1}, \dots, z_{j-(\lambda'_i - i + j)})\right]_{1 \le i,j \le \lambda_1},$$

where $\zeta(\dots) = 1$ if $\lambda'_i - i + j = 0$ and 0 if $\lambda'_i - i + j < 0$.

(2) Assume that $\Re(s_{i,\lambda_i}) > 1$ for all $1 \le i < {\lambda_1}'$. Then, we have

$$\zeta_\lambda(\boldsymbol{s}) = \det\left[\zeta^\star(z_{-j+1}, \dots, z_{-j+(\lambda_i - i + j)})\right]_{1 \le i,j \le \lambda'_1},$$

where$\zeta^\star(\dots) = 1$ if $\lambda_i - i + j = 0$ and 0 if $\lambda_i - i + j < 0$.

New relation that results from this : Example 1

For $\lambda = (2,2,1)$, $z_1, z_{-1} \geq 1$ and $z_0, z_{-2} > 1$, we have

$$\zeta_\lambda\left(\begin{array}{|c|c|}\hline z_0 & z_1 \\ \hline z_{-1} & z_0 \\ \hline z_{-2} & \\ \cline{1-1}\end{array}\right)$$

$$= \begin{vmatrix} \zeta(z_0, z_{-1}, z_{-2}) & \zeta(z_1, z_0, z_{-1}, z_{-2}) \\ \zeta(z_0) & \zeta(z_1, z_0) \end{vmatrix}$$

$$= \begin{vmatrix} \zeta^\star(z_0, z_1) & \zeta^\star(z_{-1}, z_0, z_1) & \zeta^\star(z_{-2}, z_{-1}, z_0, z_1) \\ \zeta^\star(z_0) & \zeta^\star(z_{-1}, z_0) & \zeta^\star(z_{-2}, z_{-1}, z_0) \\ 0 & 1 & \zeta^\star(z_{-2}) \end{vmatrix}.$$

Example 2

For $\lambda = (2)$, $z_1 \geq 1$ and $z_1 > 1$, we have

$$\zeta_\lambda\left(\boxed{z_0 \mid z_1}\right) = \zeta^\star(z_0, z_1)(= \zeta(z_0, z_1) + \zeta(z_0 + z_1))$$

$$= \begin{vmatrix} \zeta(z_0) & \zeta(z_1, z_0) \\ 1 & \zeta(z_1) \end{vmatrix}$$

$$= \zeta(z_0)\zeta(z_1) - \zeta(z_1, z_0).$$

Considering the case $\lambda = (n)$, we have the following

Corollary

For $s_1, \ldots, s_n \in \mathbb{C}$ with $\Re(s_1), \ldots, \Re(s_n) > 1$, it holds that

(1) $\zeta^\star(s_1, \ldots, s_n) =$

$$\begin{vmatrix} \zeta(s_1) & \zeta(s_2, s_1) & & \cdots & \cdots & \zeta(s_n, \ldots, s_2, s_1) \\ 1 & \zeta(s_2) & & \cdots & \cdots & \zeta(s_n, \ldots, s_2) \\ & 1 & \ddots & & & \vdots \\ & & \ddots & 1 & \zeta(s_{n-1}) & \zeta(s_n, s_{n-1}) \\ 0 & & & & 1 & \zeta(s_n) \end{vmatrix}.$$

Considering the case $\lambda = (1^n)$, we have the following

Corollary

For $s_1, \ldots, s_n \in \mathbb{C}$ with $\Re(s_1), \ldots, \Re(s_n) > 1$, it holds that

(2) $$\zeta(s_1, \ldots, s_n) = \begin{vmatrix} \zeta^\star(s_1) & \zeta^\star(s_2, s_1) & \cdots & \cdots & \zeta^\star(s_n, \ldots, s_2, s_1) \\ 1 & \zeta^\star(s_2) & \cdots & \cdots & \zeta^\star(s_n, \ldots, s_2) \\ & 1 & \ddots & & \vdots \\ & & \ddots \quad 1 & \zeta^\star(s_{n-1}) & \zeta^\star(s_n, s_{n-1}) \\ 0 & & & 1 & \zeta^\star(s_n) \end{vmatrix}$$

The Pieri formula expresses product of the Schur polynomials by complete or elementary symmetric polynomials.

Pieri formula

Let s_λ be the Schur polynomial associated with a partition λ. Let $h_r = s_{(r)}$ and $e_r = s_{(1^r)}$ be the complete and symmetric polynomials, respectively. Then we have

$$s_\lambda(\mathrm{x}) h_r(\mathrm{x}) = \sum_\mu s_\mu(\mathrm{x}), \quad s_\lambda(\mathrm{x}) e_r(\mathrm{x}) = \sum_\mu s_\mu(\mathrm{x}),$$

where the sum is taken over all partitions μ obtained by adding r boxes to the diagram D_λ with no two boxes in the same column or row, respectively.

Example. When $\lambda = (2,1)$, $\mathrm{x} = (x_1, x_2, x_3)$, it holds that

$$s_{\ydiagram{2,1}}(\mathrm{x})\, s_{\ydiagram[*(gray)]{1}}(\mathrm{x}) = s_{\ydiagram[*(gray)]{0,0,1}*{2,1,1}}(\mathrm{x}) + s_{\ydiagram[*(gray)]{0,1+1}*{2,2}}(\mathrm{x}) + s_{\ydiagram[*(gray)]{2+1}*{3,1}}(\mathrm{x}),$$

where

$$s_{\ydiagram{2,1}}(\mathrm{x}) = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2 + x_3 + x_2x_3^2,$$

$$s_{\ydiagram[*(gray)]{1}}(\mathrm{x}) = x_1 + x_2 + x_3, \qquad s_{\ydiagram[*(gray)]{0,0,1}*{2,1,1}}(\mathrm{x}) = x_1x_2x_3(x_1 + x_2 + x_3),$$

$$s_{\ydiagram[*(gray)]{0,1+1}*{2,2}}(\mathrm{x}) = x_1^2x_2^2 + x_1^2x_2x_3 + x_1^2x_3^2 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_2^2x_3^2,$$

$$s_{\ydiagram[*(gray)]{2+1}*{3,1}}(\mathrm{x}) = x_2^3x_2 + x_1^2x_2^2 + 2x_1^2x_2x_3 + x_1x_2^3 + 2x_1x_2^2x_3 + 2x_1x_2x_3^2$$

Theorem(N.–Takeda, 2022)

For a positive integer ℓ and non-negative integers k and m, we further assume $\Re(y_i), \Re(t_j) > 1$ $(1 \le i \le \ell, 1 \le j \le \ell - 1)$. Then we have

$$\sum_{Sym} \zeta_{(\ell,1^k)}\left(\begin{array}{|c|c|c|}\hline y_1 & \cdots & y_\ell \\ \hline \multicolumn{1}{|c|}{x_1} \\ \cline{1-1} \multicolumn{1}{|c|}{\vdots} \\ \cline{1-1} \multicolumn{1}{|c|}{x_k} \\ \cline{1-1}\end{array}\right) \cdot \zeta_{(m)}\left(\begin{array}{|c|c|c|c|}\hline t_1 & t_2 & \cdots & t_m \\ \hline\end{array}\right)$$

$$= \sum_{Sym} \sum_{\mathrm{u}_\mu \in U_H} \zeta_\mu(\mathrm{u}_\mu),$$

where Sym means the summation over the permutation of $S = \{y_1, \ldots, y_\ell, t_1, \ldots, t_{\ell-1}\}$ as indeterminates and the inner sum in the right-hand side is takes all the term $\mathrm{u}_\mu \in U_H$ obtained by the pushing rule and μ is the shape of u_μ.

Pushing rule H

As in the Pieri formula, the Schur multiple zeta function $\zeta_{\ydiagram{3,1}}$ appears as the summand of $\zeta_{\ydiagram{2,1}}\left(\begin{ytableau} s_{11} & s_{12} \\ s_{21} \end{ytableau}\right)\zeta_{\ydiagram{1}}\left(\begin{ytableau} *(gray) t_1 \end{ytableau}\right)$. However, in general,

$$\zeta_{\ydiagram{3,1}}\left(\begin{ytableau} s_{11} & s_{12} & *(gray) t_1 \\ s_{21} \end{ytableau}\right) \neq \zeta_{\ydiagram{3,1}}\left(\begin{ytableau} *(gray) t_1 & s_{11} & s_{12} \\ s_{21} \end{ytableau}\right).$$

For $s \in T_\lambda(C)$ and $t \in T_{(r)}(C)$, we construct a new Young tableau by inserting all the components in t into s. The insetion method which we use here is called Pushing rule H.

Example. For $\zeta_{\begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}\left(\begin{array}{|c|c|}\hline s_{11} & s_{12} \\ \hline s_{21} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) \zeta_{\begin{array}{|c|}\hline \; \\ \hline\end{array}}\left(\begin{array}{|c|}\hline t_1 \\ \hline\end{array}\right)$,

$$U_H = \left\{ \zeta_{\begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \multicolumn{1}{c}{} \\ \cline{1-1} \; & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}\left(\begin{array}{|c|c|}\hline t_1 & s_{12} \\ \hline s_{11} & \multicolumn{1}{c}{} \\ \cline{1-1} s_{21} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right), \zeta_{\begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \; \\ \hline\end{array}}\left(\begin{array}{|c|c|}\hline s_{11} & t_1 \\ \hline s_{21} & s_{12} \\ \hline\end{array}\right), \zeta_{\begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \; & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}}\left(\begin{array}{|c|c|c|}\hline s_{11} & s_{12} & t_1 \\ \hline s_{21} & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\right) \right\}.$$

Example. When $\lambda = (3,2,1)$ and $r = 2$

$$\zeta_{\ydiagram{2,1}}\left(\begin{array}{|c|c|c|}\hline s_{11} & s_{12} & s_{13} \\ \hline s_{21} & s_{22} & \\ \cline{1-2} s_{31} & & \\ \cline{1-1}\end{array}\right) \zeta_{\ydiagram{2}}\left(\begin{array}{|c|c|}\hline t_1 & t_2 \\ \hline\end{array}\right), \text{ then}$$

$$U_H = \left\{ \begin{array}{|c|c|c|}\hline t_1 & t_2 & s_{13} \\ \hline s_{11} & s_{12} \\ \cline{1-2} s_{21} & s_{22} \\ \cline{1-2} s_{31} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline t_1 & s_{12} & t_2 \\ \hline s_{11} & s_{22} & s_{13} \\ \hline s_{21} \\ \cline{1-1} s_{31} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|c|}\hline t_1 & s_{12} & s_{13} & t_2 \\ \hline s_{11} & s_{22} \\ \cline{1-2} s_{21} \\ \cline{1-1} s_{31} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline s_{11} & t_1 & t_2 \\ \hline s_{21} & s_{12} & s_{13} \\ \hline s_{31} & s_{22} \\ \cline{1-2}\end{array}, \right.$$

$$\left. \begin{array}{|c|c|c|c|}\hline s_{11} & t_1 & s_{13} & t_2 \\ \hline s_{21} & s_{12} \\ \cline{1-2} s_{31} & s_{22} \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|c|}\hline s_{11} & s_{12} & t_1 & t_2 \\ \hline s_{21} & s_{22} & s_{13} \\ \cline{1-3} s_{31} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|c|c|}\hline s_{11} & s_{12} & s_{13} & t_1 & t_2 \\ \hline s_{21} & s_{22} \\ \cline{1-2} s_{31} \\ \cline{1-1}\end{array}, \right\}$$

Theorem(N.–Takeda, 2022)

For a positive integer k and non-negative integers ℓ and m, we further assume $\Re(x_i), \Re(t_j) > 1$ $(1 \leq i \leq k, 1 \leq j \leq k-1)$. Then we have

$$\sum_{Sym} \zeta_{(\ell+1,1^{k-1})}\left(\begin{array}{|c|c|c|c|}\hline x_1 & y_1 & \cdots & y_\ell \\ \hline \multicolumn{1}{|c|}{\vdots} \\ \cline{1-1} \multicolumn{1}{|c|}{x_k} \\ \cline{1-1}\end{array}\right) \cdot \zeta_{(1^m)}\left(\begin{array}{|c|}\hline t_1 \\ \hline \vdots \\ \hline t_m \\ \hline\end{array}\right) = \sum_{Sym} \sum_{\mathrm{u}_\mu \in U_E} \zeta_\mu(\mathrm{u}_\mu),$$

where *Sym* means the summation over the permutation of $S = \{x_1, \ldots, x_k, t_1, \ldots, t_{k-1}\}$ as indeterminates and the inner sum in the right-hand side is takes all the term $\mathrm{u}_\mu \in U_E$ obtained by the pushing rule and μ is the shape of u_μ.

Example. When $\lambda = (3,2,1)$ and $r = 2$

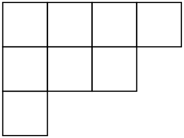
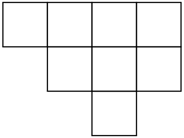
$$\zeta_{\ydiagram{2,1}}\left(\begin{array}{|c|c|c|}\hline s_{11} & s_{12} & s_{13} \\ \hline s_{21} & s_{22} & \\ \cline{1-2} s_{31} & & \\ \cline{1-1}\end{array}\right)\zeta_{\ydiagram{1,1}}\left(\begin{array}{|c|}\hline t_1 \\ \hline t_2 \\ \hline\end{array}\right),$$ then

$$U_E = \left\{ \begin{array}{|c|c|c|c|}\hline t_1 & s_{11} & s_{12} & s_{13} \\ \hline t_2 & s_{21} & s_{22} & \\ \cline{1-3} s_{31} & & & \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|c|}\hline t_1 & s_{11} & s_{12} & s_{13} \\ \hline s_{21} & s_{22} & & \\ \cline{1-2} t_2 & s_{31} & & \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|c|}\hline t_1 & s_{11} & s_{12} & s_{13} \\ \hline s_{21} & s_{22} & & \\ \cline{1-2} s_{31} & & & \\ \cline{1-1} t_2 & & & \\ \cline{1-1}\end{array}, \right.$$

$$\left. \begin{array}{|c|c|c|}\hline s_{11} & s_{12} & s_{13} \\ \hline t_1 & s_{21} & s_{22} \\ \hline t_2 & s_{31} & \\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline s_{11} & s_{12} & s_{13} \\ \hline t_1 & s_{21} & s_{22} \\ \hline s_{31} & & \\ \cline{1-1} t_2 & & \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline s_{11} & s_{12} & s_{13} \\ \hline s_{21} & s_{22} & \\ \cline{1-2} t_1 & s_{31} & \\ \cline{1-2} t_2 & & \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline s_{11} & s_{12} & s_{13} \\ \hline s_{21} & s_{22} & \\ \cline{1-2} s_{31} & & \\ \cline{1-1} t_1 & & \\ \cline{1-1} t_2 & & \\ \cline{1-1}\end{array}, \right\}$$

- We consider the shifted diagram of shape λ
$SD_\lambda = \{(i,j) \in \mathbb{Z}^2 \mid 1 \le i \le \ell, i \le j \le \lambda_i + i - 1\}$.

- **Example.** Let $\lambda = (4,3,1)$.

$D_\lambda =$  , $SD_\lambda =$ 

- Shifted tableau $T = (t_{ij})$ of shape λ over X is a filling of SD_λ obtained by putting $t_{ij} \in X$ into (i,j) box of SD_λ. $ST_\lambda(X)$ is the set of all shifted tableaux of shape λ.

Example. $\lambda = (4,3,1)$, $T =$

t_{11}	t_{12}	t_{13}	t_{14}
	t_{22}	t_{23}	t_{24}
		t_{33}	

Semi-standard marked shifted tableau

- $\mathbb{N}' = \{1', 1, 2', 2, 3', 3, \ldots\}$
  with the total ordering $1' < 1 < 2' < 2 < \cdots$.
- Semi-standard marked shifted tableau $T = (t_{ij})$ of shape λ is a filling of SD_λ obtained by putting $t_{ij} \in \mathbb{N}'$ into (i, j) box of SD_λ such that
  - the entries in each row are weakly increasing from left to right.
  - the entries in each column are weakly increasing from top to bottom.
  - each row has at most one marked i for every $i = 1, 2, \ldots$.
  - each column has at most one unmarked i for every $i = 1, 2, \ldots$.
  - there are no marked entries on the main diagonal.

1	1	2'	3'
	2	2	3
		3	

We denote by $PSSYT_\lambda$ the set of all semi-standard marked shifted tableaux of shape λ.

Tableau expression for Schur P-function

Tableau expression

$$P_\lambda(\boldsymbol{x}) = \sum_{T \in PSSYT_\lambda} \boldsymbol{x}^T,$$

where $\boldsymbol{x}^T = \prod_{(i,j)\in SD(\lambda)} x_{|t_{ij}|}$ with $|i| = |i'| = i$ for $i \in \mathbb{N}$.

Example. $\lambda = (3,1), \boldsymbol{x} = (x_1, x_2)$. Then

$$PSSYT_\lambda = \left\{ \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array}, \begin{array}{|c|c|c|} \hline 1 & 1 & 2' \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array}, \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2' & 2 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} \right\}$$

$$P_\lambda(\boldsymbol{x}) = x_1^3 x_2 + x_1^2 x_2^2 + x_1^2 x_2^2 + x_1 x_2^3.$$

Semi-standard marked shifted tableau

- $\mathbb{N}' = \{1', 1, 2', 2, 3', 3, \ldots\}$
  with the total ordering $1' < 1 < 2' < 2 < \cdots$.
- Semi-standard marked shifted tableau $T = (t_{ij})$ of shape λ is a filling of SD_λ obtained by putting $t_{ij} \in \mathbb{N}'$ into (i, j) box of SD_λ such that
  - the entries in each row are weakly increasing from left to right.
  - the entries in each column are weakly increasing from top to bottom.
  - each row has at most one marked i for every $i = 1, 2, \ldots$.
  - each column has at most one unmarked i for every $i = 1, 2, \ldots$.
  - ~~there are no marked entries on the main diagonal~~.

1	1	$2'$	$3'$
	$2'$	2	3
		3	

We denote by $QSSYT_\lambda$ the set of all semi-standard marked shifted tableaux of shape λ.

Tableau expression for Schur Q-function

Tableau expression

For $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$ and $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$,

$$Q_\lambda(\boldsymbol{x}) = \sum_{T \in QSSYT_\lambda} \boldsymbol{x}^T,$$

where $\boldsymbol{x}^T = \prod_{(i,j) \in SD(\lambda)} x_{|t_{ij}|}$ with $|i| = |i'| = i$ for $i \in \mathbb{N}$.

Example. $\lambda = (3,1), \boldsymbol{x} = (x_1, x_2)$. Then for $i{=}i$ or i'

$$QSSYT_\lambda = \left\{ \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array}, \begin{array}{|c|c|c|} \hline 1 & 1 & 2' \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array}, \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2' & 2 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} \right\}$$

$$Q_\lambda(\boldsymbol{x}) = 4(x_1^3 x_2 + x_1^2 x_2^2 + x_1^2 x_2^2 + x_1 x_2^3).$$

$$Q_\lambda(\boldsymbol{x}) = 2^\ell P_\lambda(\boldsymbol{x}).$$

Schur P-type and Q-type multiple zeta function

- $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$: strict partition
- $\boldsymbol{s} = (s_{ij}) \in ST_\lambda(\mathbb{C})$: variables

$$\boldsymbol{s} = \begin{array}{|c|c|c|c|} \hline s_{11} & s_{12} & s_{13} & s_{14} \\ \hline \multicolumn{1}{c|}{} & s_{21} & s_{22} & s_{23} \\ \cline{2-4} \multicolumn{2}{c|}{} & s_{31} & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array} \in ST_\lambda(\mathbb{C}), \qquad M = \begin{array}{|c|c|c|c|} \hline m_{11} & m_{12} & m_{13} & m_{14} \\ \hline \multicolumn{1}{c|}{} & m_{21} & m_{22} & m_{23} \\ \cline{2-4} \multicolumn{2}{c|}{} & m_{31} & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}$$

Schur P or Schur Q multiple zeta function

We define Schur P or Q multiple zeta function associated with λ by

$$\zeta_\lambda^P(\boldsymbol{s}) = \sum_{M \in PSSYT_\lambda} \frac{1}{M^{\boldsymbol{s}}}, \quad \zeta_\lambda^Q(\boldsymbol{s}) = \sum_{M \in QSSYT_\lambda} \frac{1}{M^{\boldsymbol{s}}},$$

where $M^{\boldsymbol{s}} = \prod_{(i,j) \in SD_\lambda} |m_{ij}|^{s_{ij}}$

Example

Example. $\lambda=(3,1), \boldsymbol{s}=\begin{array}{|c|c|c|}\hline s_{11}&s_{12}&s_{13}\\\hline \multicolumn{1}{c|}{}&s_{21}&\multicolumn{1}{c}{}\\\cline{2-2}\end{array}\in ST_\lambda(\mathbb{C})$.

$$PSSYT_\lambda=\left\{\begin{array}{|c|c|c|}\hline 1&1&1\\\hline \multicolumn{1}{c|}{}&2&\multicolumn{1}{c}{}\\\cline{2-2}\end{array},\ \begin{array}{|c|c|c|}\hline 1&1&2'\\\hline \multicolumn{1}{c|}{}&2&\multicolumn{1}{c}{}\\\cline{2-2}\end{array},\ \begin{array}{|c|c|c|}\hline 1&1&2\\\hline \multicolumn{1}{c|}{}&2&\multicolumn{1}{c}{}\\\cline{2-2}\end{array},\ \begin{array}{|c|c|c|}\hline 1&2'&2\\\hline \multicolumn{1}{c|}{}&2&\multicolumn{1}{c}{}\\\cline{2-2}\end{array},\ \cdots\right\}$$

$$QSSYT_\lambda=\left\{\begin{array}{|c|c|c|}\hline {\color{red}1}&1&1\\\hline \multicolumn{1}{c|}{}&{\color{red}2}&\multicolumn{1}{c}{}\\\cline{2-2}\end{array},\ \begin{array}{|c|c|c|}\hline {\color{red}1}&1&2'\\\hline \multicolumn{1}{c|}{}&{\color{red}2}&\multicolumn{1}{c}{}\\\cline{2-2}\end{array},\ \begin{array}{|c|c|c|}\hline {\color{red}1}&1&2\\\hline \multicolumn{1}{c|}{}&{\color{red}2}&\multicolumn{1}{c}{}\\\cline{2-2}\end{array},\ \begin{array}{|c|c|c|}\hline {\color{red}1}&2'&2\\\hline \multicolumn{1}{c|}{}&{\color{red}2}&\multicolumn{1}{c}{}\\\cline{2-2}\end{array},\ \cdots\right\}$$

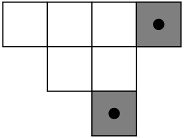
$$\zeta_\lambda^P(\boldsymbol{s})=$$

$$\frac{1}{1^{s_{11}}1^{s_{12}}1^{s_{13}}2^{s_{21}}}+\frac{1}{1^{s_{11}}1^{s_{12}}2^{s_{13}}2^{s_{21}}}+\frac{1}{1^{s_{11}}1^{s_{12}}2^{s_{13}}2^{s_{21}}}+\frac{1}{1^{s_{11}}2^{s_{12}}2^{s_{13}}2^{s_{21}}}+$$

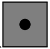
$$\zeta_\lambda^Q(\boldsymbol{s})={\color{red}4}\times$$

$$\left(\frac{1}{1^{s_{11}}1^{s_{12}}1^{s_{13}}2^{s_{21}}}+\frac{1}{1^{s_{11}}1^{s_{12}}2^{s_{13}}2^{s_{21}}}+\frac{1}{1^{s_{11}}1^{s_{12}}2^{s_{13}}2^{s_{21}}}+\frac{1}{1^{s_{11}}2^{s_{12}}2^{s_{13}}2^{s_{21}}}+\right.$$

For a strict partition $\lambda = (\lambda_1, \cdots, \lambda_\ell)$,
$C_\lambda \subset SD_\lambda$: the set of all corners of λ.



Example.

$C_{(4,2,1)} = \{(1,4), \ \ (3,3)\}$. ( $\in C_\lambda$)

Lemma

$W_\lambda^Q :=$

$$\left\{ \boldsymbol{s} = (s_{ij}) \in ST_\lambda(\mathbb{C}) \;\middle|\; \begin{array}{l} \Re(s_{ij}) \geq 1 \text{ for } {}^\forall(i,j) \in SD_\lambda \setminus C_\lambda \\ \Re(s_{ij}) > 1 \text{ for } {}^\forall(i,j) \in C_\lambda \end{array} \right\}.$$

Then, $\zeta_\lambda^P(\boldsymbol{s})$ and $\zeta_\lambda^Q(\boldsymbol{s})$ converge absolutely if $\boldsymbol{s} \in W_\lambda^Q$.

Pfaffian expression for ζ_λ^Q

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a strict partition.

$$W_\lambda^{Q,\mathrm{diag}} := \{\boldsymbol{s} \in W_\lambda^Q | s_{i+k,j+k} = s_{i,j} \text{ for } {}^\forall k \in \mathbb{Z}\}$$

Theorem(N.–Takeda, 2025)

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a strict partition with an even integer ℓ and $\lambda_i \geq 0$. Then for $\boldsymbol{s} = (s_{ij}) \in W_\lambda^{Q,\mathrm{diag}}$,

$$\zeta_\lambda^Q(\boldsymbol{s}) = \mathrm{pf}(U_\lambda),$$

where $U_\lambda = (u_{ij})$ is an $\ell \times \ell$ upper triangular matrix with

$u_{ij} = \zeta_{(\lambda_i,\lambda_j)}^Q(\boldsymbol{s}_{(\lambda_i,\lambda_j)})$ and $\boldsymbol{s}_{(\lambda_i,\lambda_j)} = \begin{array}{|c|c|c|c|c|} \hline s_{ii} & \cdots & \cdots & \cdots & s_{it_i} \\ \hline \multicolumn{1}{c|}{} & s_{jj} & \cdots & s_{jt_j} & \multicolumn{1}{c}{} \\ \cline{2-4} \end{array}$ where

$t_i = i + \lambda_i - 1$.

Example

Example. $\lambda = (3,2,1)$. Assume that $\boldsymbol{s} = (s_{ij}) \in W_\lambda^{Q,\mathrm{diag}}$. Let $s_{ij} = z_{j-i}$ by a given sequence $(z_k)_{k\in\mathbb{Z}}$. Then

$$\zeta_\lambda^Q(\boldsymbol{s}) = \zeta^Q_{(3,2,1)}\left(\begin{array}{|c|c|c|}\hline z_0 & z_1 & z_2 \\ \hline \multicolumn{1}{c|}{} & z_0 & z_1 \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & z_0 \\ \cline{3-3}\end{array}\right)$$

$$= \mathrm{pf}\begin{pmatrix} 0 & \zeta^Q_{(3,2)}\left(\begin{array}{|c|c|c|}\hline z_0 & z_1 & z_2 \\ \hline \multicolumn{1}{c|}{} & z_0 & z_1 \\ \cline{2-3}\end{array}\right) & \zeta^Q_{(3,1)}\left(\begin{array}{|c|c|c|}\hline z_0 & z_1 & z_2 \\ \hline \multicolumn{1}{c|}{} & z_0 & \multicolumn{1}{|c}{} \\ \cline{2-2}\end{array}\right) & \zeta^Q_{(3)}\left(\begin{array}{|c|c|c|}\hline z_0 & z_1 & z_2 \\ \hline\end{array}\right) \\ 0 & 0 & \zeta^Q_{(2,1)}\left(\begin{array}{|c|c|}\hline z_0 & z_1 \\ \hline \multicolumn{1}{c|}{} & z_0 \\ \cline{2-2}\end{array}\right) & \zeta^Q_{(2)}\left(\begin{array}{|c|c|}\hline z_0 & z_1 \\ \hline\end{array}\right) \\ 0 & 0 & 0 & \zeta^Q_{(1)}\left(\begin{array}{|c|}\hline z_0 \\ \hline\end{array}\right) \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For an *admissible piece* $A(a,b)=(\underbrace{1,\dots,1}_{a-1},b+1)$, the dual index set $A(a,b)^\dagger$ is defined $A(b,a)=(\underbrace{1,\dots,1}_{b-1},a+1)$.

When we write an index set k as

$$\mathrm{k}=(A_1,A_2,\dots,A_m),$$

we define the *dual* index set of k:

$$\mathrm{k}^\dagger=(A_m^\dagger,A_{m-1}^\dagger,\dots,A_1^\dagger).$$

Theorem (Duality formula for Multiple zeta values)

$$\zeta(\mathrm{k})=\zeta(\mathrm{k}^\dagger).$$

Duality formula for Schur multiple zeta values

Use an admissible piece $A = (\underbrace{1,\ldots,1}_{a-1}, b+1)^t$. Let

$$I_\delta^D = \{\boldsymbol{s} \in W_\delta^{diag} | \text{write } \boldsymbol{s} \text{ in tems of } A_{ij}, A_{ij} = A_{pq} \text{ if } j-i = q-p\}.$$

Example. $\mathrm{k} = \begin{array}{|c|c|c|}\hline A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & \\ \cline{1-2} A_{31} & & \\ \cline{1-1} \end{array} \leftrightarrow \mathrm{k}^\dagger = \begin{array}{ccc} & & A_{31}^\dagger \\ & A_{22}^\dagger & A_{21}^\dagger \\ A_{13}^\dagger & A_{12}^\dagger & A_{11}^\dagger \end{array}$

Theorem(N.–Ohno)

For $\boldsymbol{k} \in I_\delta^D$, there exists $\boldsymbol{k}^\dagger$ such that

$$\zeta_\delta(\boldsymbol{k}) = \zeta_{\delta^\dagger}(\boldsymbol{k}^\dagger),$$

where $\delta^\dagger$ is a shape of $\boldsymbol{k}^\dagger$

Example. Let $\lambda = (3, 2, 1)$.

$$\zeta_\lambda\left(\begin{array}{|c|c|c|}\hline 2 & 2 & 3 \\ \hline 4 & 2 & \multicolumn{1}{|c}{} \\ \cline{1-2} 4 & \multicolumn{2}{|c}{} \\ \cline{1-1}\end{array}\right) = \zeta_\lambda\left(\begin{array}{ccc} & & \multicolumn{1}{|c|}{1} \\ \cline{3-3} & & \multicolumn{1}{|c|}{1} \\ \cline{3-3} & & \multicolumn{1}{|c|}{2} \\ \cline{3-3} & & \multicolumn{1}{|c|}{1} \\ \cline{3-3} & & \multicolumn{1}{|c|}{1} \\ \cline{2-3} & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\ \cline{1-3} \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\ \cline{1-3} \multicolumn{1}{|c|}{2} & & \\ \cline{1-1}\end{array}\right).$$

The proof strategy for the example

Example. Let $\lambda = (3,2,1)$ and
$$\boldsymbol{k} = \begin{array}{|c|c|c|}\hline 2 & 2 & 3 \\ \hline 4 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,.$$

Then, we have

$$\zeta_\lambda(\boldsymbol{k}) = \begin{vmatrix} \begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline 4 \\ \hline\end{array} & \begin{array}{|c|}\hline 2 \\ \hline 2 \\ \hline 4 \\ \hline 4 \\ \hline\end{array} & \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 2 \\ \hline 4 \\ \hline 4 \\ \hline\end{array} \\ \begin{array}{|c|}\hline 2 \\ \hline\end{array} & \begin{array}{|c|}\hline 2 \\ \hline 2 \\ \hline\end{array} & \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 2 \\ \hline\end{array} \\ 0 & 1 & \begin{array}{|c|}\hline 3 \\ \hline\end{array} \end{vmatrix}.$$

We write $\begin{array}{|c|}\hline a_0 \\ \hline \vdots \\ \hline\end{array}$ shortly for

$$\zeta_\lambda\left(\begin{array}{|c|}\hline a_0 \\ \hline \vdots \\ \hline\end{array}\right) = \zeta(a_0, \ldots).$$

Applying the Duality formulafor multiple zeta values, we have

$$A = \begin{array}{|c|}\hline 1\\\hline 1\\\hline 2\\\hline 1\\\hline 1\\\hline 2\\\hline 2\\\hline\end{array}\,, \quad B = \begin{array}{|c|}\hline 1\\\hline 1\\\hline 2\\\hline 1\\\hline 1\\\hline 2\\\hline 2\\\hline 2\\\hline\end{array}\,, \quad C = \begin{array}{|c|}\hline 1^2\\\hline 2\\\hline 1^2\\\hline 2\\\hline 2\\\hline 2\\\hline 1\\\hline 2\\\hline\end{array}\,,$$

$$\zeta_\lambda(\boldsymbol{k}) = \begin{vmatrix} A & B & C \\ \begin{array}{|c|}\hline 2\\\hline\end{array} & \begin{array}{|c|}\hline 2\\\hline 2\\\hline\end{array} & \begin{array}{|c|}\hline 2\\\hline 2\\\hline 1\\\hline 2\\\hline\end{array} \\ 0 & 1 & \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \end{vmatrix}.$$

From the property of determinant of matrix and the Jacobi-Trudi formula for skew type, we have

$$
\zeta_\lambda\left(\begin{array}{|c|c|c|}\hline 2&2&3\\\hline 4&2&\multicolumn{1}{c}{}\\\cline{1-2} 4&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\right)
=
\left|\begin{array}{ccc}
\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} & \begin{array}{|c|}\hline 2\\\hline 2\\\hline 1\\\hline 2\\\hline\end{array} & C\\
1 & \begin{array}{|c|}\hline 2\\\hline 2\\\hline\end{array} & B\\
0 & \begin{array}{|c|}\hline 2\\\hline\end{array} & A
\end{array}\right|
=
\zeta_\lambda\left(\begin{array}{ccc}
\cline{3-3}
 & & \multicolumn{1}{|c|}{1} \\ \cline{3-3}
 & & \multicolumn{1}{|c|}{1} \\ \cline{3-3}
 & & \multicolumn{1}{|c|}{2} \\ \cline{3-3}
 & & \multicolumn{1}{|c|}{1} \\ \cline{3-3}
 & & \multicolumn{1}{|c|}{1} \\ \cline{2-3}
 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\ \cline{1-3}
\multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\ \cline{1-3}
\multicolumn{1}{|c|}{2} & & \\ \cline{1-1}
\end{array}\right).
$$

$$s_\lambda(1^{-k}, 2^{-k,\cdots}) = \zeta_\lambda\left(\begin{array}{|c|c|c|}\hline k & k & k \\ \hline k & k & \\ \cline{1-2} k & & \\ \cline{1-1}\end{array}\right) = \zeta_\lambda\left(\begin{array}{ccc} & & 1 \\ & & \vdots \\ & 1 & 2 \\ & \vdots & 1 \\ & 2 & \vdots \\ & 1 & 2 \\ & \vdots & 1 \\ & 2 & \vdots \\ & 1 & 2 \\ 1 & \vdots & \\ \vdots & 2 & \\ 2 & & \end{array}\right).$$

Thank you