Newton Polytopes of Dual Schubert Polynomials

Serena An, Katherine Tung, and Yuchong Zhang

Dual Schubert Polynomials

In the (strong) Bruhat order on the symmetric group S_n, let the edge $u \lessdot ut_{ab}$ have weight

$$m(u \lessdot ut_{ab}) := x_a + x_{a+1} + \cdots + x_{b-1},$$

and let the chain $C = (u_0 \lessdot u_1 \lessdot \cdots \lessdot u_\ell)$ have weight

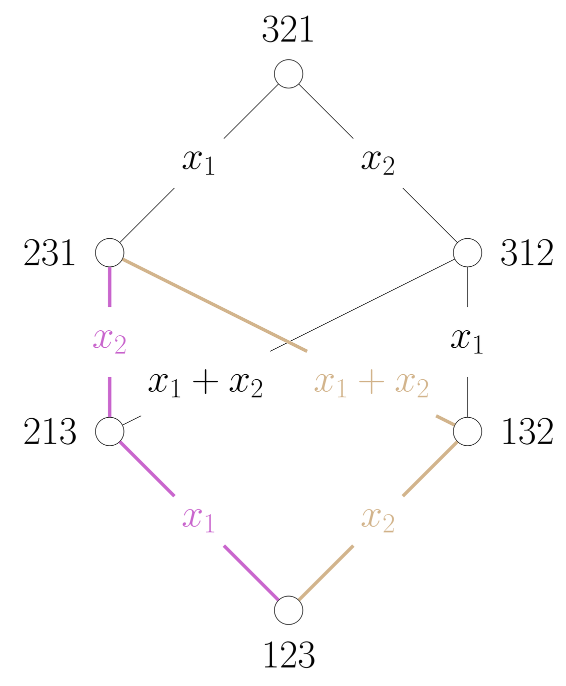
$$m_C := m(u_0 \lessdot u_1) m(u_1 \lessdot u_2) \cdots m(u_{\ell-1} \lessdot u_\ell).$$

Definition [BGG73, PS09]

For $w \in S_n$, the *dual Schubert polynomial* D^w is defined by

$$D^w(x_1, \ldots, x_{n-1}) := \frac{1}{\ell(w)!} \sum_C m_C(x_1, \ldots, x_{n-1}),$$

where $\ell(w)$ denotes the Coxeter length of w, and the sum is over all saturated chains C from id to w.



$$D^{231} = \frac{1}{2!}(x_1 x_2 + x_2(x_1 + x_2)).$$

Newton Polytopes

- For a tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, let x^α denote the monomial $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{R}[x_1, \ldots, x_n]$. We call α the *exponent vector* of x^α.
- Let $f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha x^\alpha \in \mathbb{R}[x_1, \ldots, x_n]$ be a polynomial. The *support* of f, denoted supp(f), is the set of exponent vectors α of the nonzero terms of f.

Definition

The *Newton polytope* of a polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$, denoted Newton$(f)$, is the convex hull of supp(f) in $\mathbb{R}^n$.

- [MTY19] A polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ has *saturated Newton polytope (SNP)* if supp(f) consists of all integer points in Newton(f).
- Example: Since $D^{231} = x_1 x_2 + 0.5x_2^2 = x^{(1,1)} + 0.5x^{(0,2)}$, Newton$(D^{231})$ is the line segment from $(1, 1)$ to $(0, 2)$ in $\mathbb{R}^2$ so D^{231} has SNP.

Previous Results on SNP

Many polynomials are known to have SNP, such as

- Schur polynomials [Rad52],
- resultants [GKZ90],
- cycle index polynomials and Reutenauer's symmetric polynomials and Stembridge's symmetric polynomials and symmetric Macdonald polynomials [MTY19],
- key polynomials and Schubert polynomials [FMD18], and
- double Schubert polynomials [CRMM23].

Work of Huh, Matherne, Mészáros, and St. Dizier [HMMSD22] proved Lorentzian-ness, which implies SNP, for dual Schubert polynomials. We offer the first elementary proof of SNP for dual Schuberts by fully characterizing their supports.

Main Theorem (ATZ '24)

The support of the dual Schubert polynomial D^w is

$$\text{supp}(D^w) = \sum_{(a,b) \in \text{Inv}(w)} \{e_a, e_{a+1}, \ldots, e_{b-1}\},$$

where the right-hand side is a Minkowski sum of sets of elementary basis vectors. The sum is over pairs of indices (a, b) for which there is an inversion in w.

Proof Outline

We say that D^w has *single chain Newton polytope (SCNP)* if there exists a saturated chain C in the interval $[\text{id}, w]$ such that supp(m_C) = supp(D^w). Such a saturated chain C is called a *dominant chain* of the interval $[\text{id}, w]$. We show that for each $w \in S_n$, there exists a dominant chain, so D^w has SCNP. We also show that SCNP implies SNP, completing the proof of SNP. Any dominant chain has weight $\prod_{(a,b) \in \text{Inv}(w)} (x_a + x_{a+1} + \cdots + x_{b-1})$, yielding the characterization in our theorem.

Corollaries of the Main Theorem

Corollary 1. D^w has SNP.

The *generalized permutahedron* $P^z_n(\{z_I\})$ associated to the collection of real numbers $\{z_I\}$ for $I \subseteq [n]$, is given by

$$P^z_n(\{z_I\}) = \left\{ t \in \mathbb{R}^n : \sum_{i \in I} t_i \geq z_I \text{ for } I \neq [n], \sum_{i=1}^n t_i = z_{[n]} \right\}.$$

Corollary 2. The Newton polytope of D^w is a generalized permutahedron.

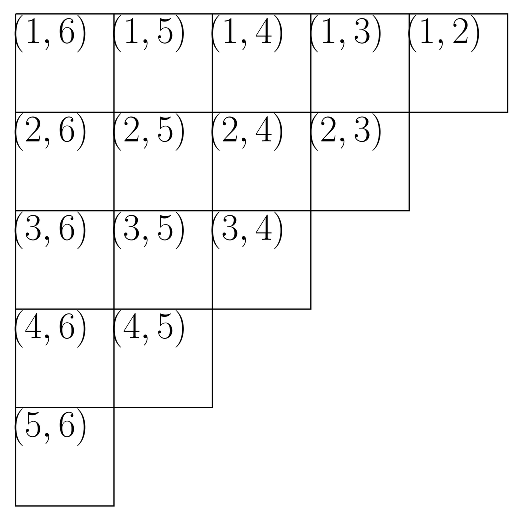
Proposition. ([Mur03, Theorem 4.15], [HMMSD22]) A homogeneous polynomial f has M-convex support if and only if f has SNP and Newton(f) is a generalized permutahedron.

Corollary 3. D^w has M-convex support.

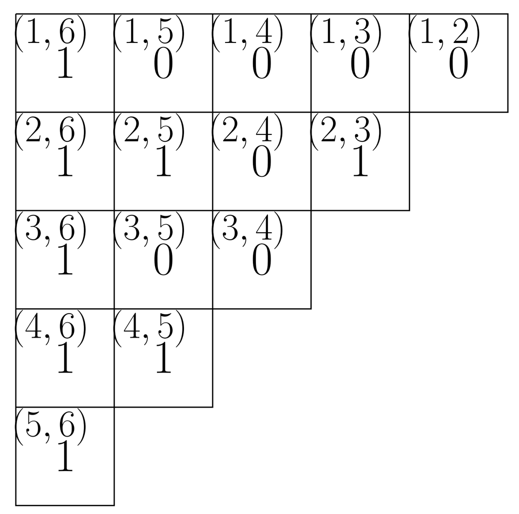
Corollary 4 (ATZ '24). The vertices of Newton(D^w) are $\{\alpha \in \mathbb{Z}^{n-1}_{\geq 0} \mid x^\alpha$ has coeff. 1 in $\prod_{(a,b) \in \text{Inv}(w)} (x_a + x_{a+1} + \cdots + x_{b-1})\}$.

Characterizing Vertices of Newton(D^w)

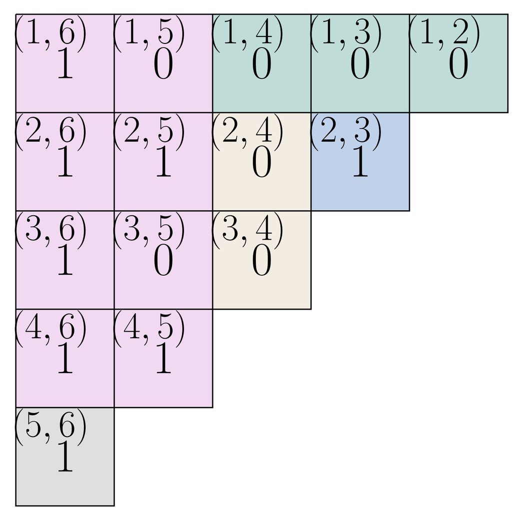
1. Construct a Young diagram $(n-1, n-2, \ldots, 1)$, and in the ith row of the diagram for $1 \leq i \leq n-1$, label the boxes from left to right by $(i, n), (i, n-1), \ldots, (i, i+1)$.
2. In each box (i, j), write $\mathbb{1}_{(i,j) \in \text{Inv}(w)}$.
3. Construct all tilings of the staircase by $n-1$ rectangles.
4. For each tiling, sum the entries of each rectangle and write the sum at the bottom right corner. Reading the summands from top to bottom gives a vertex of Newton(D^w).



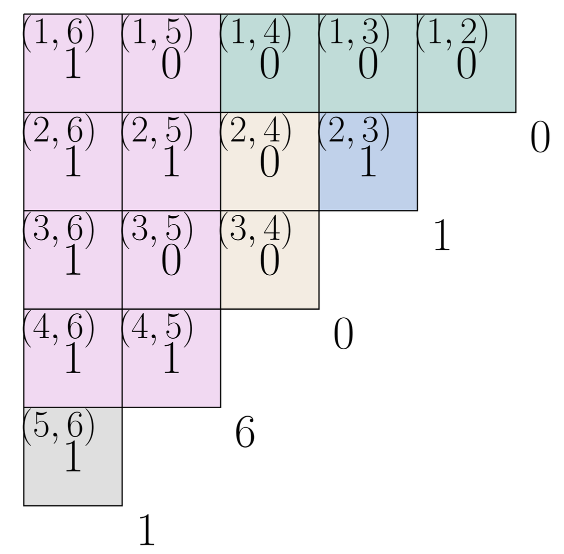
Step 1: Build a staircase Young diagram with $n = 6$.



Step 2: When $w = 253641$, the above boxes are filled with 1's.



Step 3: We consider a tiling by $n - 1$ rectangles.



Step 4: We find that Newton(D^{253641}) has vertex $(0, 1, 0, 6, 1)$.

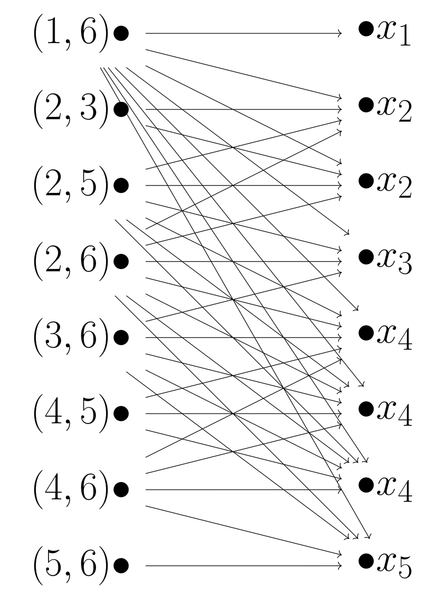
The Vanishing Problem for D^w

Given a Schubert polynomial $\mathfrak{S}_w = \sum_{\alpha \in \mathbb{Z}_{\geq 0}} c_{\alpha,w} x^\alpha$ for $w \in S_n$, Adve, Robichaux, and Yong give a polynomial-time algorithm to determine, given some α, whether $c_{\alpha,w} = 0$ [ARY21]. We prove an analogous result for dual Schubert polynomials.

Theorem (ATZ '24)

For $w \in S_n$ and $\alpha \in \mathbb{Z}^{n-1}_{\geq 0}$, there is an $O(n^5)$ algorithm to determine whether $\alpha \in \text{supp}(D^w)$.

We construct a certain bipartite graph with inversions of w as left vertices and variables of D^w as right vertices. The vanishing problem reduces to determining if a maximum matching with $\ell(w)$ edges exists, which we can do in $O(n^5)$ time.



The network testing the term $x_1 x_2^2 x_3 x_4^3 x_5$ in D^{253641}.

References

[ARY21] Anshul Adve, Colleen Robichaux, and Alexander Yong. An efficient algorithm for deciding vanishing of Schubert polynomial coefficients. *Adv. Math.*, 383:Paper No. 107669, 38, 2021.

[BGG73] Bernstein, Gelfand, and Gelfand. Schubert cells and cohomology of the spaces G/P. *Russian Mathematical Surveys*, 28:1–26, 1973.

[CRMM23] Castillo, Cid Ruiz, Mohammadi, and Montano. Double Schubert polynomials do have saturated Newton polytopes. *Forum Math Sigma*, 11:e100, 2023.

[FMD18] Fink, Mézśaros, and St. Dizier. Schubert polynomials as integer point transforms of generalized permutahedra. *Advances in Math*, 332:465–475, 2018.

[GKZ90] Gelfand, Kapranov, and Zelevinsky. Newton polytopes of the classical resultant and discriminant. *Advances in Math*, 84:237–254, 1990.

[HMMSD22] June Huh, Jacob P. Matherne, Karola Mészáros, and Avery St. Dizier. Logarithmic concavity of Schur and related polynomials. *Trans. Amer. Math. Soc.*, 375(6):4411–4427, 2022.

[MTY19] Cara Monical, Neriman Tokcan, and Alexander Yong. Newton polytopes in algebraic combinatorics. *Selecta Mathematica*, 25, 2019.

[Mur03] Kazuo Murota. *Discrete Convex Analysis*. Society for Industrial and Applied Mathematics, 2003.

[PS09] Alexander Postnikov and Richard P. Stanley. Chains in the Bruhat order. *J. Algebraic Combin.*, 29(2):133–174, 2009.

[Rad52] Richard Rado. An inequality. *J. London Math. Soc.*, 27:1–6, 1952.

Acknowledgements

This research was conducted through the University of Minnesota Combinatorics and Algebra REU and supported by NSF grant DMS-2053288 and the D.E. Shaw group. We thank Ayah Almousa, Casey Appleton, Meagan Kenney, Pavlo Pylyavskyy, Vic Reiner, Shiyun Wang, and Alex Yong.