Colored hooks and poset structure of cylindric diagrams

Kento Nakada [1] Takeshi Suzuki [2] Yoshitaka Toyosawa [3]

Cylindric diagrams

Fix $\omega = (m, -\ell) \in \mathbb{Z}_{\geqq 1} \times \mathbb{Z}_{\leqq -1}$ and define $\mathcal{C}_\omega = \mathbb{Z}^2/\mathbb{Z}\omega$.

The "cylinder" $\mathcal{C}_\omega$ admits a poset structure induced from the order on $\mathbb{Z}^2$ given by

$$(a,b) \leq (a',b') \iff a \geqq a' \text{ and } b \geqq b' \text{ (as integers).}$$

Definition 1.

(i) A non-trivial order filter of $\mathcal{C}_\omega$ is called a *cylindric diagram*.

(ii) A finite order ideal of a cylindric diagram is called a *cylindric skew diagram*. Equivalently, a skew diagram is a set difference θ/η of two diagrams $\theta \supset \eta$.

Remark 2. For a partition $\lambda = (\lambda_1, \dots, \lambda_m)$ with $\lambda_1 - \lambda_m \leqq \ell$, one can associate a cylindric diagram

$$\mathring{\lambda} := \pi(\boldsymbol{\lambda}),$$

where $\pi : \mathbb{Z}^2 \to \mathcal{C}_\omega$ is the natural projection and $\boldsymbol{\lambda}$ denotes the semi-infinite Young diagram

$$\boldsymbol{\lambda} = \{(a,b) \in \mathbb{Z}^2 \mid 1 \leqq a \leqq m,\ b \leqq \lambda_i\}.$$

Conversely, any cylindric dagram can be obtained this way. Fig.1 indeicates s cylindric diagram associated with $\lambda = (5,3,3,1)$.

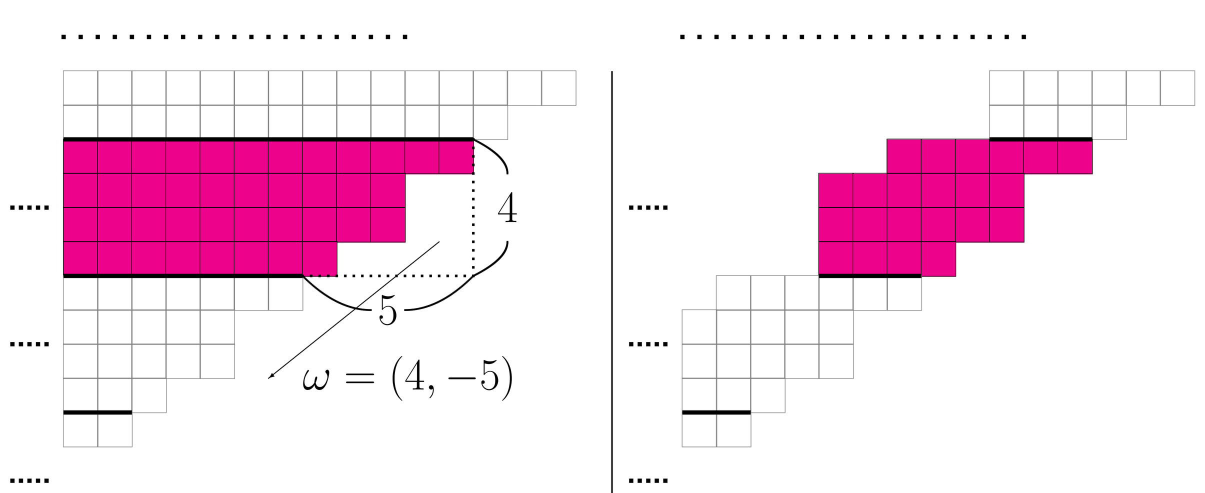


Fig.1 A cylindric diagram | A cylindric skew diagram.

Content map and bottom set

Let $\omega = (m, -\ell) \in \mathbb{Z}_{\geqq 1} \times \mathbb{Z}_{\leqq -1}$ and put $\kappa = m + \ell$. Let $\theta \subset \mathcal{C}_\omega$ be a cylindric diagram.

Define the *content map* (or *coloring map*) by

$$\mathbf{c} : \theta \to \mathbb{Z}/\kappa\mathbb{Z}, \quad \mathbf{c}(a,b) = b - a \pmod{\kappa}.$$

For $i \in \Gamma$, let b_i denote the minimum element in $\mathbf{c}^{-1}(i)$ and define the *bottom set* Γ of θ by

$$\Gamma = \{b_i \mid i \in \mathbb{Z}/\kappa\mathbb{Z}\}.$$

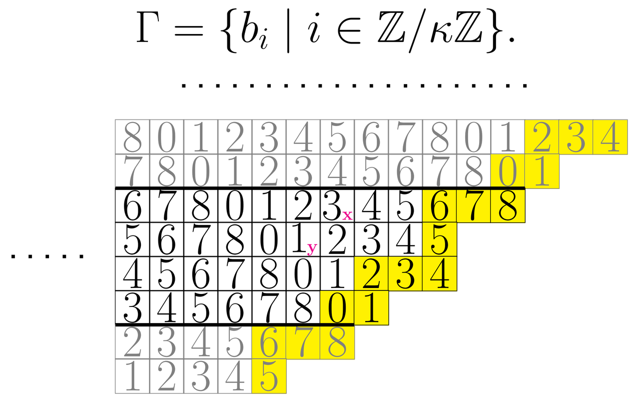


Fig. 1. Contents and bottom sets

We identify the set $\mathbb{Z}/\kappa\mathbb{Z}$ with the Dynkin diagram of type $A^{(1)}_{\kappa-1}$, and consider the associated root system:

- $R = R(A^{(1)}_{\kappa-1})$: the set of real roots, for which we have a decomposition $R = R_+ \sqcup R_-$.
- $W = W(A^{(1)}_{\kappa-1}) = \langle s_i \mid i \in \mathbb{Z}/\kappa\mathbb{Z}\rangle$: the (affine) Weyl group
- $\Pi = \{\alpha_i \mid i \in \mathbb{Z}/\kappa\mathbb{Z}\}$: the set of simple roots.

Colored hook length

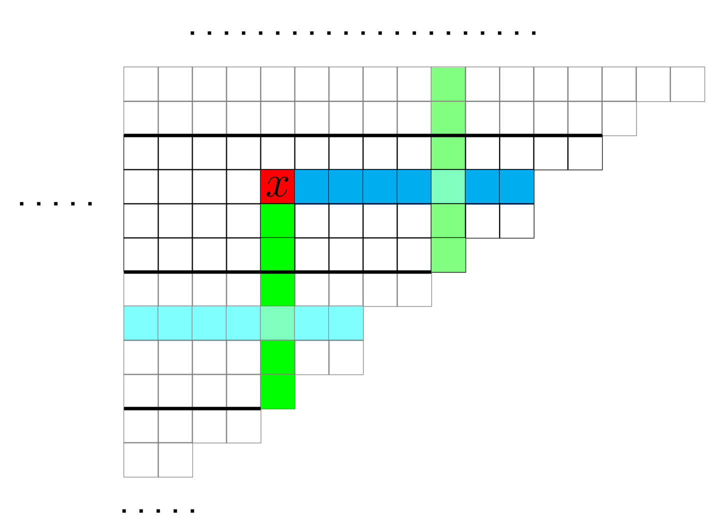
Definition 3. For $x \in \theta$, put

$$\begin{aligned} \mathrm{Hook}(x) &= \{x\} \sqcup \mathrm{Arm}(x) \sqcup \mathrm{Leg}(x) \quad \text{(a multiset), where} \\ \mathrm{Arm}(x) &= \{x + (0,k) \in \theta \mid k \in \mathbb{Z}_{\geqq 1}\}, \\ \mathrm{Leg}(x) &= \{x + (k,0) \in \theta \mid k \in \mathbb{Z}_{\geqq 1}\}, \end{aligned}$$

and define

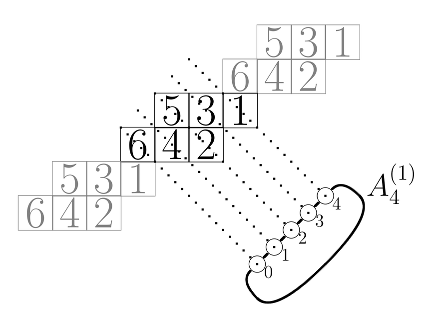
$$\mathbf{h}(x) = \sum_{y \in \mathrm{Hook}(x)} \alpha_{\mathbf{c}(y)}, \qquad h(x) = |\mathrm{Hook}(x)|.$$

We call $\mathbf{h}(x)$ the *colored hook length* and $h(x)$ the *hook length* at x. It is straightforward to see that the colored hook length is a positive real root $\mathbf{h}(x) \in R_+$.



Standard tableaux

For a cylindric skew diagram θ/η with n elements, an order preserving bijection $\mathbf{t} : \theta/\eta \longrightarrow \{1, 2, \cdots, n\}$ is called a (reverse) *standard tableau*. The set of standard tableaux is denoted by $\mathrm{ST}(\theta/\eta)$.



Representation-theoretic background

- $\mathrm{Irr}(\mathbb{C}\mathfrak{S}_n\text{-mod}) \leftrightarrow$ {Young diagrams}
- $\mathrm{Irr}(H^{\mathbb{C}}_n\text{-mod}^{cs}) \leftrightarrow$ {skew Young diagrams}
- $\mathrm{Irr}(H^K_n\text{-mod}^{cs}) \leftrightarrow$ {cylindric skew Young diagrams}, where $\mathrm{ch}K = \kappa$ (prime).
- $\dim D(\theta/\eta) = \sharp\mathrm{ST}(\theta/\eta)$.

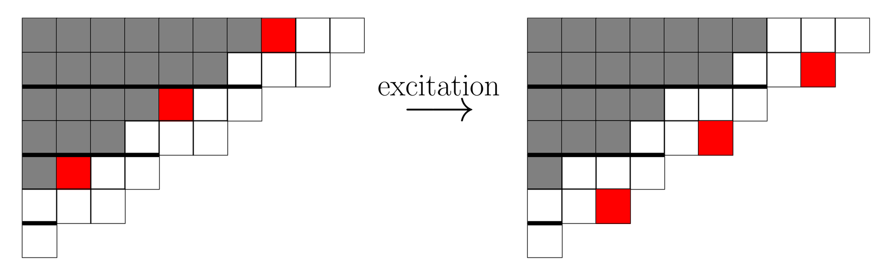
Here, H^K_n stands for the degenerate affine Hecke algebra over the field K, $H^K_n\text{-mod}^{cs}$ for the the category of completely splittable modules of H^K_n and $D(\theta/\eta)$ for the simple module corresponding to the cylindric skew Young diagram θ/η.

Excited diagrams

Let θ be a cylindric diagram and $D \subset \theta$.

If $(i,j) \in D$ and $(i+1,j), (i,j+1), (i+1,j+1) \in \theta \setminus D$ (set-minus), then an *elementary excitation* with respect to θ is the replacement:

$$D \longmapsto D \setminus \{(i,j)\} \cup \{(i+1,j+1)\}.$$



Let θ/η be a cylindric skew diagram. An *excited diagram* of θ/η is a subset of θ obtained from $D = \eta$ after a sequence of elementary excitations. We denote the set of the excited diagrams by $\mathcal{E}(\theta/\eta)$.

Cylindric hook formula

Conjecture 4 ([2]). *Let θ/η be a cylindric skew diagram with $|\theta/\eta| = n$. Then*

$$|\mathrm{ST}(\theta/\eta)| = n! \sum_{D \in \mathcal{E}(\theta/\eta)} \prod_{x \in \theta\setminus D} \frac{1}{h(x)},$$

where $h(x)$ is hook length on θ.

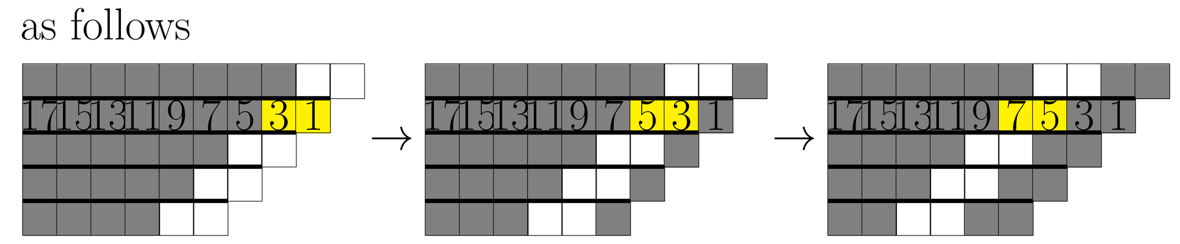
Remark 5. Cylindric hook formula is confirmed in the following cases:

(i) $\theta = \mathring{\lambda}, \eta = \mathring{\mu}$ with $\lambda = (n)$, $\mu = (0)$, and $\omega = (1, -\ell)$.

(ii) $\theta = \mathring{\lambda}, \eta = \mathring{\mu}$ with $\lambda = (\overbrace{\ell+1, \ell+1, \dots, \ell+1}^{m})$, $\mu = (\underbrace{\ell, \ell, \dots, \ell}_{m-1}, 0)$, and $\omega = (m, -\ell)$.

Remark 6. In general, $\mathcal{E}(\theta/\eta)$ is an infinite set, and hence RHS is an infinite sum. On the other hand, if ℓ is large enough, then θ/η can be regarded as a classical (finite) skew Young diagram, and the set $\mathcal{E}(\theta/\eta)$ is finite. In this case, our cojectural formula coincides with skew hook formula (Naruse-Okada, Morales-Pak-Panova).

Example 7. Let $\lambda = (2)$, $\mu = (0)$, $l = 1$ $(n = 2)$. Then we have just one element $\boxed{2}\boxed{1}$ in $\mathrm{ST}(\mathring{\lambda}/\mathring{\mu})$. The RHS is computed as follows



$$(\mathrm{RHS}) = 2!\left(\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots\right)$$
$$= 2\cdot\frac{1}{2}\left(\left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots\right) = 1.$$

Root realization via colored hook length

Let $\theta/\eta \subset \mathcal{C}_\omega$ be a cylindric skew diagram. For $\mathbf{t} \in \mathrm{ST}(\theta/\eta)$, we associate a Weyl group element

$$w_{\theta/\eta} = s(\boxed{1})s(\boxed{2})\dots s(\boxed{n}),$$

where $s(\boxed{k}) = s_{\mathbf{c}(\mathbf{t}^{-1}(k))}$. It turns out that $w_{\theta/\eta}$ is independent of the chioce of $\mathbf{t}$.

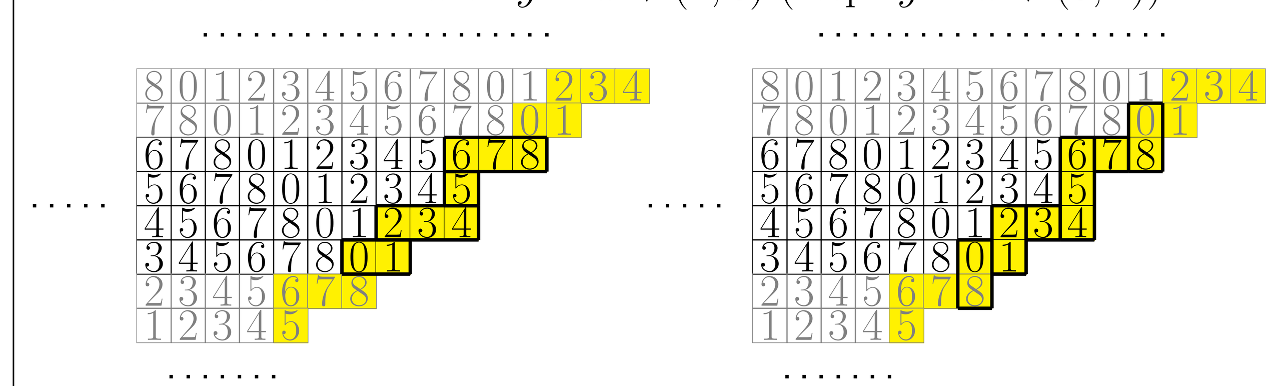
Define $R(w_{\theta/\eta}) := R_+ \cap w_{\theta/\eta}R_-$ and $R(w_\theta) := \bigcup_{\eta\subset\theta} R(w_{\theta/\eta})$

Theorem 8 ([1]). *The correspondence $x \mapsto \mathbf{h}(x)$ induces the following bijections:*

$$\theta/\eta \xrightarrow{\cong} R(w_{\theta/\eta}), \qquad \theta \xrightarrow{\cong} R(w_\theta).$$

Poset structure of cylindric diagrams

Put $\Pi_\theta = \Pi^{\rightarrow}_\theta \sqcup \Pi^{\downarrow}_\theta$, where $\Pi^{\rightarrow}_\theta$ (resp. $\Pi^{\downarrow}_\theta$) is the set consisting of the elements of the form $\mathbf{h}(x) - \mathbf{h}(y)$ for an adjacent pair (x,y) in the bottom set Γ with $y = x + (1,0)$ (resp. $y = x + (0,1)$).



$$\Pi^{\rightarrow}_\theta = \left\{\begin{matrix}\alpha_0 + \alpha_1, \alpha_2 + \alpha_3 + \alpha_4, \\ \alpha_5, \alpha_6 + \alpha_7 + \alpha_8\end{matrix}\right\}, \Pi^{\downarrow}_\theta = \left\{\begin{matrix}\alpha_8 + \alpha_0, \alpha_1 + \alpha_2, \alpha_3, \\ \alpha_4 + \alpha_5 + \alpha_6, \alpha_7\end{matrix}\right\}.$$

Definition 9. Define the partial order $\trianglelefteq$ on $R(w_\theta)$ by

$$\alpha \trianglelefteq \beta \iff \beta - \alpha \in \mathbb{Z}_{\geqq 0}\Pi_\theta = \left\{\sum_{\gamma\in\Pi_\theta} k_\gamma\gamma \;\middle|\; k_\gamma \in \mathbb{Z}_{\geqq 0}\ (\forall\gamma \in \Pi_\theta)\right\}.$$

Theorem 10 ([1]). *For $x, y \in \theta$, we have*

$$x < y \iff \mathbf{h}(x) \trianglelefteq \mathbf{h}(y)$$

In other words, the maps in Theorem 8 induces poset isomorphisms $(\theta/\eta, \leq) \cong (R(w_{\theta/\eta}), \trianglelefteq)$ and $(\theta, \leq) \cong (R(w_\theta), \trianglelefteq)$.

For the incomparable cells **x** and **y** in Fig. 1, we have $\mathbf{h}(x) - \mathbf{h}(y) = \alpha_6 + \alpha_8 \notin \mathbb{Z}_{\geq 0}\Pi_\theta$, and hence, $\mathbf{h}(y) \ntrianglelefteq \mathbf{h}(y)$, while we have $\mathbf{h}(y) <^{\mathrm{or}} \mathbf{h}(y)$.

Remark 11. The order $\trianglelefteq$ gives an alternative description of the heap order defined by Stembridge and Nakada.

References

[1] K. Nakada, T. Suzuki, and Y. Toyosawa, *Poset structure concerning cylindric diagrams*, Electron. J. Comb. **31** (2024), P1.56.

[2] T. Suzuki and Y. Toyosawa, *On hook formulas for cylindric skew diagrams*, Math J. Okayama Univ. **64** (2022), 191–213.