Permutation representations of classical Weyl groups on mod q lattices

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§1. Quasi-polynomials

- R : a commutative ring.

Definition (quasi-polynomial).

A map $f : \mathbb{Z}_{(>0)} \longrightarrow R$ is a **quasi-polynomial** if there exist a period $\tilde{n} \in \mathbb{Z}_{>0}$ and constituents $g_1, \ldots, g_{\tilde{n}} \in R[t]$ such that

$$f(q) = g_r(q) \ \text{ if } \ q \equiv r \mod \tilde{n} \quad (1 \le r \le \tilde{n}).$$

- A quasi-polynomial F has **gcd-property** if it satisfies

$$\gcd\{\tilde{n}, i\} = \gcd\{\tilde{n}, j\} \implies g_i = g_j.$$

Example. —Some counting function are quasi-polynomials

- For a rational polytope $\mathcal{P}$ in $\mathbb{R}^\ell$,

$$L_{\mathcal{P}}(q) := \#(q\mathcal{P} \cap \mathbb{Z}^\ell)$$

  is a quasi-polynomial. (**Ehrhart quasi-polynomial**)

- For a hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$,

$$\chi^{\text{quasi}}_{\mathcal{A}}(q) := \#\left\{ \bar{x} \in (\mathbb{Z}/q\mathbb{Z})^\ell \;\middle|\; s_H(x) \not\equiv 0 \pmod{q} \text{ for all } H \in \mathcal{A} \right\}$$

  is a quasi-polynomial with gcd-property. ($H = \ker s_H$ for $s_H \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^\ell, \mathbb{Z})$) (**characteristic quasi-polynomial**)

§2. Equivariant versions

Equivariant Ehrhart theory (Stapledon, Adv. Math. (2011))

- A generalization of Ehrhart theory.
- Study polytopes that exhibit symmetries.
- Counting lattice points in polytopes with group representations.

- Γ : a finite group acting linearly on a lattice L.
- $\mathcal{P}$: a Γ-invariant lattice polytope.

Theorem (Stapledon, Adv. Math. (2011)).

The permutation character

$$\chi_{q\mathcal{P}}(\gamma) = \#(q\mathcal{P} \cap L)^\gamma = \#\left\{ x \in q\mathcal{P} \cap L \;\middle|\; \gamma x = x \right\}.$$

is a quasi-polynomial in q.

- We want to introduce an equivariant theory of characteristic quasi-polynomial.
  $\longrightarrow$ Here, we consider the case $\mathcal{A} = \emptyset$.

- For a group Γ and a lattice L, let $\rho : \Gamma \longrightarrow \mathrm{GL}(L)$ be a group homomorphism.
- R_γ : the representation matrix of $\rho(\gamma)$:

$$\rho(\gamma) : L \longrightarrow L; \quad x \longmapsto xR_\gamma.$$

- For $q \in \mathbb{Z}_{>0}$. let $L_q := L/qL$ and consider the prmutation character χ_{L_q} of L_q:

$$\chi_{L_q}(\gamma) = \#L_q^\gamma = \#\left\{ x \in L_q \;\middle|\; \rho(\gamma)(x) = x \text{ in } L_q \right\}.$$

- $\chi_1, \ldots, \chi_k$: irreducible characters of Γ.
- $m_L(\chi_i; q) := (\chi_i, \chi_{L_q})$: the multiplicity of χ_i in χ_{L_q}.

Theorem (U.–Yoshinaga, arXiv:2409.01084).

The permutation character χ_{L_q} is a quasi-polynomial in q with gcd-property. More explicitly,

$$m_L(\chi_i; q) = \frac{1}{\#\Gamma} \sum_{\gamma \in \Gamma} \chi_i(\gamma) \left(\prod_{j=1}^{r(\gamma)} \gcd\{e_{\gamma,j}, q\} \right) q^{\ell - r(\gamma)},$$

where

- $r(\gamma) = \mathrm{rank}(R_\gamma - I)$ (I : the identity matrix);
- $e_{\gamma,1}, \ldots, e_{\gamma,r(\gamma)}$: the elementary divisors of $R_\gamma - I$.

§3. Results for standard lattices

- $B := \{e_1, \ldots, e_\ell\}$: the standard basis for $\mathbb{R}^\ell$.
- Φ : a root system of type $A_{\ell-1}$, B_ℓ, C_ℓ or D_ℓ.
- $L := \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_\ell$.
- $\Gamma := W(\Phi)$: the Weyl group of Φ, acting as signed permutations for B.
  - $W(A_{\ell-1})$: the symmetric group $\mathfrak{S}_\ell$.
  - $W(B_\ell) = W(C_\ell)$: the hyperoctahedral group $\mathfrak{H}_\ell$.
  - $W(D_\ell)$: a subgroup $\widetilde{\mathfrak{H}}_\ell$ of index 2 of $\mathfrak{H}_\ell$.
- (λ_w, μ_w) : the cycle type of $w \in \Gamma$, the pair of integer partitions satisfying $|\lambda_w| + |\mu_w| = \ell$. ※Note thet if $w \in \mathfrak{S}_\ell$, then $\mu_w = \emptyset$.

Theorem (Theorem 2.3 in the Extended Abstract).

For any $w \in \Gamma$,

$$\chi_{L_q}(w) = g(q)^{\ell(\mu_w)} q^{\ell(\lambda_w)}.$$

Furthermore, for an irreducible character χ of Γ,

$$m_L(\chi; q) = \frac{1}{\#\Gamma} \sum_{w \in \Gamma} \chi(w) g(q)^{\ell(\mu_w)} q^{\ell(\lambda_w)},$$

where $\ell(\cdot)$ denotes the length of an integer partition, and $g(q) := \gcd\{2, q\}$.

- χ^λ : the irreducible character of $\mathfrak{S}_\ell$ defined by an integer partition λ of ℓ.
- $\chi^{\lambda,\mu}$: the irreducible character of $\mathfrak{H}_\ell$ defined by integer partitions λ, μ satisfying $|\lambda| + |\mu| = \ell$.
- $\widetilde{\chi}^{\lambda,\mu}$: the irreducible character of $\widetilde{\mathfrak{H}}_\ell$ defined by integer partitions λ, μ satisfying $|\lambda| + |\mu| = \ell$.

Theorem (Cor. 2.5 & Thm. 2.6, 2.7 & 2.8 in the Extended Abstract).

- (type $A_{\ell-1}$, (Littlewood, Molchanov)) $m_L(\chi^\lambda; q) = \dfrac{\chi^\lambda(1)}{\ell!} \displaystyle\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (q - i + j)$.
- (type B_ℓ, C_ℓ) $m_L(\chi^{\lambda,\mu}; q) = m\left(\chi^\lambda; \dfrac{q + g(q)}{2}\right) \cdot m\left(\chi^\mu; \dfrac{q - g(q)}{2}\right)$.
- (type D_ℓ) $m_L(\widetilde{\chi}^{\lambda,\mu}; q) = \begin{cases} m_L(\chi^{\lambda,\mu}; q) + m_L(\chi^{\mu,\lambda}; q) & \text{if } \lambda \neq \mu; \\ m_L(\chi^{\lambda,\lambda}; q) & \text{if } \lambda = \mu. \end{cases}$

§4. Results for coroot lattices

- $\check{Q} := \check{Q}(\Phi)$: the coroot lattice of Φ.
  - $\check{Q}(A_{\ell-1}) = \mathbb{Z}(e_1 - e_2) \oplus \cdots \oplus \mathbb{Z}(e_{\ell-1} - e_\ell)$.
  - $\check{Q}(B_\ell) = \mathbb{Z}(e_1 - e_2) \oplus \cdots \oplus \mathbb{Z}(e_{\ell-1} - e_\ell) \oplus \mathbb{Z}(2e_\ell)$.
  - $\check{Q}(C_\ell) = \mathbb{Z}(e_1 - e_2) \oplus \cdots \oplus \mathbb{Z}(e_{\ell-1} - e_\ell) \oplus \mathbb{Z}e_\ell = L$.
  - $\check{Q}(D_\ell) = \mathbb{Z}(e_1 - e_2) \oplus \cdots \oplus \mathbb{Z}(e_{\ell-1} - e_\ell) \oplus \mathbb{Z}(e_{\ell-1} + e_\ell) = \check{Q}(B_\ell)$.

Theorem (It follows from Lemma 3.1 & 3.3).

Let $w \in \Gamma$ and $\lambda_w = (\lambda_{w,1}, \ldots, \lambda_{w,\ell(\lambda_w)})$.

- (type $A_{\ell-1}$) $\chi_{\check{Q}_q}(w) = \gcd\{\lambda_{w,1}, \ldots, \lambda_{w,\ell(\lambda_w)}, q\} \cdot q^{-1} \cdot \chi_{L_q}(w)$.
- (type C_ℓ) $\chi_{\check{Q}_q}(w) = \chi_{L_q}(w)$.
- (type B_ℓ, D_ℓ)

$$\chi_{\check{Q}_q}(w) = \begin{cases} \chi_{L_q}(w) & \text{if } \lambda_w \text{ has at least an odd part } \& \ \mu_w = \emptyset; \\ g(q) \cdot \chi_{L_q}(w) & \text{if } \lambda_w \text{ has all even parts } \& \ \mu_w = \emptyset; \\ \dfrac{1}{g(q)} \cdot \chi_{L_q}(w) & \text{if } \lambda_w \text{ has at least an odd part } \& \ \mu_w \neq \emptyset; \\ \chi_{L_q}(w) & \text{if } \lambda_w \text{ has all even parts } \& \ \mu_w \text{ has all even/odd parts}; \\ \dfrac{\gcd\{4, q\}}{g(q)^2} \cdot \chi_{L_q}(w) & \text{if } \lambda_w \text{ has all even parts } \& \ \mu_w \text{ has both even and odd parts.} \end{cases}$$