Indecomposable polytopes & Rays of the submodular cone

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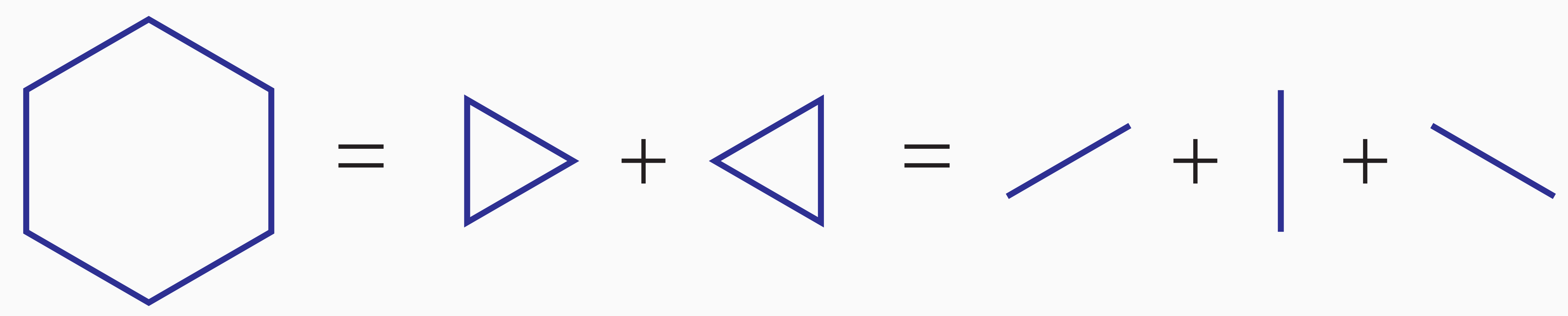
(In)decomposability of polytopes

Minkowski sum: $Q + R = \{q + r \,;\, q \in Q, r \in R\}$

Indecomposable: If $P = Q + R$, then $Q = \lambda P + t$ for some $\lambda > 0$, t



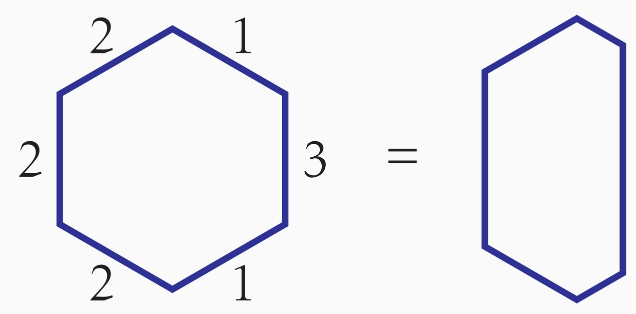
Question A: How to check if P is indecomposable?

Parameterizing deformations of P

Deformations of P: any Q s.t. $\lambda P = Q + R$ for some $\lambda > 0$ and R

$\longleftrightarrow$ edges of Q parallel to edges of P, but not same length

$\longleftrightarrow$ *edge-length vector*: λ_e for each edge e of P, $\lambda_e(Q) = \frac{\text{length}(e \in Q)}{\text{length}(e \in P)}$

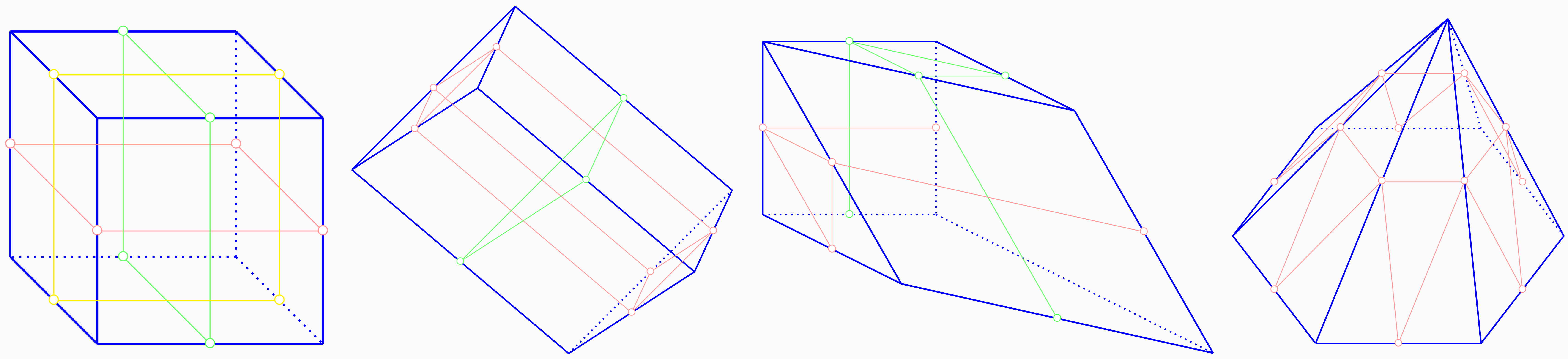


Cycle eqn.: if cycle $e_1, \dots, e_r$, then for all Q: $\lambda_{e_1}(Q)\vec{e}_1 + \dots + \lambda_{e_r}(Q)\vec{e}_r = 0$

Edge-length dependencies graph: new criteria for indecomposability

Dependent edges e, f: for all deformations Q of P, $\lambda_e(Q) = \lambda_f(Q)$ informally: "the length of e can be deduced from the length of f"

Examples of dependent edges: edges of a triangle; edges of a cycle whose convex hull is a simplex; edges opposite in a parallelogram or a trapezoid...



Graph of edge-length dependencies $ED(P)$: nodes = edges of P; arcs ef if e and f dependent edges

THM. $ED(P)$ is a clique $\iff$ $ED(P)$ is connected $\iff$ P is indecomposable $\dim \mathbb{DC}(P) \leq |cc(ED(P))|$

THM. $\exists$ subset S of dependent edges, forming a connected sub-graph of the graph of P, such that S touches every facet $\implies$ P indecomposable

Application: Re-deduce almost all previous indecomposability criteria **Application:** Study decomposability products of polytopes $P \times P'$

Edmonds's problem

Deformed permutahedron: all edge directions $e_i - e_j$

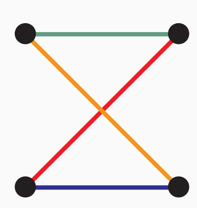
Edmonds '70 problem: Find (all) indecomposable deformed permutahedra!

Nguyen '86: give $2^{2^{n-3/2\log n + O(1)}}$ indecomposable deform. permutahedra via connected matroids

Question B: How to create indecomposable deformed permutahedra, not matroid polytopes?

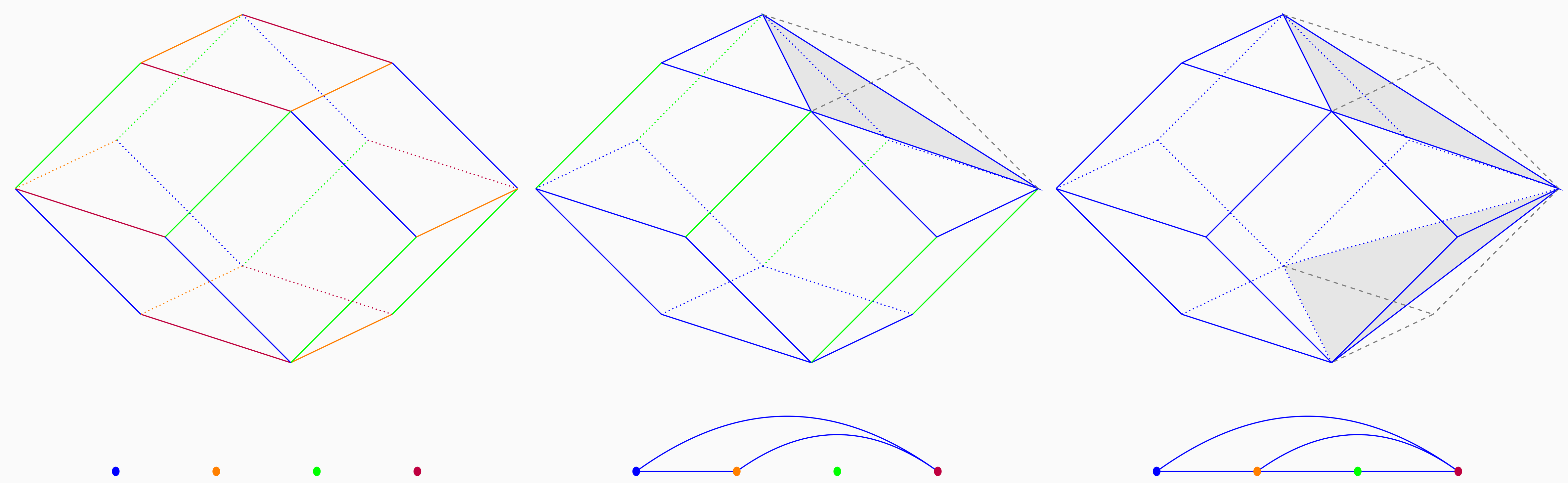
Idea: use *graphical zonotope* of graph G:

$$Z_G := \sum_{ij \in E} [e_i, e_j] \qquad K_{2,2} =$$



Vertices $Z_G \longleftrightarrow$ acyclic orientations of G

Truncating zonotopes



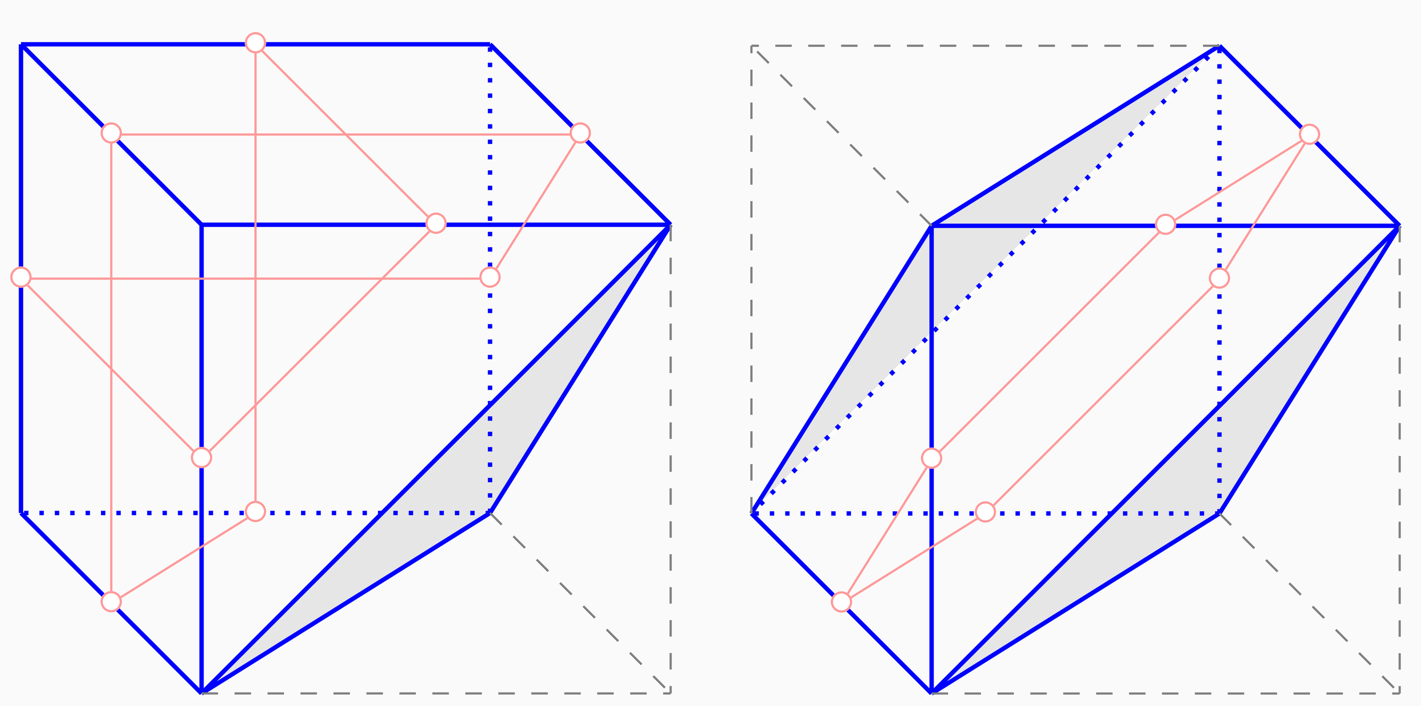
Application: Pick a zonotope Z + remove vertices = Z′ $\to$ easy to tell if Z′ is indecomposable

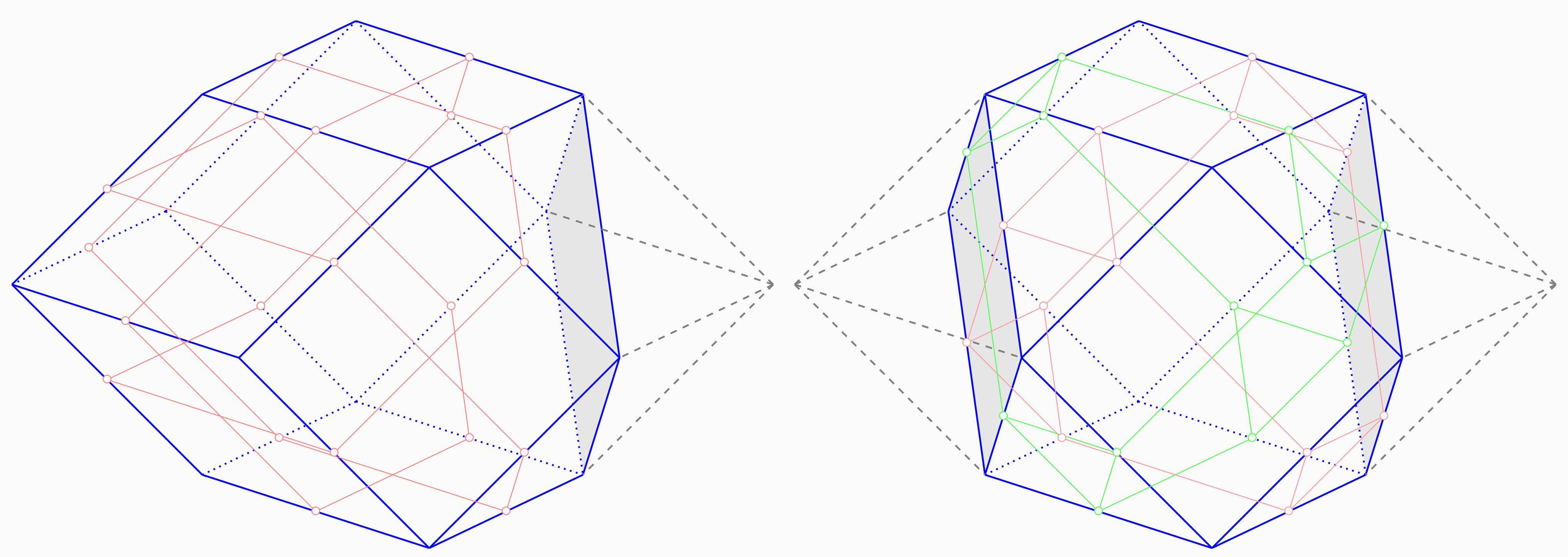
Idem for stacking vertices

New rays of the submodular cone: truncated graphical zonotopes of complete bipartite graphs

Works *only* when $G = K_{n,m}$ complete bipartite graph $Z^{-\circ}_{n,m}$, $Z^{\circ-\circ}_{n,m}$: graphical zonotope $Z_{K_{n,m}}$, remove 1 or 2 specific vertices

$$Z^{-\circ}_{n,m} := Z_{K_{n,m}} \cap \left\{x \in \mathbb{R}^{n+m} \,;\, \sum_{j=1}^{m} x_{b_j} \leq nm - 1\right\} \quad \text{and} \quad Z^{\circ-\circ}_{n,m} := Z^{-\circ}_{n,m} \cap \left\{x \in \mathbb{R}^{n+m} \,;\, \sum_{i=1}^{n} x_{a_i} \leq nm - 1\right\}$$





THM. $Z^{-\circ}_{n,m}$ and $Z^{\circ-\circ}_{n,m}$ are indecomposable deformed $(n+m)$-permutahedra, not matroid polytope except 5 cases

THM. There are $\geq 2\left\lfloor \frac{d-1}{2} \right\rfloor$ such indecomposable deformed d-permutahedra Soon on ArXiv: $\geq 2^{2^n}$ (Loho–Padrol–Poullot, other method)