Strongly nice property and Schur positivity of graphs

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Chromatic symmetric function

Definition 1 (Stanley, 1995)

Let G be a graph with vertex set $V(G) = \{v_1, \ldots, v_d\}$. Then the chromatic symmetric function of G is defined by Stanley as

$$X_G = \sum_{\kappa} x_{\kappa(v_1)} \cdots x_{\kappa(v_d)},$$

where $\kappa : V(G) \to \{1, 2, \ldots\}$ ranges over all proper colorings of G, i.e., $\kappa(u) \neq \kappa(v)$ for any edge $uv \in E(G)$.

- A graph G is Schur positive (or s-positive) if X_G is a nonnegative linear combination of Schur functions.

Schur positivity of claw-free graphs

Theorem 1 (Gasharov, 1996)

Incomparability graphs of $(3+1)$-free posets are Schur positive.

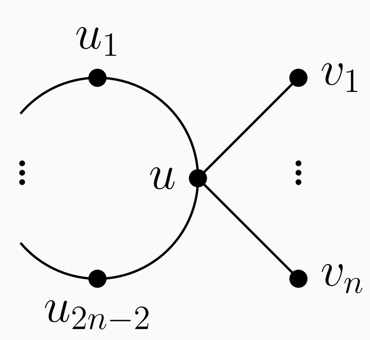
Conjecture 1 (Stanley, 1998)

All claw-free graphs (containing no induced subgraph isomorphic to the claw $K_{1,3}$) are Schur positive.

- Incomparability graphs of $(3+1)$-free posets are claw-free.

Non-Schur positivity of Squid graphs

- The squid graphs $Sq(2n-1; 1^n)$ defined by attaching n leaves to one vertex of a cycle C_{2n-1}.



Conjecture 2 (Wang and Wang, 2020)

The squid graph $Sq(2n-1; 1^n)$ is not Schur positive for $n \geq 3$.

Nice property

- A stable partition π of $G = (V, E)$ is a set partition of V such that each block of π is a stable set.
- The type of π is a partition of $|V|$ whose parts are the sizes of the blocks of π.
- A graph G is said to be nice if G has a stable partition of type λ, then G has a stable partition of type μ for each $\mu \leq \lambda$ (dominance order).

Theorem 2 (Stanley, 1998)

- Schur positive $\Longrightarrow$ nice;
- G is claw-free $\Longleftrightarrow$ G and all its induced subgraphs are nice.

- one can use the above result to prove the non-Schur positivity of certain chromatic symmetric functions by showing that they are not nice.

Strongly nice property

- A semi-ordered stable partition of $G = (V, E)$ is a stable partition such that the parts of the same size are ordered.

Theorem 3 (Stanley, 1995)

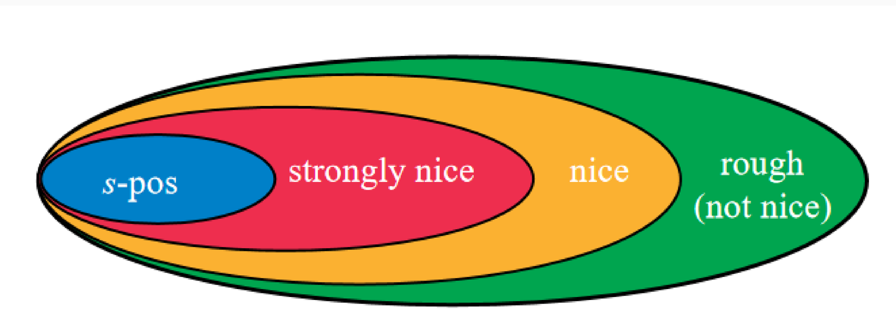
Let $\tilde{a}_\lambda$ be the number of semi-ordered stable partitions of G of type λ. Then

$$X_G = \sum_{\lambda} \tilde{a}_\lambda m_\lambda.$$

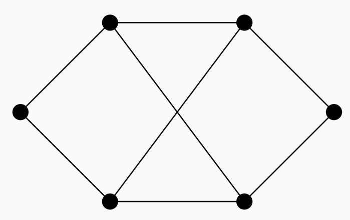
- A symmetric function f is nice if for any pair of partitions $\mu \leq \lambda$ in dominance order with $[m_\lambda]f > 0$ we have $[m_\mu]f > 0$.
- The definition coincides with the above one if f is a chromatic symmetric function.

Definition 2

A symmetric function f is said to be strongly nice if $[m_\mu]f \geq [m_\lambda]f$ whenever $\mu \leq \lambda$ in dominance order. A graph G is strongly nice if X_G is strongly nice.



- The following graph is strongly nice but is not Schur positive.



Proof of non-Schur positivity of squid graphs

Theorem 4

For $n \geq 3$, the squid graph $Sq(2n-1; 1^n)$ is not strongly nice. Moreover, $Sq(2n-1; 1^n)$ is not Schur-positive.

Proof(sketch).

- We show that

$$[m_{(n,n,n-1)}]X_{Sq(2n-1;1^n)} = 4\binom{2n-2}{n-1}$$

and

$$[m_{(n+1,n-1,n-1)}]X_{Sq(2n-1;1^n)} = 8\binom{2n-2}{n}.$$

- Thus for $n \geq 3$,

$$\frac{[m_{(n,n,n-1)}]X_{Sq(2n-1;1^n)}}{[m_{(n+1,n-1,n-1)}]X_{Sq(2n-1;1^n)}} = \frac{n}{2(n-1)} < 1.$$

- The following result shows that it is not enough to prove this result by means of the nice property.

Theorem 5

The squid graph $Sq(2n-1; 1^n)$ is nice for $n \geq 3$.

Proof(sketch).

- $\lambda = (2n-1, n-1, 1)$ is maximum in the types of all stable partitions of $Sq(2n-1; 1^n)$.
- There exists a stable partition of $Sq(2n-1; 1^n)$ with type μ for all $\mu \leq \lambda$.

Strongly nice property of claw-free graphs

Theorem 3

A graph G is claw-free if and only if G and all its induced subgraphs are strongly nice.

Proof(sketch).

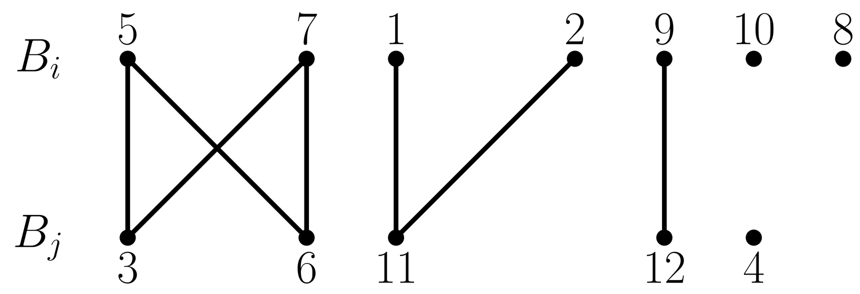
- The sufficiency follows directly from Stanley's result.
- Any induced subgraph of a claw-free graph is also claw-free.
- The necessity can be deduced from the following claim.

Claim. If a graph G is claw-free, then it is strongly nice.

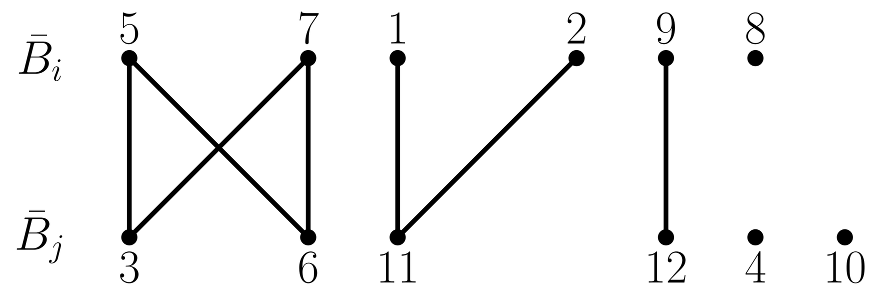
- To prove the strongly nice property of claw-free graphs, we only need to show that $\tilde{a}_\lambda \leq \tilde{a}_\mu$ for all partitions μ, λ with λ covering μ in dominance order.
- When restricting to two blocks of a stable partition, the induced graph is a bipartite graph.
- Any claw-free bipartite graph is composed by even cycles and odd paths.
- Let $\tilde{A}_\lambda$ denote the set of semi-ordered stable partitions of type λ.
- We construct an injection ϕ from $\tilde{A}_\lambda$ to $\tilde{A}_\mu$ by reversing one odd path.

Example 1

Let the semi-ordered stable partition B be shown as in the following figure (we only present B_i and B_j for convenience).



Then $W(B) = 1211$ and its image $\bar{B} = \phi(B)$ is exactly the semi-ordered stable partition shown in following figure. It follows that $W(\bar{B}) = 1212$ and $W(\hat{B}) = 1211 = W(B)$, which implies $\varphi(\bar{B}) = B$.



One open problem

- Let $\mathrm{inc}(B_n)$ be the incomparability graph of the Boolean lattice B_n.

Conjecture 3 (Griggs, 1988)

$\mathrm{inc}(B_n)$ is nice.

Conjecture 4 (Stanley, 1988)

$\mathrm{inc}(B_n)$ is Schur positive.

- Conjecture 3 is verified for $n \leq 5$ and Conjecture 4 is vefiried for $n \leq 4$ by computer.

Problem 1

Is $\mathrm{inc}(B_n)$ strongly nice?